FLORIDA INTERNATIONAL UNIVERSITY

Miami, Florida

# SOME ONE AND TWO PARAMTER ESTIMATORS FOR THE MULTICOLLINEAR LOGISTIC REGRESSION MODEL: THEORY, SIMULATION, AND APPLICATIONS

A thesis submitted in partial fulfillment of the
requirements for the degree of
MASTER OF SCIENCE
in
STATISTICS
by
Md Ariful Hoque

2023

To: Dean Michael Heithaus
College of Arts, Sciences and Education

This thesis, written by Md Ariful Hoque, and entitled Some One and Two Parameter Estimators for the Multicollinear Logistic Regression Model: Theory, Simulation, and Applications, having been approved in respect to style and intellectual content, is referred to you for judgment.

We have read this thesis and recommend that it be approved.

Florence George

Wensong Wu

B. M. Golam Kibria, Major Professor

Date of Defense: June 16, 2023

The thesis of Md Ariful Hoque is approved.

Dean Michael Heithaus
College of Arts, Sciences and Education

Andrés G. Gil
Senior Vice President for Research and Economic
Development and Dean of the University Graduate School

Florida International University, 2023

# DEDICATION

I dedicate this thesis to my parents. Without their patience, understanding, support, and most of all love, the completion of this work would not have been possible.

# ACKNOWLEDGMENTS

I wish to thank the members of my committee Dr. Florence George, Dr. Wensong Wu and Dr. B. M. Golam Kibria for their support, patience, and good humor. Their gentle but firm direction has been most appreciated. I would like to thank my major professor, Dr. B. M. Golam Kibria, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing my thesis. I could not have imagined having a better advisor and mentor for my thesis.

I have found my coursework throughout the Curriculum and Instruction program to be stimulating and thoughtful, providing me with the tools with which to explore both past and present ideas and issues.

## ABSTRACT OF THE THESIS

# SOME ONE AND TWO PARAMETER ESTIMATORS FOR THE MULTICOLLINEAR LOGISTIC REGRESSION MODEL: THEORY, SIMULATION, AND APPLICATIONS

by

Md Ariful Hoque

Florida International University, 2023

Professor B. M. Golam Kibria, Major Professor

The logistic regression model (LRM) is one of the most widely used models and plays an important role in analyzing and making predictions about binary data. The most popularly used estimation technique is the maximum likelihood estimator (MLE) in LRM. However, the MLE becomes unstable and gives misleading result in the presence of multicollinearity among the regressors. The ridge regression estimator has been used as an alternative method to solve the multicollinearity problem for both linear and non-linear regression models. The objective of this thesis is to propose some new estimators, namely Stein's estimators for ridge regression and Kibria and Lukman estimator (KLE) and compare their performance with some existing estimators, namely maximum likelihood estimator, ridge regression estimator, Liu estimator, almost unbiased ridge and Liu estimators, adjusted Liu estimator, James stein's estimator, Kibria and Lukman estimator, Dorugade estimator and Modified ridge estimator for the logistic regression model to solve the multicollinearity problem. The bias, covariance matrix and mean square error matrix for each of the estimators are provided. A Monte Carlo simulation has been conducted to compare the performance of different estimators. We consider the smaller MSE value as a performance criterion. From simulation study it is evident that all proposed estimators performed better than the maximum likelihood estimator. Finally, real life data is analyzed to illustrate the findings of the thesis. Some promising estimators are recommended for the practitioners.

## TABLE OF CONTENTS

CHAPTER PAGE

# LIST OF TABLES

TABLE PAGE

# LIST OF FIGURES

FIGURE PAGE

# ABBREVIATIONS AND ACRONYMS

| | |
|---|---|
| ALLE | Adjusted Logistic Liu Estimator |
| AULE | Almost Unbiased Liu Estimator |
| AURE | Almost Unbiased Ridge Estimator |
| LDE | Logistic Dorugade Estimator |
| LKLE | Logistic Kibria and Lukman Estimator |
| LLE | Logistic Liu Estimator |
| LMRT | Logistic Modified Ridge Estimator |
| LRE | Logistic Ridge Estimator |
| LRM | Linear Regression Model |
| LSE | Logistic James Stein's Estimator |
| LSKE | Logistic Stein's Kibria and Lukman Estimator |
| LSRE | Logistic Stein's Ridge Estimator |
| MLE | Maximum Likelihood Estimator |
| MMSE | Matrix of Mean Squared Error |
| MSE | Mean Squared Error |
| OLS | Ordinary Least Squares |

# Chapter 1

# Introduction

Multicollinearity is a problem that occurs in a model when the explanatory variables are linearly related which is described by Frisch (1934). It affects various regression models, including the linear regression model (LRM), logistic regression model, Poisson regression model, and Gamma regression model. In the linear regression model, we assume independence among predictor variables and the parameters are estimated using the ordinary least squares (OLS) estimator. However, when multicollinearity occurs, the OLS estimator yields incorrect signs with large standard errors and imprecise confidence intervals (Saleh and Kibria, 1993 and Kibria, 2003).

The logistic regression model, widely used in areas such as biostatistics, epidemiology, medicine, finance, and medical sciences, seeks to establish the relationship between a categorical dependent variable and one or more independent variables. The method of maximum likelihood estimator (MLE) is commonly employed to estimate coefficients in the logistic regression model. In practice, we assume orthogonality among the regressors or explanatory variables. Nevertheless, multicollinearity between the explanatory variables renders the MLE inefficient, leading to high standard errors and incorrect sign estimates.

To address the issue of multicollinearity, several methods have been proposed. One popular and recommended approach is the construction of ridge regression by Hoerl and Kennard (1970) for the linear regression model. The logistic regression model, it was first involving biasing parameters to overcome this multicollinearity problem. Noteworthy contributions include Kibria et al. (2012), Inan and Erdogan (2013), Nagarajah and Wijekoon (2015), Asar et al. (2017), Asar and Genc (2017), Varathan and Wijekoon (2018), and Lukman et al. (2020). Liu (1993) discussed another estimator, known as Liu estimator, which utilizes a parameter called $d$ to tackle multicollinearity issues. For the logistic regression models, the logistic Liu estimator was introduced by Mansson et al. (2011). Several authors have further enhanced the efficiency of the Liu estimators in logistic regression models, including Inan and Erdogan (2013), Siray et al. (2015), Asar and Genc (2016), Wu (2016), and Wu and Asar (2017). Lukman et al. (2020) introduced the modified one-parameter Liu estimator for a linear regression model to overcome the limitations of existing Liu estimators, while Amin et al. (2023) proposed a new adjusted logistic Liu estimator. Abonazel and Farghali (2019) developed new two-parameter estimators for the multinomial logistic models, Farghali et al. (2021) introduced two generalized estimators for logistic regression models with two parameters, and Yang and Chang (2010) proposed a new two-parameter estimator based on the Liu estimator and ridge regression estimator.

This thesis introduces two new Stein estimators for the logistic regression model. One estimator is based on the ridge regression estimator, and the other is based on K-L estimator by Lukman et al. (2023). Theoretical properties are discussed, and the performance of these new estimators is evaluated through a theoretical comparison with other estimators. The performance evaluation criteria employ the matrix mean squared error (MMSE) and the scalar mean squared error (MSE).

The organization of this thesis is as follows: Chapter 2 presents the statistical methodology including a theoretical MSE comparison among the estimators and a description of the estimation of the parameters $k$ and $d$. Chapter 3 encompasses a simulation study, while Chapter 4 focuses on the analysis of real-life data. Finally, chapter 5 provides a summary and concluding remarks.

## Chapter 2

## Statistical Methodology

In this chapter, we will focus on the logistic regression model and explore various types of estimators, their bias, covariance, and mean squared error (MSE). We organize as follows: section 2.1 discusses different types of existing estimators, bias, covariance, and MSE and comparison among those estimators in section 2.2.

### 2.1 Model and Some Existing Estimators

We consider the following logistics regression model given by

$$y_i = \pi_i + \epsilon_i \tag{2.1}$$

where $\epsilon_i$ $(i = 1,2, \dots, n)$ are disturbances which are distributed with $E(\epsilon_i) = 0$ and $Cov(\epsilon_i) = \pi_i(1 - \pi_i)$ where $\pi_i$ is the expectation of $y_i$ when the $i$th value of the dependent variable $(y_i)$ of the regression model is $Be(\pi_i)$ as

$$\pi_i = \frac{exp(x_i'\beta)}{1 + exp(x_i'\beta)} \tag{2.2}$$

where $x_i$ is the $i$th row of $X$ is an $n \times (p + 1)$ data matrix with $p$ explanatory (or independent) variables and $\beta$ is a $(p + 1) \times 1$ vector of coefficients. The maximum likelihood estimator (MLE) is the most common method of estimating the coefficient $\beta$ where the following log-likelihood should be maximized:

$$l = \sum_{i=1}^{N} y_i \, log(\pi_i) + \sum_{i=1}^{N} (1 - y_i) \, log(1 - \pi_i) \tag{2.3}$$

and setting the first derivative of the above equation to be equal to zero, and ML estimate can be found by solving this equation:

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{N} (y_i - \pi_i)x_i = 0 \tag{2.4}$$

Since this equation is nonlinear in $\beta$, we can solve it with Newton- Rapshon method. We use the following iterative weighted least square (IWLS) algorithm to estimate the coefficient $\beta$:

$$\hat{\beta}_{MLE} = \left(X'\widehat{W}X\right)^{-1}\left(X'\widehat{W}\hat{z}\right) \tag{2.5}$$

where $\widehat{W} = diag\ [(\hat{\pi}_i)(1 - \hat{\pi}_i)]$ and $z$ is a vector where the $i$th element equals

$$z_i = \log(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i(1 - \hat{\pi}_i)}$$

Let $Q$ be the orthogonal matrix whose columns constitute the eigen vectors of $X'WX$. Then the asymptotic covariance matrix of the ML estimator equals the inverse of the matrix of second derivatives (most referred to as the inverse of the Hessian matrix):

$$Cov\left(\hat{\beta}_{MLE}\right) = E\left(-\frac{\partial^2 l}{\partial\beta\partial\beta^2}\right) = \left(X'\widehat{W}X\right)^{-1} \tag{2.6}$$

and the MMSE and scalar MSE of the ML estimator equals:

$$MMSE\left(\hat{\beta}_{MLE}\right) = Q\Lambda^{-1}Q' \tag{2.7}$$

$$MSE\left(\hat{\beta}_{MLE}\right) = E\left(\hat{\beta}_{MLE} - \beta\right)'\left(\hat{\beta}_{MLE} - \beta\right) = tr(Q\Lambda^{-1}Q') = \textstyle\sum_{j=1}^{p} \frac{1}{\lambda_j} \tag{2.8}$$

where $\Lambda = diag\left(\lambda_1, \lambda_2, \dots, \lambda_p\right)$ and $\lambda_j$ is the $j$th eigenvalue of the matrix $X'\widehat{W}X$ matrix. It can be easily seen that the asymptotic variance becomes inflated when the independent variable is highly correlated since some of the eigen values will be small when $X'WX$ is ill-conditioned. The logistic ridge regression estimator is defined by Schaefer et al. (1984) as a straightforward extension of Hoerl and Kennard (1970) to solve the multicollinearity problem as

$$\hat{\beta}_{LRE} = \left(X'\widehat{W}X + kI\right)^{-1}\left(X'\widehat{W}X\right)\hat{\beta}_{MLE} \tag{2.9}$$

where $k$ is the biasing parameter $(k > 0)$ and identity matrix $I$. The bias, covariance matrix, and MMSE are.

$$Bias\left(\hat{\beta}_{LRE}\right) = -kQ\Lambda_k^{-1}\alpha \tag{2.10}$$

$$Cov\left(\hat{\beta}_{LRE}\right) = Q\Lambda_k^{-1}\Lambda\Lambda_k^{-1}Q' \tag{2.11}$$

$$\begin{aligned} MMSE\left(\hat{\beta}_{LRE}\right) &= Cov\left(\hat{\beta}_{LRE}\right) + Bias\left(\hat{\beta}_{LRE}\right)Bias\left(\hat{\beta}_{LRE}\right)' \\ &= Q\Lambda_k^{-1}\Lambda\Lambda_k^{-1}Q' + k^2Q\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}Q' \end{aligned} \tag{2.12}$$

where $\Lambda_k = diag\left(\lambda_1 + k, \lambda_2 + k, \dots, \lambda_p + k\right)$. Then the asymptotic MSE of the logistic ridge regression estimator equals:

$$MSE\left(\hat{\beta}_{LRE}\right) = tr(MMSE) = \sum_{j=1}^{p}\frac{\lambda_j}{\left(\lambda_j + k\right)^2} + \sum_{j=1}^{p}\frac{k^2\alpha_j^2}{\left(\lambda_j + k\right)^2} \tag{2.13}$$

where $\alpha = Q'\beta$. Now the asymptotic variance is not inflated since $\lambda_j$ is replaced by $\lambda_j + k$ in the denominator. And, as the asymptotic variance decreases the squared bias becomes larger at the same time as $k$ increases. So, the objective of logistic regression is to choose a value of $k$ such that the reduction in the variance term is greater than the increase of the squared bias.

Another estimator following Liu (1993), Urgan and Tez (2008), and Mansson et al. (2011) for the logistic regression which is defined as:

$$\hat{\beta}_{LLE} = \left(X'\widehat{W}X + I\right)^{-1}\left(X'\widehat{W}X + dI\right)\hat{\beta}_{MLE} \tag{2.14}$$

where $d$ is another biasing parameter $(0 < d < 1)$. The bias vector, covariance matrix and MMSE are given respectively by

$$Bias\left(\hat{\beta}_{LLE}\right) = -(1-d)Q\Lambda_I^{-1}\alpha \tag{2.15}$$

$$Cov\left(\hat{\beta}_{LLE}\right) = Q\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1}Q' \tag{2.16}$$

$$MMSE\left(\hat{\beta}_{LLE}\right) = Q\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1}Q' + (1-d)^2 Q\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}Q' \tag{2.17}$$

where $\Lambda_I = diag\left(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_p + 1\right)$ and $\Lambda_d = diag\left(\lambda_1 + d, \lambda_2 + d, \ldots, \lambda_p + d\right)$.

The scalar MSE in terms of eigen values is defined as:

$$MSE\left(\hat{\beta}_{LLE}\right) = \sum_{j=1}^{p}\frac{\left(\lambda_j + d\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2} + \sum_{j=1}^{p}\frac{(1-d)^2\alpha_j^2}{\left(\lambda_j + 1\right)^2} \tag{2.18}$$

Now, the almost unbiased ridge regression estimator for the logistics regression model proposed by Chang (2015) which is based on logistic ridge regression (Schaefer et al., 1984), and obtained as

$$\hat{\beta}_{AURE} = \left(I - k^2\left(X'\widehat{W}X + kI\right)^{-2}\right)\hat{\beta}_{MLE} \tag{2.19}$$

Bias vector, covariance matrix, MMSE and scalar MSE are given respectively:

$$Bias\left(\hat{\beta}_{AURE}\right) = -k^2 \Lambda_k^{-2} \alpha \tag{2.10}$$

$$Cov\left(\hat{\beta}_{AURE}\right) = Q(I - k^2 \Lambda_k^{-2}) \Lambda^{-1} (I - k^2 \Lambda_k^{-2}) Q' \tag{2.11}$$

$$MMSE\ \left(\hat{\beta}_{AURE}\right) = Q(I - k^2 \Lambda_k^{-2}) \Lambda^{-1} (I - k^2 \Lambda_k^{-2}) Q' + k^4 Q\ \Lambda_k^{-2} \alpha \alpha' \Lambda_k^{-2} Q' \tag{2.12}$$

$$MSE\left(\hat{\beta}_{AURE}\right) = \sum_{j=1}^{p} \frac{\left(\lambda_j + 2k\right)^2 \lambda_j}{\left(\lambda_j + k\right)^4} + \sum_{j=1}^{p} \frac{k^4 \alpha_j^2}{\left(\lambda_j + k\right)^4} \tag{2.13}$$

And almost unbiased Liu estimator for logistic regression model based on the Liu estimator (Mansson et al., 2011) proposed by Chang (2015) can be written as:

$$\hat{\beta}_{AULE} = \left(1 - (1 - d)^2 \left(X^{'} \widehat{W} X + I\right)^{-2}\right) \hat{\beta}_{MLE} \tag{2.14}$$

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively:

$$Bias\ \left(\hat{\beta}_{AULE}\right) = -(1 - d)^2 \Lambda_I^{-2} \alpha \tag{2.15}$$

$$Cov\left(\hat{\beta}_{AULE}\right) = Q(I - (1 - d)^2 \Lambda_I^{-2}) \Lambda^{-1} (I - (1 - d)^2 \Lambda_I^{-2}) Q' \tag{2.16}$$

$$\begin{aligned} MMSE\ \left(\hat{\beta}_{AULE}\right) &= Q(I - (1 - d)^2 \Lambda_I^{-2}) \Lambda^{-1} (I - (1 - d)^2 \Lambda_I^{-2}) Q' \\ &+ (1 - d)^4 Q \Lambda_I^{-2} \alpha \alpha' \Lambda_I^{-2} Q' \end{aligned} \tag{2.17}$$

$$MSE\left(\hat{\beta}_{AULE}\right) = \sum_{j=1}^{p} \frac{\left(\lambda_j + d\right)^2 \left(2 + \lambda_j - d\right)^2}{\lambda_j \left(\lambda_j + 1\right)^4} + \sum_{j=1}^{p} \frac{(1 - d)^4 \alpha_j^2}{\left(\lambda_j + 1\right)^4} \tag{2.18}$$

The adjusted logistic Liu estimator proposed by Amin et al. (2023) is defined as:

$$\hat{\beta}_{ALLE} = \left(X'\widehat{W}X + I\right)^{-1}\left(X'\widehat{W}X - dI\right)\hat{\beta}_{MLE} \tag{2.19}$$

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively.

$$Bias\left(\hat{\beta}_{ALLE}\right) = -(1+d)Q\Lambda_I^{-1}\alpha \tag{2.20}$$

$$Cov\left(\hat{\beta}_{ALLE}\right) = Q\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1}Q' \qquad 21)$$

$$MMSE\left(\hat{\beta}_{ALLE}\right) = Q\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1}Q' \qquad 22)$$

$$+ (1+d)^2 Q\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}Q'$$

$$MSE\left(\hat{\beta}_{ALLE}\right) = \sum_{j=1}^{p}\frac{\left(\lambda_j - d\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2} + \sum_{j=1}^{p}\frac{(1+d)^2\alpha_j^2}{\left(\lambda_j + 1\right)^2} \qquad 23)$$

James stein’s estimator for logistic regression model is defined as:

$$\hat{\beta}_{LSE} = c_s\hat{\beta}_{MLE} \tag{2.24}$$

where $0 < c_s < 1$ and for the selection of $c_s$, it can be obtained as:

$$tr\left(MSE(LSE)\right) = c_s^2 tr(MSE(\hat{\beta}_{MLE})) + (c_s - 1)^2\hat{\beta}_{MLE}{}'\hat{\beta}_{MLE}$$

$$\frac{\partial tr\left(MSE(LSE)\right)}{\partial c_s} = 2c_s tr(MSE(\hat{\beta}_{MLE)})) + 2(c_s - 1)\hat{\beta}_{MLE}{}'\hat{\beta}_{MLE} = 0$$

$$\Leftrightarrow c_s tr(MSE(\hat{\beta}_{MLE})) + c_s\hat{\beta}_{MLE}{}'\hat{\beta}_{MLE} = \hat{\beta}_{MLE}{}'\hat{\beta}_{MLE}$$

$$\Leftrightarrow c_s = \frac{\hat{\beta}_{MLE}{}'\hat{\beta}_{MLE}}{\hat{\beta}_{MLE}{}'\hat{\beta}_{MLE} + tr(MSE(\hat{\beta}_{MLE}))}$$

So, $c_s$ can be written as:

$$c_s = \frac{\hat{\beta}'_{MLE}\hat{\beta}_{MLE}}{\hat{\beta}'_{MLE}\hat{\beta}_{MLE} + tr\left(MSE(\hat{\beta}_{MLE})\right)} \tag{2.25}$$

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively,

$$Bias(\hat{\beta}_{LSE}) = (c_s - 1)Q\alpha \tag{2.26}$$

$$Cov(\hat{\beta}_{LSE}) = Qc_s^2\Lambda^{-1}Q' \tag{2.27}$$

$$MMSE(\hat{\beta}_{LSE}) = Qc_s^2\Lambda^{-1}Q' + (c_s - 1)^2Q\alpha\alpha'Q' \qquad 28)$$

$$MSE(\hat{\beta}_{LSE}) = \sum_{j=1}^{p}\frac{c_s^2}{\lambda_j} + \sum_{j=1}^{p}(c_s - 1)^2\alpha_j^2 \qquad 29)$$

K-L estimator for the logistic regression model proposed by Lukman et al. (2023) followed by the Kibria and Lukman (2020) is defined as:

$$\hat{\beta}_{LKLE} = \left(X'\widehat{W}X + kI\right)^{-1}\left(X'\widehat{W}X - kI\right)\hat{\beta}_{MLE} \tag{2.30}$$

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively,

$$Bias\left(\hat{\beta}_{LKLE}\right) = -2k\Lambda_k^{-1}\alpha \qquad 31)$$

$$Cov(\hat{\beta}_{LKLE}) = Q\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1}Q' \qquad 32)$$

$$MMSE(\hat{\beta}_{LKLE}) = Q\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1}Q' + 4k^2Q\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}Q' \qquad 33)$$

$$MSE(\hat{\beta}_{LKLE}) = \sum_{j=1}^{p} \frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} + \sum_{j=1}^{p} \frac{4k^2\alpha_j^2}{(\lambda_j + k)^2}$$ 34)

Dorugade (D) estimator for the logistic regression model, which is the extension of Dorugade (2014) is defined as:

$$\hat{\beta}_{LDE} = (X'\hat{W}X + kdI)^{-1}(X'\hat{W}X)\hat{\beta}_{MLE}$$ 35)

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively,

$$Bias\ (\ \hat{\beta}_{LDE}) = -kdQ(\Lambda + kdI)^{-1}\alpha$$ 36)

$$Cov\ (\ \hat{\beta}_{LDE}) = Q(\Lambda + kdI)^{-1}\Lambda(\Lambda + kdI)^{-1}Q'$$ 37)

$$MMSE(\ \hat{\beta}_{LDE}) = Q(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1}Q' + k^2d^2Q(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}Q'$$ 38)

$$MSE(\ \hat{\beta}_{LDE}) = \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + kd)^2} + \sum_{j=1}^{p} \frac{k^2d^2\alpha_j^2}{(\lambda_j + kd)^2}$$ 39)

The modified ridge estimator (MRT) for the logistic regression model proposed by Lukman et al. (2020) is defined as:

$$\hat{\beta}_{LMRT} = (X'\hat{W}X + k(1+d)I)^{-1}(X'\hat{W}X)\hat{\beta}_{MLE}$$ 40)

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively,

$$Bias(\hat{\beta}_{LMRT}) = k(1+d)Q(\Lambda + k(1+d)I)^{-1}\alpha$$ 41)

$$Cov\left(\hat{\beta}_{LMRT}\right) = Q(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1}Q' \quad 42)$$

$$MMSE\left(\hat{\beta}_{LMRT}\right) = Q(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1}Q' \quad (2.43)$$
$$+ k^2(1+d)^2 Q(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}Q'$$

$$MSE\left(\hat{\beta}_{LMRT}\right) = \sum_{j=1}^{p} \frac{\lambda_j}{\left(\lambda_j + k(1+d)\right)^2} + \sum_{j=1}^{p} \frac{k^2(1+d)^2\alpha_j^2}{\left(\lambda_j + k(1+d)\right)^2} \quad 44)$$

We propose two new Stein estimators. One of them is Stein's estimator for the logistic ridge regression estimator and defined as:

$$\hat{\beta}_{LSRE} = c_r\hat{\beta}_{LRE} = c_r A\hat{\beta}_{MLE} \quad 45)$$

where $0 < c_r < 1$ and $A = \left(X'\widehat{W}X + kI\right)^{-1}X'\widehat{W}X$. For the selection of $c_r$, we consider the previous procedure used in LSE and obtained as:

$$c_r = \frac{\left(\hat{\beta}'_{MLE}A\hat{\beta}_{MLE}\right)}{\left(\hat{\beta}'_{MLE}A\hat{\beta}_{MLE}\right) + \left(A\left(X'\widehat{W}X\right)^{-1}A'\right)} \quad 46)$$

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively,

$$Bias\left(\hat{\beta}_{LSRE}\right) = (c_r A - 1)\alpha \quad 47)$$

$$Cov\left(\hat{\beta}_{LSRE}\right) = c_r^2 QA\Lambda^{-1}A'Q' \quad 48)$$

$$MMSE\left(\hat{\beta}_{LSRE}\right) = c_r^2 QA\Lambda^{-1}A'Q' + Q(c_r A - 1)\alpha\alpha'(c_r A - 1)'Q' \quad 49)$$

$$MSE\left(\hat{\beta}_{LSRE}\right) = \sum_{j=1}^{p} \frac{c_r^2 \lambda_j}{\left(\lambda_j + k\right)^2} + \sum_{j=1}^{p} \frac{\left(c_r \lambda_j - \lambda_j - k\right)^2 \alpha_j^2}{\left(\lambda_j + k\right)^2}$$

50)

The stein's estimator for the K-L estimator is defined as:

$$\hat{\beta}_{LSKE} = c_k \hat{\beta}_{LKLE} = c_k K \hat{\beta}_{MLE}$$

51)

where $0 < c_k < 1$ and $K = \left(X'\widehat{W}X + kI\right)^{-1}\left(X'\widehat{W}X - kI\right)$. Following the same procedure for the selection of $c_k$, we obtain as:

$$c_k = \frac{\left(\hat{\beta}_{MLE}' K \hat{\beta}_{MLE}\right)}{\left(\hat{\beta}_{MLE}' K \hat{\beta}_{MLE}\right) + \left(K\left(X'\widehat{W}X\right)^{-1} K'\right)}$$

52)

The bias vector, covariance matrix, MMSE and scalar MSE are given respectively,

$$Bias\left(\hat{\beta}_{LSKE}\right) = (c_k K - 1)\alpha$$

53)

$$Cov\left(\hat{\beta}_{LSKE}\right) = c_k^2 Q K \Lambda^{-1} K' Q'$$

54)

$$MMSE\left(\hat{\beta}_{LSKE}\right) = c_k^2 Q K \Lambda^{-1} K' Q' + Q(c_k K - 1)\alpha\alpha'(c_k K - 1)' Q'$$

55)

$$MSE\left(\hat{\beta}_{LSKE}\right) = \sum_{j=1}^{p} \frac{c_k^2 \left(\lambda_j - k\right)^2}{\lambda_j \left(\lambda_j + k\right)^2} + \sum_{j=1}^{p} \frac{\left(c_r(\lambda_j - k) - \lambda_j - k\right)^2 \alpha_j^2}{\left(\lambda_j + k\right)^2}$$

56)

The following notations and lemmas are needful for making comparison among these estimators:

**Lemma 2.1** (Farebrother, 1976): Let $M$ be an $n \times n$ positive definite matrix, let $\alpha$ be a non-zero $n \times 1$ column matrix and let $c$ be a positive scalar. Then $cM - \alpha\alpha' > 0$ iff $\alpha' M^{-1}\alpha < d$, and $cM - \alpha\alpha' \geq 0$ iff $\alpha' M^{-1}\alpha \leq d.$

**Lemma 2.2** (Trenkler and Toutennurg, 1990): Let $\hat{\alpha}_i = A_i y, i = 1{,}2$ be two linear estimators of $\alpha.$ Suppose that $D = Cov(\hat{\alpha}_1) - Cov(\hat{\alpha}_2) > 0$, where $Cov(\hat{\alpha}_i), i = 1{,}2$ denotes the covariance matrix $\hat{\alpha}_i$and $b_i = Bias(\hat{\alpha}_i) = (A_i X - I)\alpha, i = 1{,}2$. Consequently,

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MMSE(\hat{\alpha}_1) - MMSE(\hat{\alpha}_2) = \sigma^2 D + b_1 b_1' - b_2 b_2' > 0$$

if and only if $b_2'[\sigma^2 D + b_1 b_1']^{-1} < 1$, where $MMSE(\hat{\alpha}_i) = Cov(\hat{\alpha}_i) + b_i b_i'$

### 2.2 Theoretical Comparison among the Estimators

Following Amin et al. (2023), Lukman et al. (2023), and Awwad et al. (2022), we have made all theoretical comparisons in this chapter.

### 2.2.1 Comparison between $\hat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ and $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{MLE}})$ is given by,

$$MMSE[\hat{\beta}_{MLE}] - MMSE[\hat{\beta}_{LSRE}] = \Lambda^{-1} - c_r^2 A\Lambda^{-1}A' - (c_r A - 1)\alpha\alpha'(c_r A - 1)' \qquad 57)$$

Let $k > 0$, then we have the following theorem.

**Theorem 2.1:** If $k > 0$, $b_{LSRE} = Bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ is the bias of logistic stein ridge regression, the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\hat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] > 0$ if and only if,

$$b_{LSRE}'[\Lambda^{-1} - c_r^2 A\Lambda^{-1}A']^{-1} b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = \Lambda^{-1} - c_r^2 A\Lambda^{-1}A' - b_{LSRE}b_{LSRE}'$$

$$= diag\left\{\frac{1}{\lambda_j} - \frac{c_r^2\lambda_j}{\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} - b_{LSRE}b_{LSRE}'$$

58)

where, $\Lambda^{-1} - c_r^2 A\Lambda^{-1}A'$ is a positive definite if and only if $\left(\lambda_j + k\right)^2 - c_r^2\lambda_j^2 > 0$. For $k > 0$, we observed that $\lambda_j(1 - c_r) + k > 0$. So consequently, $\Lambda^{-1} - c_r^2 A\Lambda^{-1}A'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.2 Comparison between $\hat{\boldsymbol{\beta}}_{\boldsymbol{LRE}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ and $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LRE}})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LRE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = \Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}$$

$$-(c_r A - 1)\alpha\alpha'(c_r A - 1)' \qquad (2.69)$$

Let $k > 0$, then we have the following theorem.

**Theorem 2.2:** If $k > 0$, $b_{LRE} = Bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{LRE}})$ is the bias of logistic ridge regression, the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\hat{\boldsymbol{\beta}}_{\boldsymbol{LRE}}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LRE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] > 0$ if and only if,

$$b_{LSRE}'[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}]^{-1} b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LRE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}]$$

$$= \Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + b_{LRE}b'_{LRE} - b_{LSRE}b'_{LSRE}$$

$$= diag\left\{\frac{\lambda_j}{\left(\lambda_j + k\right)^2} - \frac{c_r^2\lambda_j}{\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LRE}b'_{LRE} - b_{LSRE}b'_{LSRE} \tag{2.70}$$

where, $\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A^{'}$ is a positive definite if and only if $\lambda_j - c_r^2\lambda_j > 0$. For $k > 0$, we observed that $\lambda_j(1 - c_r^2) > 0$. So consequently, $\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A^{'}$is positive definite. By lemma 2.1, the proof is completed.

### 2.2.3 Comparison between $\widehat{\boldsymbol{\beta}}_{LLE}$ and $\widehat{\boldsymbol{\beta}}_{LSRE}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}})$ is obtained by,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = \Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A'$$

$$+(1-d)^2 Q\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}Q' - (c_rA - 1)\alpha\alpha'(c_rA - 1)' \tag{2.71}$$

Let $k > 0$ and $0 < d < 1$, then we have the following theorem.

**Theorem 2.3:** If $k > 0$ and $0 < d < 1$, $b_{LLE} = Bias(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}})$ is the bias of logistic Liu estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] > 0$ if and only if,

$$b'_{LSRE}[\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + (1-d)^2\Lambda_I^{-1}\alpha\alpha'\Lambda_I^{-1}]^{-1}b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}]$$

$$= \Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + b_{LLE}b'_{LLE} - b_{LSRE}b'_{LSRE}$$

$$= diag\left\{\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} - \frac{c_r^2 \lambda_j}{(\lambda_j + k)^2}\right\}_{j=1}^{p} + b_{LLE}b'_{LLE} - b_{LSRE}b'_{LSRE} \tag{2.72}$$

where, $\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A'$ is a positive definite if and only if $(\lambda_j + d)^2(\lambda_j + k)^2 - c_r^2\lambda_j^2(\lambda_j + 1)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_j^2(1 - c_r) + \lambda_j(k + d - c_r) + kd > 0$. So consequently, $\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.4 Comparison between $\widehat{\boldsymbol{\beta}}_{AURE}$ and $\widehat{\boldsymbol{\beta}}_{LSRE}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{LSRE})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{AURE})$ is obtained by,

$$\begin{aligned} &MMSE[\widehat{\boldsymbol{\beta}}_{AURE}] - MMSE[\widehat{\boldsymbol{\beta}}_{LSRE}] \\ &\quad = (I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_r^2 A\Lambda^{-1}A' \\ &\quad + k^4\,\Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2} - (c_r A - 1)\alpha\alpha'(c_r A - 1)' \end{aligned} \tag{2.73}$$

Let $k > 0$, then we have the following theorem.

**Theorem 2.4:** If $k > 0$, $b_{AURE} = Bias(\widehat{\boldsymbol{\beta}}_{AURE})$ is the bias of almost unbiased ridge regression, the estimator $\widehat{\boldsymbol{\beta}}_{LSRE}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{AURE}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{AURE}] - MMSE[\widehat{\boldsymbol{\beta}}_{LSRE}] > 0$ if and only if,

$$b'_{LSRE}[(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_r^2 A\Lambda^{-1}A' + k^4\,\Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2}]^{-1}b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{AURE}] - MMSE[\widehat{\boldsymbol{\beta}}_{LSRE}] = (I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_r^2 A\Lambda^{-1}A' + b_{AURE}b'_{AURE}$$

$$-b_{LSRE}b'_{LSRE} = diag\left\{\frac{\lambda_j\left(\lambda_j+2k\right)^2}{\left(\lambda_j+k\right)^4} - \frac{c_r^2\lambda_j}{\left(\lambda_j+k\right)^2}\right\}_{j=1}^{p} + b_{AURE}b'_{AURE} - b_{LSRE}b'_{LSRE} \tag{2.74}$$

where, $(I-k^2\Lambda_k^{-2})\Lambda^{-1}(I-k^2\Lambda_k^{-2}) - c_r^2A\Lambda^{-1}A'$ is a positive definite if and only if $\lambda_j\left(\lambda_j+2k\right)^2 - c_r^2\lambda_j\left(\lambda_j+k\right)^2 > 0$. For $k > 0$, we observed that $\lambda_j(1-c_r)+k(2-c_r) > 0$. So consequently, $(I-k^2\Lambda_k^{-2})\Lambda^{-1}(I-k^2\Lambda_k^{-2}) - c_r^2A\Lambda^{-1}A'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.5 Comparison between $\hat{\boldsymbol{\beta}}_{AULE}$ and $\hat{\boldsymbol{\beta}}_{LSRE}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{LSRE})$ and $MMSE(\hat{\boldsymbol{\beta}}_{AULE})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{AULE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSRE}] = (I-(1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I-(1-d)^2\Lambda_I^{-2})$$

$$-c_r^2A\Lambda^{-1}A' + (1-d)^4\Lambda_I^{-2}\alpha\alpha'\Lambda_I^{-2} - c_r^2A\Lambda^{-1}A' - (c_rA-1)\alpha\alpha'(c_rA-1)' \tag{2.75}$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.5:** If $k > 0$ and $0 < d < 1$, $b_{AULE} = Bias(\hat{\boldsymbol{\beta}}_{AULE})$ is the bias of almost unbiased logistic liu estimator, the estimator $\hat{\boldsymbol{\beta}}_{LSRE}$ **is** better than that $\hat{\boldsymbol{\beta}}_{AULE}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{AULE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSRE}] > 0$ if and only if,

$$b'_{LSRE}[(I-(1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I-(1-d)^2\Lambda_I^{-2}) - c_r^2A\Lambda^{-1}A'$$

$$+ (1-d)^4\Lambda_I^{-1}\alpha\alpha'\Lambda_I^{-1}]^{-1}b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{AULE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSRE}]$$

$$= (I-(1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I-(1-d)^2\Lambda_I^{-2}) - c_r^2A\Lambda^{-1}A'$$

$$+b_{AULE}b'_{AULE} - b_{LSRE}b'_{LSRE}$$

$$= diag\left\{\frac{(\lambda_j + d)^2(2 + \lambda_j - d)^2}{\lambda_j(\lambda_j + 1)^4} - \frac{c_r^2\lambda_j}{(\lambda_j + k)^2}\right\}_{j=1}^{p} + b_{AULE}b'_{AULE} - b_{LSRE}b'_{LSRE} \tag{2.76}$$

where, $(I - (1 - d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1 - d)^2\Lambda_I^{-2}) - c_r^2 A\Lambda^{-1}A'$ is a positive definite if and only if $(\lambda_j + d)^2(2 + \lambda_j - d)^2(\lambda_j + k)^2 - c_r^2\lambda_j^2(\lambda_j + 1)^4 > 0$. For $k > 0$, and $0 < d < 1$ we observed that $\lambda_j^3(1 - c_r) + \lambda_j^2(2 + k - 2c_r) + \lambda_j(2k + 2d - d^2 - c_r) + kd(2 - d) > 0$. So consequently, $(I - (1 - d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1 - d)^2\Lambda_I^{-2}) - c_r^2 - c_r^2 A\Lambda^{-1}A'$is positive definite. By lemma 2.1, the proof is completed.

### 2.2.6 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}})$ is obtained by,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = \Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1}$$

$$-c_r^2 A\Lambda^{-1}A' + (1 + d)^2\Lambda_I^{-2}\alpha\alpha'\Lambda_I^{-2} - c_r^2 A\Lambda^{-1}A' - (c_r A - 1)\alpha\alpha'(c_r A - 1)' \tag{2.77}$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.6:** If $k > 0$ and $0 < d < 1$, $b_{ALLE} = Bias(\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}})$ is the bias of adjusted logistic Liu estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] > 0$ if and only if,

$$b'_{LSRE}\left[\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + (1 + d)^2\Lambda_I^{-1\prime}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}\right]-MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right]= \Lambda_I^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)\Lambda_I^{-1}-c_r^2 A\Lambda^{-1}A'$$

$$+b_{ALLE}b'_{ALLE}-b_{LSRE}b'_{LSRE}$$

$$=diag\left\{\frac{\left(\lambda_j-d\right)^2}{\lambda_j\left(\lambda_j+1\right)^2}-\frac{c_r^2\lambda_j}{\left(\lambda_j+k\right)^2}\right\}_{j=1}^{p}+b_{ALLE}b'_{ALLE}-b_{LSRE}b'_{LSRE} \tag{2.78}$$

where, $\Lambda_I^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)\Lambda_I^{-1}-c_r^2A\Lambda^{-1}A^{'}$ is a positive definite if and only if $\left(\lambda_j-d\right)^2\left(\lambda_j+k\right)^2-c_r^2\lambda_j^2\left(\lambda_j+1\right)^2>0$. For $k>0$, and $0<d<1$ we observed that $\lambda_j^2(1-c_r)+\lambda_j(k-d-c_r)-kd>0$. So consequently, $\Lambda_I^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)\Lambda_I^{-1}--c_r^2A\Lambda^{-1}A^{'}$is positive definite. By lemma 2.1, the proof is completed.

### 2.2.7 Comparison between $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$

The difference between $MMSE\left(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right)$ and $MMSE\left(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}\right)$ is obtained by,

$$MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}\right]-MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right]= c_s^2\Lambda^{-1}-c_r^2A\Lambda^{-1}A'+(c_s-1)^2\alpha\alpha'$$

$$-(c_rA-1)\alpha\alpha'(c_rA-1)' \tag{2.79}$$

Let $k>0$, and $0<c_s<1$ then we have the following theorem.

**Theorem 2.7:** If $k>0$ and $0<c_s<1$, $b_{LSE}=Bias\left(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}\right)$ is the bias of logistic stein's estimator, the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}$ using the criterion of MMSE, that is $MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}\right]-MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right]>0$ if and only if,

$$b'_{LSRE}\left[\,c_s^2\Lambda^{-1}-c_r^2A\Lambda^{-1}A'+(c_s-1)^2\alpha\alpha^{'}\right]^{-1}b_{LSRE}<1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = c_s^2\Lambda^{-1} - c_r^2 A\Lambda^{-1}A' + b_{LSE}b'_{LSE} - b_{LSRE}b'_{LSRE}$$

$$= diag\left\{\frac{c_s^2}{\lambda_j} - \frac{c_r^2\lambda_j}{\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LSE}b'_{LSE} - b_{LSRE}b'_{LSRE} \tag{2.80}$$

where, $c_s^2\Lambda^{-1} - c_r^2 A\Lambda^{-1}A'$ is a positive definite if and only if $c_s^2\left(\lambda_j + k\right)^2 - c_r^2\lambda_j^2 > 0$. For $k > 0$, and $0 < c_s < 1$ we observed that $\lambda_j(c_s - c_r) + c_s k > 0$. So consequently, $c_s^2\Lambda^{-1} - c_r^2 A\Lambda^{-1}A'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.8 Comparison between $\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ and $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - kI)\Lambda_k^{-1}$$

$$-c_r^2 A\Lambda^{-1}A' + 4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1} - (c_r A - 1)\alpha\alpha'(c_r A - 1)' \tag{2.81}$$

Let $k > 0$ then we have the following theorem.

**Theorem 2.8:** If $k > 0$, $b_{LKLE} = Bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}})$ is the bias of logistic K-L estimator, the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] > 0$ if and only if,

$$b'_{LSRE}\left[\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + 4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A'$$

$$+b_{LKLE}b'_{LKLE} - b_{LSRE}b'_{LSRE}$$

$$= diag\left\{\frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} - \frac{c_r^2\lambda_j}{(\lambda_j + k)^2}\right\}_{j=1}^{p} + b_{LLKE}b'_{LKLE} - b_{LSRE}b'_{LSRE} \tag{2.82}$$

where, $\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1}$is a positive definite if and only if $(\lambda_j - k)^2 - c_r^2\lambda_j^2 > 0$. For $k > 0$, we observed that $\lambda_j(1 - c_r) + k > 0$. So consequently, $\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A'$is positive definite. By lemma 2.1, the proof is completed.

### 2.2.9 Comparison between $\hat{\boldsymbol{\beta}}_{LDE}$ and $\hat{\boldsymbol{\beta}}_{LSRE}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{LSRE})$ and $MMSE(\hat{\boldsymbol{\beta}}_{LDE})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{LDE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSRE}] = (\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_r^2 A\Lambda^{-1}A'$$

$$+k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1} - (c_rA - 1)\alpha\alpha'(c_rA - 1)' \tag{2.83}$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.9:** If $k > 0$ and $0 < d < 1$, $b_{LDE} = Bias(\hat{\boldsymbol{\beta}}_{LDE})$ is the bias of logistic D estimator, the estimator $\hat{\boldsymbol{\beta}}_{LSRE}$ **is** better than that $\hat{\boldsymbol{\beta}}_{LDE}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{LDE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSRE}] > 0$ if and only if,

$$b'_{LSRE}[\,(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_r^2 A\Lambda^{-1}A'$$
$$+ k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}]^{-1}b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{LDE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSRE}] = \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A'$$

$$+b_{LDE}b'_{LDE} - b_{LSRE}b'_{LSRE} \tag{2.84}$$

$$= diag\left\{\frac{\lambda_j}{\left(\lambda_j + kd\right)^2} - \frac{c_r^2\lambda_j}{\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LDE}b'_{LDE} - b_{LSRE}b'_{LSRE}$$

where, $(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_r^2 A\Lambda^{-1}A^{'}$ is a positive definite if and only if $\lambda_j\left(\lambda_j + k\right)^2 - c_r^2\lambda_j\left(\lambda_j + kd\right)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_j(1 - c_r) + k(1 - c_r d) > 0$. So consequently, $(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_r^2 A\Lambda^{-1}A^{'}$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.10 Comparison between $\hat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$

The difference between $MMSE\left(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right)$ and $MMSE\left(\hat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}\right)$ is obtained by,

$$MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}\right] - MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right] = (\Lambda + k(1 + d)I)^{-1}\Lambda(\Lambda + k(1 + d)I)^{-1}$$

$$-c_r^2 A\Lambda^{-1}A^{'} + k^2(1 + d)^2(\Lambda + k(1 + d)I)^{-1}\alpha\alpha'(\Lambda + k(1 + d)I)^{-1}$$

$$-(c_r A - 1)\alpha\alpha'(c_r A - 1)' \tag{2.85}$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.10:** If $k > 0$ and $0 < d < 1$, $b_{LMRT} = Bias\left(\hat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}\right)$ is the bias of logistic Modified ridge estimator, the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ **is** better than that $\hat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}$ using the criterion of MMSE, that is $MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}\right] - MMSE\left[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}\right] > 0$ if and only if,

$$b'_{LSRE}[(\Lambda + k(1 + d)I)^{-1}\Lambda(\Lambda + k(1 + d)I)^{-1} - c_r^2 A\Lambda^{-1}A'$$

$$+ k^2(1 + d)^2(\Lambda + k(1 + d)I)^{-1}\alpha\alpha'(\Lambda + k(1 + d)I)^{-1}]^{-1}b_{LSRE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] = (\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1}$$

$$-c_r^2 A\Lambda^{-1}A' + b_{LMRT}b'_{LMRT} - b_{LSRE}b'_{LSRE}$$

$$= diag\left\{\frac{\lambda_j}{\left(\lambda_j + k(1+d)\right)^2} - \frac{c_r^2\lambda_j}{\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LMRT}b'_{LMRT} - b_{LSRE}b'_{LSRE} \qquad (2.86)$$

where, $(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_r^2 A\Lambda^{-1}A^{'}$ is a positive definite if and only if $\lambda_j\left(\lambda_j + k\right)^2 - c_r^2\lambda_j\left(\lambda_j + k(1+d)\right)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_j(1 - c_r) + k\ (1 - c_r - c_r d) > 0$. So consequently, $(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_r^2 A\Lambda^{-1}A^{'}$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.11 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{MLE}})$ is obtained by,

$$MMSE[\hat{\beta}_{MLE}] - MMSE[\hat{\beta}_{LSKE}] = \Lambda^{-1} - c_k^2 K\Lambda^{-1}K' - (c_k K - 1)\alpha\alpha^{'}(c_k K - 1)^{'} \qquad 59)$$

Let $k > 0$, then we have the following theorem.

**Theorem 2.11:** If $k > 0$, $b_{LSKE} = Bias(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}})$ is the bias of logistic stein K-L regression estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] > 0$ if and only if,

$$b'_{LSKE}\left[\Lambda^{-1} - c_{\mathrm{k}}^2 \mathrm{K}\Lambda^{-1}\mathrm{K}'\right]^{-1} b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{MLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = \Lambda^{-1} - c_k^2 K\Lambda^{-1}K' - b_{LSKE}b'_{LSKE}$$

$$= diag\left\{\frac{1}{\lambda_j} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} - b_{LSKE}b'_{LSKE}$$

60)

where, $\Lambda^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}^{'}$ is a positive definite if and only if $\left(\lambda_j + k\right)^2 - c_k^2\left(\lambda_j - k\right)^2 > 0$. For $k > 0$, we observed that $\lambda_j(1 - c_k) + k(1 + c_k) > 0$. So consequently, $\Lambda^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.12 Comparison between $\widehat{\boldsymbol{\beta}}_{LRE}$ and $\widehat{\boldsymbol{\beta}}_{LSKE}$

The difference between $MMSE\left(\widehat{\boldsymbol{\beta}}_{LSKE}\right)$ and $MMSE\left(\widehat{\boldsymbol{\beta}}_{LRE}\right)$ is obtained by,

$$MMSE\left[\widehat{\boldsymbol{\beta}}_{LRE}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{LSKE}\right] = \Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_k^2K\Lambda^{-1}K' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}$$
$$-(c_kK - 1)\alpha\alpha^{'}(c_kK - 1)^{'} \tag{2.89}$$

Let $k > 0$, then we have the following theorem.

**Theorem 2.12:** If $k > 0$, $b_{LRE} = Bias\left(\widehat{\boldsymbol{\beta}}_{LRE}\right)$ is the bias of logistic ridge regression, the estimator $\widehat{\boldsymbol{\beta}}_{LSKE}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{LRE}$ using the criterion of MMSE, that is $MMSE\left[\widehat{\boldsymbol{\beta}}_{LRE}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{LSKE}\right] > 0$ if and only if,

$$b'_{LSKE}\left[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1}b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE\left[\widehat{\boldsymbol{\beta}}_{LRE}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{LSKE}\right] = \Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2A\Lambda^{-1}A' + b_{LRE}b'_{LRE}$$

$$-b_{LSKE}b'_{LSKE} = diag\left\{\frac{\lambda_j}{\left(\lambda_j + k\right)^2} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LRE}b'_{LRE} - b_{LSKE}b'_{LSKE} \tag{2.90}$$

where, $\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}'$ is a positive definite if and only if $\lambda_j^2 - c_k^2\left(\lambda_j - k\right)^2 > 0$. For $k > 0$, we observed that $\lambda_j(1 - c_k) + c_k k > 0$. So consequently, $\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.13 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right)$ and $MMSE\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}\right)$ is obtained by,

$$\begin{aligned} MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] = {} & \Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_k^2 K\Lambda^{-1}K' \\ & +(1-d)^2 Q\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}Q' - (c_kK - 1)\alpha\alpha'(c_kK - 1)' \end{aligned} \tag{2.91}$$

Let $k > 0$ and $0 < d < 1$, then we have the following theorem.

**Theorem 2.13:** If $k > 0$ and $0 < d < 1$, $b_{LLE} = Bias\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}\right)$ is the bias of logistic liu estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}$ using the criterion of MMSE, that is $MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] > 0$ if and only if,

$$b_{LSKE}'\left[\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}' + (1-d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$\begin{aligned} MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LLE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] &= \Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_k^2K\Lambda^{-1}K' + b_{LLE}b_{LLE}' \\ & \quad - b_{LSKE}b_{LSKE}' \\ &= diag\left\{\frac{\left(\lambda_j + d\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LLE}b_{LLE}' - b_{LSKE}b_{LSKE}' \end{aligned} \tag{2.92}$$

where, $\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $\left(\lambda_j + d\right)^2\left(\lambda_j + k\right)^2 - c_r^2\left(\lambda_j - k\right)^2\left(\lambda_j + 1\right)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_j^2(1 - c_k) + \lambda_j(k + d - c_k + c_k k) + k(c_k + d) > 0$. So consequently, $\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.14 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right)$ and $MMSE\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}\right)$ is obtained by,

$$MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] = (I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2})$$

$$-c_k^2 K\Lambda^{-1}K' + k^4\,\Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2} - (c_k K - 1)\alpha\alpha'(c_k K - 1)' \qquad (2.93)$$

Let $k > 0$, then we have the following theorem.

**Theorem 2.14:** If $k > 0$, $b_{AURE} = Bias\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}\right)$ is the bias of almost unbiased ridge regression, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}$ using the criterion of MMSE, that is $MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] > 0$ if and only if,

$$b_{LSKE}'[(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_k^2 K\Lambda^{-1}K' + k^4\,\Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2}]^{-1}b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{AURE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] = (I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_k^2 K\Lambda^{-1}K'$$

$$+b_{AURE}b_{AURE}' - b_{LSKE}b_{LSKE}'$$

$$= diag\left\{\frac{\lambda_j\left(\lambda_j + 2k\right)^2}{\left(\lambda_j + k\right)^4} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{AURE}b_{AURE}' - b_{LSKE}b_{LSKE}' \qquad (2.94)$$

where, $(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $\lambda_j^2(\lambda_j + 2k)^2 - c_r^2(\lambda_j - k)^2\ (\lambda_j + k)^2 > 0$. For $k > 0$, we observed that $\lambda_j^2(1 - c_k) + k(2\lambda_j - c_k k) > 0$. So consequently, $(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.15 Comparison between $\hat{\boldsymbol{\beta}}_{AULE}$ and $\hat{\boldsymbol{\beta}}_{LSKE}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{LSKE})$ and $MMSE(\hat{\boldsymbol{\beta}}_{AULE})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{AULE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSKE}] = (I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_k^2 K\Lambda^{-1}K' + (1-d)^4\Lambda_I^{-2}\alpha\alpha'\Lambda_I^{-2} - (c_k K - 1)\alpha\alpha'(c_k K - 1)' \qquad (2.95)$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.15:** If $k > 0$ and $0 < d < 1$, $b_{AULE} = Bias(\hat{\boldsymbol{\beta}}_{AULE})$ is the bias of almost unbiased logistic liu estimator, the estimator $\hat{\boldsymbol{\beta}}_{LSKE}$ **is** better than that $\hat{\boldsymbol{\beta}}_{AULE}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{AULE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSKE}] > 0$ if and only if,

$$b'_{LSKE}\left[(I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_k^2 K\Lambda^{-1}K' + (1-d)^4\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{AULE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSKE}] = (I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_k^2 K\Lambda^{-1}K' + b_{AULE}b'_{AULE} - b_{LSKE}b'_{LSKE} \qquad (2.96)$$

$$= diag\left\{\frac{\left(\lambda_j + d\right)^2\left(2 + \lambda_j - d\right)^2}{\lambda_j\left(\lambda_j + 1\right)^4} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{AULE}b'_{AULE} - b_{LSKE}b'_{LSKE}$$

where, $(I - (1 - d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1 - d)^2\Lambda_I^{-2}) - c_k^2K\Lambda^{-1}K'$ is a positive definite if and only if $\left(\lambda_j + d\right)^2\left(2 + \lambda_j - d\right)^2\left(\lambda_j + k\right)^2 - c_k^2\left(\lambda_j - k\right)^2\left(\lambda_j + 1\right)^4 > 0$. For $k > 0$, and $0 < d < 1$ we observed that $\lambda_j^3(1 - c_k^2) + \lambda_j^2(2 + k - 2c_k^2 + c_k^2k) + \lambda_j(2k + 2d - c_r^2 + 2kc_k^2) + k(2d - d^2 + c_k^2) > 0$. So consequently, $(I - (1 - d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1 - d)^2\Lambda_I^{-2}) - c_r^2 - c_k^2K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.16 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right)$ and $MMSE\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}\right)$ is obtained by,

$$MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] = \Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1}$$

$$-c_k^2K\Lambda^{-1}K' + (1 + d)^2\Lambda_I^{-2}\alpha\alpha'\Lambda_I^{-2} - (c_kK - 1)\alpha\alpha'(c_kK - 1)' \tag{2.97}$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.16:** If $k > 0$ and $0 < d < 1$, $b_{ALLE} = Bias\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}\right)$ is the bias of adjusted logistic Liu estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}$ using the criterion of MMSE, that is $MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{AULE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] > 0$ if and only if,

$$b'_{LSKE}\left[\,\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_k^2K\Lambda^{-1}K' + (1 + d)^2\Lambda_I^{-1\prime}\alpha\alpha'\Lambda_I^{-1}\right]^{-1}b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{ALLE}}\right] - MMSE\left[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}\right] = \Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_k^2K\Lambda^{-1}K'$$

$$+b_{ALLE}b'_{ALLE} - b_{LSKE}b'_{LSKE}$$

$$= diag\left\{\frac{(\lambda_j - d)^2}{\lambda_j(\lambda_j+1)^2} - \frac{c_k^2(\lambda_j - k)^2}{\lambda_j(\lambda_j+k)^2}\right\}_{j=1}^{p} + b_{ALLE}b'_{ALLE} - b_{LSKE}b'_{LSKE} \quad (2.98)$$

where, $\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $(\lambda_j - d)^2(\lambda_j + k)^2 - c_r^2(\lambda_j - k)^2(\lambda_j + 1)^2 > 0$. For $k > 0$, and $0 < d < 1$ we observed that $\lambda_j^2(1 - c_k) + \lambda_j(k - d - c_k + c_k k) + k(c_k - d) > 0$. So consequently, $\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.17 Comparison between $\hat{\boldsymbol{\beta}}_{LSE}$ and $\hat{\boldsymbol{\beta}}_{LSKE}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{LSKE})$ and $MMSE(\hat{\boldsymbol{\beta}}_{LSE})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{LSE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSKE}] = c_s^2\Lambda^{-1} - c_k^2 K\Lambda^{-1}K' + (c_s - 1)^2\alpha\alpha'$$

$$-(c_k K - 1)\alpha\alpha'(c_k K - 1)' \quad (2.99)$$

Let $k > 0$, and $0 < c_s < 1$ then we have the following theorem.

**Theorem 2.17:** If $k > 0$ and $0 < c_s < 1$, $b_{LSE} = Bias(\hat{\boldsymbol{\beta}}_{LSE})$ is the bias of logistic stein's estimator, the estimator $\hat{\boldsymbol{\beta}}_{LSKE}$ **is** better than that $\hat{\boldsymbol{\beta}}_{LSE}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{LSE}] - MMSE[\hat{\boldsymbol{\beta}}_{LSKE}] > 0$ if and only if,

$$b'_{LSKE}\left[c_s^2\Lambda^{-1} - c_k^2 K\Lambda^{-1}K' + (c_s - 1)^2\alpha\alpha'\right]^{-1} b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = c_s^2\Lambda^{-1} - c_r^2 A\Lambda^{-1}A' + b_{LSE}b_{LSE}' - b_{LSKE}b_{LSKE}'$$

$$= diag\left\{\frac{c_s^2}{\lambda_j} - \frac{c_k^2(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2}\right\}_{j=1}^{p} + b_{LSE}b_{LSE}' - b_{LSKE}b_{LSKE}' \quad (2.100)$$

where, $c_s^2\Lambda^{-1} - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $c_s^2(\lambda_j + k)^2 - c_k^2(\lambda_j - k)^2 > 0$. For $k > 0$, and $0 < c_s < 1$ we observed that $\lambda_j(c_s - c_r) + k(c_s + c_k) > 0$. So consequently, $c_s^2\Lambda^{-1} - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.18 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}})$ is obtained by,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}]$$

$$= \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K' \quad (2.101)$$

$$+4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1} - (c_k K - 1)\alpha\alpha'(c_k K - 1)'$$

Let $k > 0$ then we have the following theorem.

**Theorem 2.18:** If $k > 0$, $b_{LKLE} = Bias(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}})$ is the bias of logistic K-L estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] > 0$ if and only if,

$$b_{LSKE}'\left[\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K' + 4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LKLE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K'$$

$$+b_{LKLE}b'_{LKLE} - b_{LSKE}b'_{LSKE}$$

$$= diag\left\{\frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} - \frac{c_k^2(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2}\right\}_{j=1}^{p} + b_{LKLE}b'_{LKLE} - b_{LSKE}b'_{LSKE} \qquad (2.102)$$

where, $\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $(\lambda_j - k)^2 - c_k^2(\lambda_j - k)^2 > 0$. For $k > 0$, we observed that $\lambda_j(1 - c_k) + k(c_k - 1) > 0$. So consequently, $\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.19 Comparison between $\hat{\boldsymbol{\beta}}_{\boldsymbol{LDE}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}})$ and $MMSE(\hat{\boldsymbol{\beta}}_{\boldsymbol{LDE}})$ is obtained by,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LDE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = (\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_k^2 K\Lambda^{-1}K'$$

$$+k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha^{'}(\Lambda + kdI)^{-1} - (c_kK - 1)\alpha\alpha'(c_kK - 1)' \qquad (2.103)$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.19:** If $k > 0$ and $0 < d < 1$, $b_{LDE} = Bias(\hat{\boldsymbol{\beta}}_{\boldsymbol{LDE}})$ is the bias of logistic D estimator, the estimator $\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\hat{\boldsymbol{\beta}}_{\boldsymbol{LDE}}$ using the criterion of MMSE, that is $MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LDE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] > 0$ if and only if,

$$b'_{LSKE}\Big[(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_k^2 K\Lambda^{-1}K^{'}$$

$$+ k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}\Big]^{-1} b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LDE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K'$$

$$+b_{LDE}b'_{LDE} - b_{LSKE}b'_{LSKE}$$

$$= diag\left\{\frac{\lambda_j}{\left(\lambda_j + kd\right)^2} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LDE}b'_{LDE} - b_{LSKE}b'_{LSKE} \qquad (2.104)$$

where, $(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $\lambda_j^2\left(\lambda_j + k\right)^2 - c_k^2\left(\lambda_j - k\right)^2\left(\lambda_j + kd\right)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_j^2(1 - c_k) + \lambda_j(k - c_k kd + c_k k) + c_k k^2 d > 0$. So consequently, $(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.20 Comparison between $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}$ and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}})$ is obtained by,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = (\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1}$$

$$-c_k^2 K\Lambda^{-1}K' + k^2(1+d)^2(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}$$

$$-(c_k K - 1)\alpha\alpha'(c_k K - 1)' \qquad (2.105)$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.20:** If $k > 0$ and $0 < d < 1$, $b_{LMRT} = Bias(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}})$ is the bias of logistic Modified ridge estimator, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] > 0$ if and only if,

$$b'_{LSKE}[(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_k^2 K\Lambda^{-1}K'$$
$$+ k^2(1+d)^2(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}]^{-1}b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = (\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1}$$

$$-c_k^2 K\Lambda^{-1}K' + b_{LMRT}b'_{LMRT} - b_{LSKE}b'_{LSKE}$$

$$= diag\left\{\frac{\lambda_j}{\left(\lambda_j + k(1+d)\right)^2} - \frac{c_k^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p} + b_{LMRT}b'_{LMRT} - b_{LSKE}b'_{LSKE} \tag{2.106}$$

where, $(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $\lambda_j^2\left(\lambda_j + k\right)^2 - c_k^2\left(\lambda_j - k\right)^2\left(\lambda_j + k(1+d)\right)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_j^2(1 - c_k) + \lambda_j(k - c_k kd) + c_k k^2\ (1+d) > 0$. So consequently, $(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

### 2.2.21 Comparison between $\widehat{\boldsymbol{\beta}}_{LSRE}$ and $\widehat{\boldsymbol{\beta}}_{LSKE}$

The difference between $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}})$ and $MMSE(\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}})$ is obtained by,

$$MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = c_r^2 A\Lambda^{-1}A' - c_k^2 K\Lambda^{-1}K'$$

$$+(c_rA - 1)\alpha\alpha'(c_rA - 1)' - (c_kK - 1)\alpha\alpha'(c_kK - 1)' \tag{2.107}$$

Let $k > 0$, and $0 < d < 1$ then we have the following theorem.

**Theorem 2.21:** If $k > 0$, the estimator $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}$ **is** better than that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}$ using the criterion of MMSE, that is $MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LMRT}}] - MMSE[\widehat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] > 0$ if and only if,

$$b'_{LSKE}[c_r^2 A\Lambda^{-1}A' - c_k^2 K\Lambda^{-1}K' + (c_rA - 1)\alpha\alpha'(c_rA - 1)']^{-1}b_{LSKE} < 1$$

**Proof:** Using the difference between MMSE,

$$MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSRE}}] - MMSE[\hat{\boldsymbol{\beta}}_{\boldsymbol{LSKE}}] = c_r^2 A\Lambda^{-1}A' - c_k^2 K\Lambda^{-1}K' + b_{LSRE}b'_{LSRE}$$

$$-b_{LSKE}b'_{LSKE} = diag\left\{\frac{c_r^2\lambda_j}{(\lambda_j + k)^2} - \frac{c_k^2(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2}\right\}_{j=1}^{p} + b_{LSRE}b'_{LSRE}$$

$$- b_{LSKE}b'_{LSKE} \tag{2.108}$$

where, $c_r^2 A\Lambda^{-1}A' - c_k^2 K\Lambda^{-1}K'$ is a positive definite if and only if $c_r^2\lambda_j^2 - c_k^2(\lambda_j - k)^2 > 0$. For $k > 0$, we observed that $\lambda_j(c_r - c_k) + c_k k > 0$. So consequently, $c_r^2 A\Lambda^{-1}A' - c_k^2 K\Lambda^{-1}K'$ is positive definite. By lemma 2.1, the proof is completed.

## 2.3 Biasing Parameters $k$ and $d$

In this section, we will discuss about different biasing parameters $k$ and $d$. Following Hoerl et al. (1975), and based on the study of Mermi et al. (2021), Lukman and Olatunji (2018), Shabbir et al. (2023), Kibria et al. (2011), Lukman et al. (2023), Amin et al. (2023), we suggest the following estimators for the biasing parameters $k$ and $d$ for the logistic regression model:

$$\hat{k}_1 = p$$

$$\hat{k}_2 = p^{1+\frac{1}{p}}$$

$$\hat{k}_3 = \prod_{i=1}^{p}\left(\frac{1}{q_i}\right)^{\frac{1}{p}}$$

$$\hat{k}_4 = \max\left(\frac{1}{q_i}\right)$$

$$\hat{d}_1 = \min\left(\frac{\alpha_i^2}{\frac{1}{\lambda_j} + \alpha_j^2}\right)$$

$$\hat{d}_2 = \min\left(\frac{\lambda_j(1+\alpha_j^2)}{1+\alpha_j^2\lambda_j}\right)$$

where, $q_i = \frac{\hat{\lambda}_{max}}{(n-p)+\hat{\lambda}_{max}\alpha_j^2}$ and $\hat{\alpha} = Q'\hat{\beta}$ where $Q$ is the eigen vector of $X'WX$. Then we suggest the following biasing parameters $k$ and $d$ for the different logistic regression estimators as follows:

a. LRE1: $\hat{k}_1$
b. LRE2: $\hat{k}_2$
c. LRE3: $\hat{k}_3$
d. LRE4: $\hat{k}_4$
e. LLE1: $\hat{d}_1$
f. LLE2: $\hat{d}_2$
g. AURE1: $\hat{k}_1$
h. AURE2: $\hat{k}_2$
i. AURE3: $\hat{k}_3$
j. AURE4: $\hat{k}_4$
k. AULE1: $\hat{d}_1$
l. AULE2: $\hat{d}_2$
m. ALLE1: $\hat{d}_1$
n. ALLE2: $\hat{d}_2$

o. LKLE1: $\hat{k}_1$

p. LKLE2: $\hat{k}_2$

q. LKLE3: $\hat{k}_3$

r. LKLE4: $\hat{k}_4$

s. LDE11: $\hat{k}_1$, $\hat{d}_1$

t. LDE12: $\hat{k}_1$, $\hat{d}_2$

u. LDE21: $\hat{k}_2$, $\hat{d}_1$

v. LDE22: $\hat{k}_2$, $\hat{d}_2$

w. LDE31: $\hat{k}_3$, $\hat{d}_1$

x. LDE32: $\hat{k}_3$, $\hat{d}_2$

y. LDE41: $\hat{k}_4$, $\hat{d}_1$

z. LDE42: $\hat{k}_4$, $\hat{d}_2$

aa. LMRT11: $\hat{k}_1$, $\hat{d}_1$

bb. LMRT12: $\hat{k}_1$, $\hat{d}_2$

cc. LMRT21: $\hat{k}_2$, $\hat{d}_1$

dd. LMRT22: $\hat{k}_2$, $\hat{d}_2$

ee. LMRT31: $\hat{k}_3$, $\hat{d}_1$

ff. LMRT32: $\hat{k}_3$, $\hat{d}_2$

gg. LMRT41: $\hat{k}_4$, $\hat{d}_1$

hh. LMRT42: $\hat{k}_4$, $\hat{d}_2$

ii. LSRE1: $\hat{k}_1$

jj. LSRE2: $\hat{k}_2$

kk. LSRE3: $\hat{k}_3$

ll. LSRE4: $\hat{k}_4$

mm. LSKE1: $\hat{k}_1$

nn. LSKE2: $\hat{k}_2$

oo. LSKE3: $\hat{k}_3$

pp. LSKEE4: $\hat{k}_4$

# Chapter 3

# Monte Carlo Simulation

In this chapter, we will discuss simulation technique that is used to analyze the logistic regression model. The organize of this chapter is as follows. In section 3.1, we will provide an overview of the simulation employed in our study. We will describe the methodology and steps involved in generating simulated data for the logistic regression model and section 3.2 will present and discuss the results obtained from the simulations. We will analyze the simulated data and evaluate the performance of the estimators under different scenarios.

## 3.1 Simulation Technique

In this section, we conducted a Monte Carlo simulation study to examine the performance of different logistic regression estimators. A substantial number of simulations were carried out to compare the performance of these estimators. This simulation procedure follows the approach outlined by McDonalds and Galarneau (1975) and Gibbons (1981).

The correlated explanatory variables were generated using the following formula:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{ip}, i = 1,2,\ldots,n;\ \ j = 1,2,\ldots,p \tag{3.1}$$

where $z_{ij}$ represents independent standard normal pseudo-random numbers, $\rho$ is the correlation between the explanatory variables and $p$ presents the number of explanatory variables. Different set of values of $\rho$ corresponding 0.90, 0.95 and 0.99 were chosen. The sample sizes used in the simulation were 30, 50, 100 and 200. Then the entire experiment is replicated 2000 times.

Furthermore, we also considered different numbers of explanatory variables, $p$: 3, 5 and 10. The estimated MSE value was calculated from the following equations:

$$MSE(\hat{\beta}) = \frac{1}{2000}\sum_{i=1}^{2000}\left(\hat{\beta}_{ij} - \beta_i\right)'\left(\hat{\beta}_{ij} - \beta_i\right) \tag{3.2}$$

where $\hat{\beta}_{ij}$ represents the estimate of the $i$th parameters in the $j$th replication and $\beta_i$ is the vector of true parameter values (selected as the eigen vector of $X'WX$ corresponding to the largest eigenvalues ensuring $\beta'\beta = 1$). The response variable $y_i$ is generated from a Bernoulli distribution, i.e., $y_i \sim Be\ (\pi_i)$, where $\pi_i = \frac{\exp\ (x_i'\beta)}{1+\exp\ (x_i'\beta)}$ , $i = 1,2,\dots,n$ such that the data matrix $X$ was represented as $(x_i')_{i=1,2,\dots,n}$. All computations were performed using the R programming language. The simulated MSE values are presented in Table 3.1, Table 3.2, and Table 3.3 for $p = 3, 5$ and 10, respectively. Also, relative efficiency for different estimators compared to MLE are shown in Table 3.4-3.6. For a better presentation, we make Figure 3.1, 3.2 and 3.3 to find the MSE vs $n$, $p$ and $\rho$.

**Table 3.1**: Estimated MSE values of the estimators for $p = 3$.

| n | | 30 | | | 50 | | | 100 | | | 200 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 |
| MLE | 4.678 | 9.949 | 51.243 | 2.863 | 5.421 | 27.734 | 1.469 | 2.741 | 13.504 | 0.841 | 0.380 | 6.576 |
| LRE1 | 0.606 | 0.561 | 0.528 | 0.556 | 0.511 | 0.462 | 0.515 | 0.476 | 0.392 | 0.479 | 0.469 | 0.382 |
| LRE2 | 0.596 | 0.568 | 0.557 | 0.528 | 0.498 | 0.480 | 0.471 | 0.440 | 0.389 | 0.440 | 0.419 | 0.356 |
| LRE3 | 0.594 | 0.590 | 0.674 | 0.518 | 0.500 | 0.557 | 0.459 | 0.427 | 0.416 | 0.433 | 0.405 | 0.348 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LRE4 | 0.614 | 0.640 | 0.776 | 0.520 | 0.527 | 0.656 | 0.447 | 0.425 | 0.474 | 0.424 | 0.393 | 0.361 |
| LLE1 | 1.439 | 1.956 | 5.054 | 1.236 | 1.525 | 3.107 | 0.940 | 1.147 | 2.109 | 0.700 | 0.909 | 1.402 |
| AURE1 | 0.769 | 0.655 | 0.504 | 0.770 | 0.672 | 0.472 | 0.738 | 0.695 | 0.473 | 0.638 | 0.701 | 0.549 |
| AURE2 | 0.641 | 0.562 | 0.489 | 0.620 | 0.542 | 0.435 | 0.608 | 0.548 | 0.394 | 0.565 | 0.573 | 0.427 |
| AURE3 | 0.585 | 0.518 | 0.555 | 0.562 | 0.474 | 0.447 | 0.571 | 0.486 | 0.349 | 0.551 | 0.533 | 0.353 |
| AURE4 | 0.554 | 0.535 | 0.667 | 0.512 | 0.453 | 0.531 | 0.524 | 0.437 | 0.369 | 0.532 | 0.490 | 0.325 |
| AULE1 | 2.507 | 3.809 | 11.071 | 1.979 | 2.813 | 6.770 | 1.282 | 1.859 | 4.403 | 0.821 | 1.248 | 2.776 |
| AULE2 | 3.945 | 6.458 | 14.232 | 2.754 | 4.673 | 9.771 | 1.440 | 2.586 | 7.299 | 0.808 | 1.412 | 4.859 |
| ALLE1 | 0.754 | 0.841 | 3.627 | 0.723 | 0.673 | 1.754 | 0.679 | 0.648 | 0.873 | 0.582 | 0.635 | 0.568 |
| ALLE2 | 0.660 | 1.151 | 4.648 | 0.469 | 0.699 | 2.405 | 0.410 | 0.407 | 1.328 | 0.405 | 0.372 | 0.701 |
| LSE | 2.393 | 5.100 | 22.329 | 1.702 | 2.971 | 12.913 | 1.103 | 1.627 | 6.424 | 0.807 | 1.094 | 3.107 |
| LKLE1 | 2.875 | 7.916 | 47.162 | 1.333 | 3.594 | 25.600 | 0.499 | 1.166 | 11.035 | 0.352 | 0.441 | 4.126 |
| LKLE2 | 3.458 | 8.726 | 48.141 | 1.739 | 4.221 | 26.510 | 0.660 | 1.517 | 11.819 | 0.377 | 0.581 | 4.706 |
| LKLE3 | 3.843 | 9.618 | 50.014 | 1.960 | 4.770 | 28.022 | 0.721 | 1.728 | 12.864 | 0.384 | 0.643 | 5.281 |
| LKLE4 | 4.302 | 10.266 | 50.722 | 2.242 | 5.264 | 28.749 | 0.821 | 1.967 | 13.465 | 0.399 | 0.726 | 5.695 |
| LDE11 | 2.159 | 3.290 | 7.741 | 1.545 | 2.332 | 5.779 | 0.983 | 1.424 | 3.448 | 0.669 | 0.938 | 2.359 |
| LDE12 | 0.714 | 0.812 | 1.339 | 0.591 | 0.650 | 1.045 | 0.492 | 0.517 | 0.741 | 0.433 | 0.443 | 0.585 |
| LDE21 | 1.925 | 2.872 | 6.661 | 1.387 | 2.055 | 5.014 | 0.902 | 1.274 | 2.938 | 0.629 | 0.856 | 2.051 |
| LDE22 | 0.633 | 0.671 | 0.971 | 0.540 | 0.555 | 0.775 | 0.464 | 0.459 | 0.567 | 0.420 | 0.405 | 0.469 |
| LDE31 | 1.763 | 2.380 | 4.336 | 1.300 | 1.813 | 3.661 | 0.875 | 1.189 | 2.243 | 0.621 | 0.827 | 1.746 |
| LDE32 | 0.601 | 0.591 | 0.623 | 0.522 | 0.510 | 0.541 | 0.455 | 0.437 | 0.442 | 0.416 | 0.394 | 0.400 |
| LDE41 | 1.546 | 1.904 | 2.920 | 1.188 | 1.551 | 2.648 | 0.836 | 1.088 | 1.716 | 0.610 | 0.792 | 1.480 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LMRT11 | 0.596 | 0.558 | 0.534 | 0.537 | 0.500 | 0.465 | 0.487 | 0.455 | 0.387 | 0.450 | 0.438 | 0.368 |
| LMRT12 | 0.595 | 0.563 | 0.537 | 0.526 | 0.496 | 0.468 | 0.457 | 0.432 | 0.386 | 0.414 | 0.392 | 0.355 |
| LMRT21 | 0.598 | 0.575 | 0.566 | 0.525 | 0.500 | 0.487 | 0.458 | 0.433 | 0.392 | 0.420 | 0.401 | 0.352 |
| LMRT22 | 0.621 | 0.592 | 0.572 | 0.547 | 0.517 | 0.495 | 0.471 | 0.440 | 0.399 | 0.423 | 0.391 | 0.352 |
| LMRT31 | 0.603 | 0.605 | 0.687 | 0.522 | 0.510 | 0.572 | 0.450 | 0.427 | 0.428 | 0.415 | 0.392 | 0.352 |
| LMRT32 | 0.633 | 0.628 | 0.694 | 0.555 | 0.538 | 0.583 | 0.472 | 0.445 | 0.442 | 0.422 | 0.390 | 0.361 |
| LMRT41 | 0.627 | 0.656 | 0.785 | 0.530 | 0.542 | 0.670 | 0.444 | 0.430 | 0.489 | 0.409 | 0.384 | 0.369 |
| LMRT42 | 0.666 | 0.682 | 0.792 | 0.575 | 0.578 | 0.682 | 0.479 | 0.461 | 0.509 | 0.424 | 0.393 | 0.387 |
| LSRE1 | 1.208 | 1.813 | 2.530 | 0.949 | 1.271 | 2.404 | 0.702 | 0.806 | 1.690 | 0.581 | 0.638 | 1.047 |
| LSRE2 | 1.045 | 1.446 | 1.698 | 0.820 | 1.025 | 1.571 | 0.614 | 0.660 | 1.131 | 0.521 | 0.535 | 0.736 |
| LSRE3 | 0.903 | 0.930 | 0.639 | 0.736 | 0.737 | 0.467 | 0.585 | 0.561 | 0.358 | 0.513 | 0.498 | 0.368 |
| LSRE4 | 0.770 | 0.717 | 0.554 | 0.640 | 0.549 | 0.357 | 0.540 | 0.459 | 0.203 | 0.498 | 0.451 | 0.204 |
| LSKE1 | 2.758 | 5.495 | 22.697 | 2.091 | 3.424 | 13.344 | 1.113 | 2.058 | 6.944 | 0.348 | 1.121 | 3.642 |
| LSKE2 | 2.753 | 5.469 | 22.664 | 2.136 | 3.414 | 13.317 | 1.383 | 2.104 | 6.925 | 0.539 | 1.382 | 3.639 |
| LSKE3 | 2.736 | 5.416 | 22.525 | 2.130 | 3.389 | 13.215 | 1.412 | 2.103 | 6.868 | 0.584 | 1.416 | 3.622 |
| LSKE4 | 2.700 | 5.353 | 22.439 | 2.116 | 3.345 | 13.115 | 1.447 | 2.093 | 6.796 | 0.646 | 1.447 | 3.595 |

**Table 3.2**: Estimated MSE values of the estimators for $p = 5$.

| n | | 30 | | | 50 | | | 100 | | | 200 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 |
| MLE | 11.631 | 25.291 | 128.278 | 6.208 | 13.062 | 66.229 | 3.029 | 6.072 | 32.242 | 1.571 | 0.502 | 14.747 |
| LRE1 | 0.638 | 0.588 | 0.565 | 0.588 | 0.533 | 0.487 | 0.556 | 0.500 | 0.414 | 0.533 | 0.494 | 0.382 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LRE2 | 0.628 | 0.620 | 0.584 | 0.569 | 0.540 | 0.503 | 0.511 | 0.477 | 0.415 | 0.482 | 0.445 | 0.367 |
| LRE3 | 0.610 | 0.598 | 0.667 | 0.585 | 0.540 | 0.555 | 0.563 | 0.491 | 0.432 | 0.551 | 0.514 | 0.370 |
| LRE4 | 0.646 | 0.692 | 0.839 | 0.575 | 0.564 | 0.720 | 0.534 | 0.476 | 0.521 | 0.525 | 0.468 | 0.378 |
| LLE1 | 2.209 | 2.758 | 9.435 | 1.891 | 2.212 | 5.734 | 1.444 | 1.825 | 3.173 | 1.069 | 1.454 | 2.209 |
| LLE2 | 4.445 | 5.320 | 11.614 | 3.887 | 5.049 | 62.948 | 2.873 | 3.969 | 5.617 | 1.793 | 2.797 | 5.971 |
| AURE1 | 0.727 | 0.617 | 0.492 | 0.755 | 0.636 | 0.442 | 0.806 | 0.705 | 0.445 | 0.786 | 0.772 | 0.515 |
| AURE2 | 0.637 | 0.571 | 0.478 | 0.614 | 0.534 | 0.427 | 0.656 | 0.568 | 0.386 | 0.659 | 0.625 | 0.410 |
| AURE3 | 0.717 | 0.572 | 0.559 | 0.795 | 0.592 | 0.447 | 0.895 | 0.702 | 0.380 | 0.833 | 0.820 | 0.439 |
| AURE4 | 0.610 | 0.579 | 0.744 | 0.631 | 0.513 | 0.588 | 0.733 | 0.545 | 0.406 | 0.763 | 0.702 | 0.360 |
| AULE1 | 4.468 | 6.170 | 22.328 | 3.407 | 4.775 | 11.274 | 2.278 | 3.272 | 6.911 | 1.436 | 2.240 | 5.031 |
| AULE2 | 7.542 | 11.540 | 26.772 | 5.482 | 8.498 | 18.054 | 3.072 | 5.412 | 14.076 | 1.545 | 3.047 | 9.902 |
| ALLE1 | 1.221 | 1.390 | 6.253 | 1.151 | 1.156 | 3.690 | 1.149 | 1.134 | 1.445 | 0.946 | 1.094 | 1.054 |
| ALLE2 | 1.299 | 2.100 | 8.867 | 0.860 | 1.472 | 4.584 | 0.606 | 0.813 | 2.443 | 0.543 | 0.607 | 1.859 |
| LSE | 5.247 | 9.917 | 53.133 | 2.825 | 5.268 | 26.827 | 1.671 | 2.833 | 12.496 | 1.100 | 1.677 | 6.475 |
| LKLE1 | 9.040 | 20.226 | 123.573 | 4.091 | 10.350 | 65.417 | 1.352 | 3.587 | 28.242 | 0.522 | 1.266 | 12.715 |
| LKLE2 | 10.093 | 22.235 | 124.008 | 4.704 | 10.876 | 68.320 | 1.640 | 4.234 | 29.422 | 0.654 | 1.548 | 13.256 |
| LKLE3 | 9.359 | 22.179 | 130.037 | 3.986 | 10.556 | 66.991 | 1.254 | 3.825 | 29.981 | 0.480 | 1.168 | 13.026 |
| LDE11 | 4.707 | 6.689 | 16.556 | 3.228 | 4.921 | 12.061 | 1.916 | 3.053 | 7.747 | 1.223 | 1.903 | 5.323 |
| LDE12 | 0.892 | 1.129 | 2.494 | 0.729 | 0.871 | 1.614 | 0.579 | 0.653 | 1.101 | 0.483 | 0.532 | 0.812 |
| LDE21 | 4.165 | 6.482 | 13.383 | 2.850 | 4.390 | 10.614 | 1.867 | 2.838 | 7.140 | 1.179 | 1.800 | 4.656 |
| LDE22 | 0.773 | 0.920 | 1.739 | 0.640 | 0.704 | 1.284 | 0.528 | 0.556 | 0.841 | 0.459 | 0.463 | 0.629 |
| LDE31 | 4.433 | 6.031 | 9.989 | 3.209 | 4.627 | 8.060 | 1.998 | 3.020 | 6.279 | 1.237 | 1.978 | 4.522 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| LDE32 | 0.898 | 1.019 | 1.103 | 0.730 | 0.837 | 0.958 | 0.599 | 0.657 | 0.830 | 0.504 | 0.557 | 0.715 |
| LDE41 | 3.679 | 4.233 | 4.956 | 2.866 | 3.526 | 4.662 | 1.890 | 2.590 | 4.620 | 1.207 | 1.790 | 3.768 |
| LDE42 | 0.724 | 0.731 | 0.672 | 0.644 | 0.636 | 0.606 | 0.538 | 0.561 | 0.541 | 0.467 | 0.502 | 0.522 |
| LMRT11 | 0.637 | 0.604 | 0.550 | 0.583 | 0.530 | 0.486 | 0.537 | 0.484 | 0.405 | 0.509 | 0.485 | 0.372 |
| LMRT12 | 0.634 | 0.611 | 0.560 | 0.580 | 0.540 | 0.483 | 0.502 | 0.476 | 0.413 | 0.445 | 0.415 | 0.366 |
| LMRT21 | 0.648 | 0.630 | 0.605 | 0.560 | 0.540 | 0.508 | 0.510 | 0.465 | 0.420 | 0.466 | 0.440 | 0.369 |
| LMRT22 | 0.666 | 0.641 | 0.616 | 0.590 | 0.566 | 0.524 | 0.512 | 0.483 | 0.437 | 0.448 | 0.423 | 0.373 |
| LMRT31 | 0.613 | 0.602 | 0.703 | 0.581 | 0.538 | 0.571 | 0.558 | 0.485 | 0.430 | 0.525 | 0.494 | 0.373 |
| LMRT32 | 0.643 | 0.622 | 0.704 | 0.560 | 0.536 | 0.588 | 0.493 | 0.467 | 0.441 | 0.446 | 0.429 | 0.365 |
| LMRT41 | 0.646 | 0.699 | 0.855 | 0.578 | 0.567 | 0.730 | 0.511 | 0.471 | 0.541 | 0.512 | 0.451 | 0.393 |
| LMRT42 | 0.675 | 0.713 | 0.858 | 0.596 | 0.606 | 0.743 | 0.493 | 0.480 | 0.548 | 0.444 | 0.415 | 0.412 |
| LSRE1 | 1.851 | 2.397 | 3.816 | 1.267 | 1.736 | 3.003 | 0.848 | 1.056 | 2.416 | 0.643 | 0.755 | 1.585 |
| LSRE2 | 1.625 | 2.062 | 2.663 | 1.070 | 1.402 | 2.200 | 0.725 | 0.839 | 1.609 | 0.574 | 0.610 | 1.028 |
| LSRE3 | 1.759 | 1.789 | 1.055 | 1.229 | 1.385 | 0.890 | 0.882 | 1.045 | 0.954 | 0.689 | 0.773 | 0.879 |
| LSRE4 | 1.271 | 1.211 | 0.811 | 0.947 | 0.902 | 0.435 | 0.794 | 0.781 | 0.237 | 0.635 | 0.667 | 0.324 |
| LSKE2 | 10.093 | 22.235 | 124.008 | 4.704 | 10.876 | 68.320 | 1.640 | 4.234 | 29.422 | 0.654 | 1.548 | 13.256 |
| LSKE3 | 9.359 | 22.179 | 130.037 | 3.986 | 10.556 | 66.991 | 1.254 | 3.825 | 29.981 | 0.480 | 1.168 | 13.026 |
| LSKE4 | 10.611 | 24.086 | 135.735 | 4.800 | 11.855 | 69.916 | 1.546 | 4.498 | 31.173 | 0.539 | 1.494 | 14.818 |

**Table 3.3**: Estimated MSE values of the estimators for $p = 10$.

| n | | 30 | | | 50 | | | 100 | | | 200 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 | 0.90 | 0.95 | 0.99 |
| MLE | 47.954 | 106.361 | 585.462 | 20.999 | 44.501 | 258.327 | 8.729 | 18.143 | 103.653 | 3.988 | 0.855 | 46.025 |
| LRE1 | 0.691 | 0.669 | 0.658 | 0.642 | 0.584 | 0.563 | 0.561 | 0.512 | 0.459 | 0.548 | 0.489 | 0.393 |
| LRE2 | 0.724 | 0.711 | 0.692 | 0.639 | 0.608 | 0.600 | 0.553 | 0.529 | 0.489 | 0.514 | 0.473 | 0.403 |
| LRE3 | 0.695 | 0.640 | 0.736 | 0.714 | 0.589 | 0.602 | 0.796 | 0.632 | 0.441 | 0.857 | 0.746 | 0.406 |
| LRE4 | 0.721 | 0.785 | 0.927 | 0.634 | 0.659 | 0.838 | 0.634 | 0.536 | 0.641 | 0.738 | 0.572 | 0.447 |
| LLE1 | 3.273 | 3.558 | 10.102 | 3.006 | 3.165 | 3.966 | 2.591 | 2.944 | 2.824 | 2.029 | 2.588 | 2.853 |
| LLE2 | 7.472 | 8.509 | 20.515 | 7.402 | 8.286 | 201.690 | 5.777 | 7.663 | 8.004 | 3.764 | 5.873 | 9.271 |
| AURE1 | 0.699 | 0.622 | 0.543 | 0.692 | 0.597 | 0.457 | 0.754 | 0.615 | 0.413 | 0.849 | 0.736 | 0.424 |
| AURE2 | 0.662 | 0.623 | 0.556 | 0.612 | 0.552 | 0.463 | 0.637 | 0.537 | 0.396 | 0.711 | 0.608 | 0.387 |
| AURE3 | 0.957 | 0.656 | 0.614 | 1.229 | 0.775 | 0.472 | 1.565 | 1.158 | 0.421 | 1.576 | 1.525 | 0.581 |
| AURE4 | 0.677 | 0.684 | 0.866 | 0.719 | 0.580 | 0.728 | 1.055 | 0.660 | 0.503 | 1.293 | 1.010 | 0.387 |
| AULE1 | 8.042 | 9.780 | 27.340 | 6.504 | 7.632 | 11.417 | 4.779 | 6.426 | 8.307 | 3.032 | 4.566 | 7.701 |
| AULE2 | 17.640 | 22.244 | 44.033 | 13.178 | 17.739 | 21.094 | 8.065 | 13.111 | 20.605 | 4.059 | 7.596 | 21.557 |
| ALLE1 | 2.638 | 2.385 | 8.448 | 2.365 | 2.222 | 2.116 | 2.371 | 2.422 | 1.691 | 1.899 | 2.321 | 1.998 |
| LSE | 19.191 | 41.701 | 221.925 | 7.830 | 15.881 | 90.868 | 3.458 | 6.812 | 35.861 | 1.865 | 3.380 | 17.522 |
| LKLE1 | 45.251 | 103.522 | 582.275 | 17.992 | 40.498 | 237.973 | 6.069 | 15.444 | 95.195 | 2.023 | 5.486 | 41.830 |
| LKLE2 | 47.518 | 99.996 | 585.014 | 18.378 | 42.187 | 238.984 | 6.564 | 15.831 | 97.587 | 2.314 | 6.154 | 42.835 |
| LKLE3 | 44.177 | 98.575 | 568.866 | 15.070 | 39.545 | 254.657 | 3.888 | 12.351 | 96.150 | 1.158 | 3.871 | 40.069 |
| LKLE4 | 45.471 | 103.236 | 584.609 | 18.303 | 42.265 | 253.287 | 5.373 | 15.335 | 98.604 | 1.419 | 4.948 | 44.927 |
| LDE11 | 16.626 | 25.633 | 70.556 | 10.127 | 17.744 | 48.724 | 5.723 | 9.897 | 28.485 | 3.192 | 5.365 | 17.978 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LDE12 | 1.760 | 2.507 | 6.100 | 1.106 | 1.496 | 3.290 | 0.765 | 0.950 | 2.025 | 0.598 | 0.724 | 1.304 |
| LDE21 | 16.453 | 23.317 | 66.603 | 9.689 | 16.034 | 47.060 | 5.407 | 9.149 | 27.351 | 3.011 | 5.168 | 16.391 |
| LDE22 | 1.518 | 1.992 | 4.841 | 0.917 | 1.217 | 2.860 | 0.661 | 0.813 | 1.482 | 0.544 | 0.630 | 1.068 |
| LDE31 | 18.921 | 28.326 | 49.805 | 11.889 | 19.050 | 40.643 | 6.520 | 11.012 | 29.972 | 3.500 | 6.373 | 20.325 |
| LDE32 | 3.067 | 3.602 | 3.766 | 2.023 | 2.594 | 2.992 | 1.370 | 1.706 | 2.724 | 0.982 | 1.322 | 2.235 |
| LDE41 | 12.897 | 15.882 | 20.113 | 9.618 | 13.086 | 17.737 | 6.103 | 8.804 | 16.355 | 3.346 | 5.579 | 13.820 |
| LDE42 | 1.366 | 1.241 | 0.883 | 1.174 | 1.125 | 0.817 | 1.006 | 1.055 | 0.875 | 0.824 | 0.971 | 0.998 |
| LMRT11 | 0.697 | 0.683 | 0.658 | 0.628 | 0.584 | 0.566 | 0.559 | 0.511 | 0.456 | 0.546 | 0.484 | 0.394 |
| LMRT12 | 0.714 | 0.693 | 0.652 | 0.646 | 0.611 | 0.573 | 0.561 | 0.523 | 0.475 | 0.488 | 0.461 | 0.399 |
| LMRT21 | 0.719 | 0.697 | 0.685 | 0.638 | 0.618 | 0.599 | 0.553 | 0.523 | 0.477 | 0.507 | 0.459 | 0.397 |
| LMRT22 | 0.741 | 0.727 | 0.694 | 0.662 | 0.641 | 0.604 | 0.576 | 0.539 | 0.495 | 0.491 | 0.466 | 0.409 |
| LMRT31 | 0.698 | 0.624 | 0.732 | 0.713 | 0.593 | 0.596 | 0.796 | 0.618 | 0.461 | 0.852 | 0.720 | 0.409 |
| LMRT32 | 0.682 | 0.641 | 0.736 | 0.670 | 0.588 | 0.609 | 0.651 | 0.556 | 0.452 | 0.612 | 0.571 | 0.400 |
| LMRT41 | 0.725 | 0.790 | 0.930 | 0.627 | 0.657 | 0.836 | 0.636 | 0.530 | 0.646 | 0.726 | 0.568 | 0.460 |
| LMRT42 | 0.745 | 0.800 | 0.931 | 0.652 | 0.680 | 0.847 | 0.583 | 0.546 | 0.660 | 0.554 | 0.496 | 0.479 |
| LSRE2 | 3.870 | 5.270 | 6.486 | 2.103 | 2.937 | 4.641 | 1.183 | 1.504 | 2.863 | 0.725 | 0.891 | 1.866 |
| LSRE3 | 4.452 | 5.504 | 4.359 | 2.972 | 4.083 | 3.703 | 1.832 | 2.609 | 4.019 | 1.188 | 1.643 | 3.365 |
| LSRE4 | 3.020 | 3.388 | 2.593 | 2.108 | 2.211 | 1.440 | 1.395 | 1.590 | 0.808 | 1.026 | 1.206 | 0.857 |
| LSKE1 | 45.251 | 103.522 | 582.275 | 17.992 | 40.498 | 237.973 | 6.069 | 15.444 | 95.195 | 2.023 | 5.486 | 41.830 |
| LSKE2 | 47.518 | 99.996 | 585.014 | 18.378 | 42.187 | 238.984 | 6.564 | 15.831 | 97.587 | 2.314 | 6.154 | 42.835 |
| LSKE3 | 44.177 | 98.575 | 568.866 | 15.070 | 39.545 | 254.657 | 3.888 | 12.351 | 96.150 | 1.158 | 3.871 | 40.069 |
| LSKE4 | 45.471 | 103.236 | 584.609 | 18.303 | 42.265 | 253.287 | 5.373 | 15.335 | 98.604 | 1.419 | 4.948 | 44.927 |

**Table 3.4:** Relative efficiency of the estimators compares to MLE for p = 3 and $\rho = 0.90$

| n | | 30 | | 50 | | 100 | | 200 |
|---|---|---|---|---|---|---|---|---|
| | MSE | | MSE | | MSE | | MSE | |
| MLE | 4.678 | 1 | 2.863 | 1 | 1.469 | 1 | 0.841 | 1 |
| LRE1 | 0.606 | 7.719472 | 0.556 | 5.149281 | 0.515 | 2.8524272 | 0.479 | 1.755741 |
| LRE2 | 0.596 | 7.848993 | 0.528 | 5.422348 | 0.471 | 3.118896 | 0.44 | 1.911364 |
| LRE3 | 0.594 | 7.875421 | 0.518 | 5.527027 | 0.459 | 3.2004357 | 0.433 | 1.942263 |
| AURE3 | 0.585 | 7.996581 | 0.562 | 5.094306 | 0.571 | 2.5726795 | 0.551 | 1.526316 |
| AURE4 | 0.554 | 8.444043 | 0.512 | 5.591797 | 0.524 | 2.8034351 | 0.532 | 1.580827 |
| LDE32 | 0.601 | 7.783694 | 0.522 | 5.484674 | 0.455 | 3.2285714 | 0.416 | 2.021635 |
| LMRT11 | 0.596 | 7.848993 | 0.537 | 5.331471 | 0.487 | 3.0164271 | 0.45 | 1.868889 |
| LMRT12 | 0.595 | 7.862185 | 0.526 | 5.442966 | 0.457 | 3.214442 | 0.414 | 2.031401 |
| LMRT21 | 0.598 | 7.822742 | 0.525 | 5.453333 | 0.458 | 3.2074236 | 0.42 | 2.002381 |
| LMRT31 | 0.603 | 7.757877 | 0.522 | 5.484674 | 0.45 | 3.2644444 | 0.415 | 2.026506 |

**Table 3.5:** Relative efficiency of the estimators compares to MLE for p = 5 and ρ = 0.90

| n | | 30 | | 50 | | 100 | | 200 |
|---|---|---|---|---|---|---|---|---|
| | MSE | | MSE | | MSE | | MSE | |
| MLE | 11.631 | 1 | 6.208 | 1 | 3.029 | 1 | 1.571 | 1 |
| LRE1 | 0.638 | 18.23041 | 0.588 | 10.55782 | 0.556 | 5.4478417 | 0.533 | 2.947467 |
| LRE2 | 0.628 | 18.5207 | 0.569 | 10.91037 | 0.511 | 5.927593 | 0.482 | 3.259336 |
| LRE3 | 0.61 | 19.06721 | 0.585 | 10.61197 | 0.563 | 5.3801066 | 0.551 | 2.85118 |
| AURE2 | 0.637 | 18.25903 | 0.614 | 10.11075 | 0.656 | 4.617378 | 0.659 | 2.383915 |
| AURE4 | 0.61 | 19.06721 | 0.631 | 9.838352 | 0.733 | 4.1323329 | 0.763 | 2.058978 |
| LMRT11 | 0.637 | 18.25903 | 0.583 | 10.64837 | 0.537 | 5.6405959 | 0.509 | 3.086444 |
| LMRT12 | 0.634 | 18.34543 | 0.58 | 10.70345 | 0.502 | 6.0338645 | 0.445 | 3.530337 |
| LMRT31 | 0.613 | 18.9739 | 0.581 | 10.68503 | 0.558 | 5.4283154 | 0.525 | 2.992381 |
| LMRT32 | 0.643 | 18.08865 | 0.56 | 11.08571 | 0.493 | 6.1440162 | 0.446 | 3.522422 |
| LMRT41 | 0.646 | 18.00464 | 0.578 | 10.74048 | 0.511 | 5.927593 | 0.512 | 3.068359 |

**Table 1.6:** Relative efficiency of the estimators compares to MLE for p = 10 and ρ = 0.90

| n | | 30 | | 50 | | 100 | | 200 |
|---|---|---|---|---|---|---|---|---|
| | MSE | | MSE | | MSE | | MSE | |
| MLE | 47.954 | 1 | 20.999 | 1 | 8.729 | 1 | 3.988 | 1 |
| LRE1 | 0.691 | 69.39797 | 0.642 | 32.70872 | 0.561 | 15.559715 | 0.548 | 7.277372 |
| LRE2 | 0.724 | 66.23481 | 0.639 | 32.86228 | 0.553 | 15.78481 | 0.514 | 7.758755 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| LRE3 | 0.695 | 68.99856 | 0.714 | 29.41036 | 0.796 | 10.96608 | 0.857 | 4.653442 |
| AURE1 | 0.699 | 68.60372 | 0.692 | 30.34538 | 0.754 | 11.576923 | 0.849 | 4.697291 |
| AURE2 | 0.662 | 72.43807 | 0.612 | 34.31209 | 0.637 | 13.703297 | 0.711 | 5.609001 |
| AURE4 | 0.677 | 70.83309 | 0.719 | 29.20584 | 1.055 | 8.2739336 | 1.293 | 3.0843 |
| LMRT11 | 0.697 | 68.80057 | 0.628 | 33.4379 | 0.559 | 15.615385 | 0.546 | 7.304029 |
| LMRT21 | 0.719 | 66.69541 | 0.638 | 32.91379 | 0.553 | 15.78481 | 0.507 | 7.865878 |
| LMRT31 | 0.698 | 68.70201 | 0.713 | 29.45161 | 0.796 | 10.96608 | 0.852 | 4.680751 |
| LMRT32 | 0.682 | 70.31378 | 0.67 | 31.34179 | 0.651 | 13.408602 | 0.612 | 6.51634 |

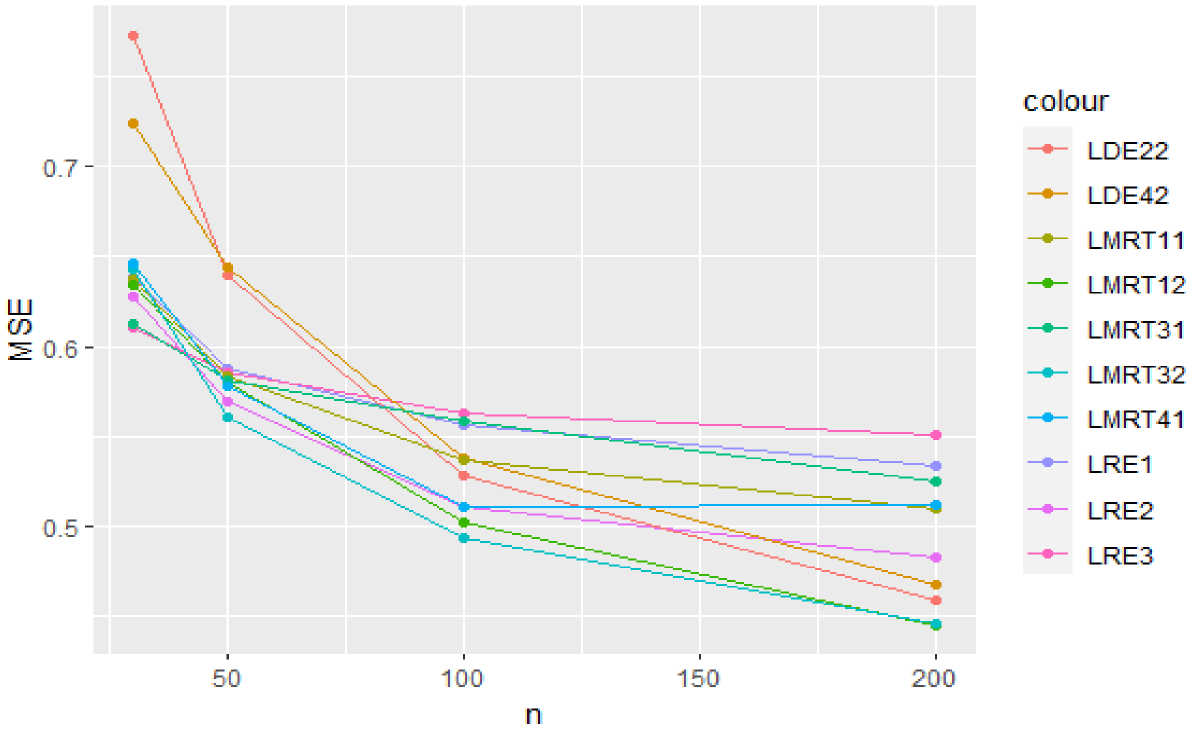

**Figure 3.1**:MSE values of the estimators versus $\boldsymbol{n}$ for $p = 5$ and $\rho = 0.90$

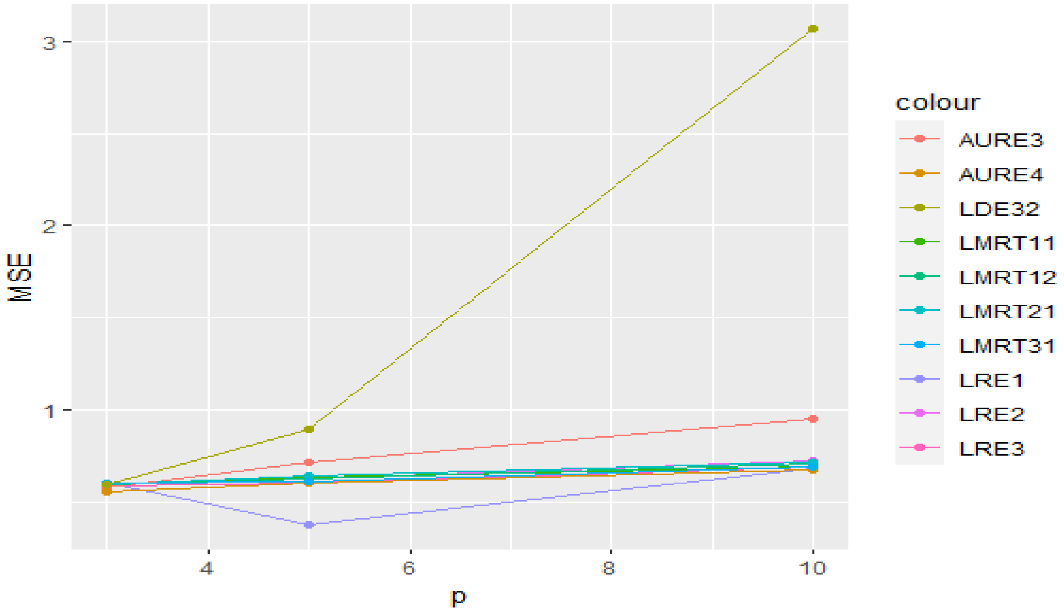

**Figure 3.2**: MSE values of the estimators versus $\boldsymbol{p}$ for $n = 30$ and $\rho = 0.90$

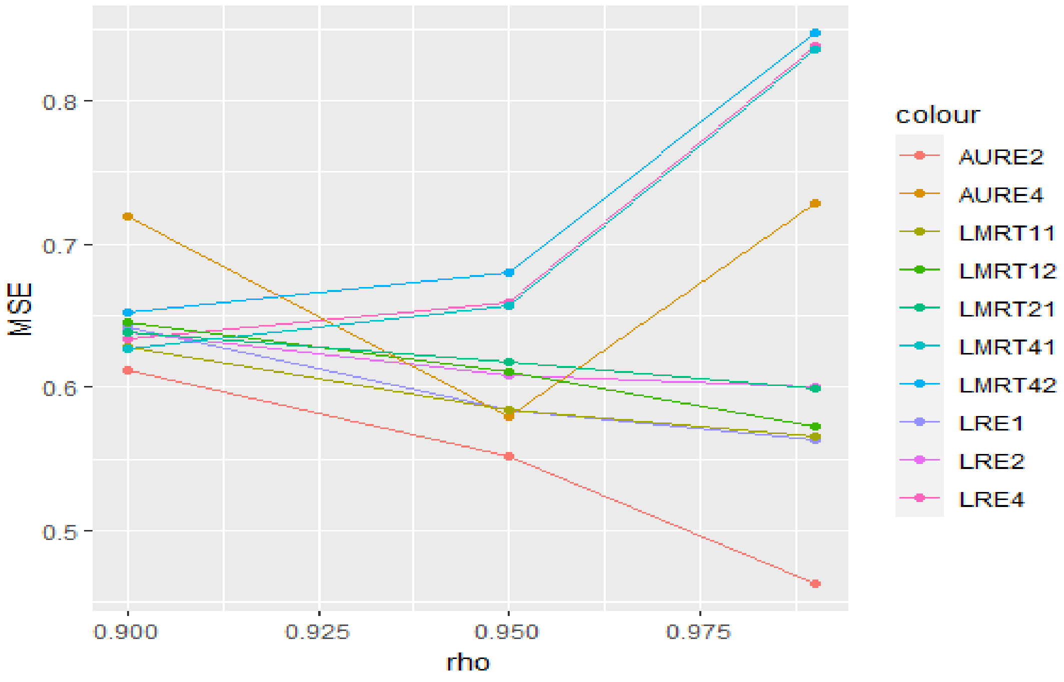

**Figure 3.3**: MSE values of the estimators versus $\boldsymbol{\rho}$ for $p = 10$ and $n = 50$

## 3.2 Simulation Results Discussion

In this section, we will discuss the simulations results and the performance of the estimators will be compared based on the estimated MSE values.

The simulated MSE values of the logistic regression model estimators under different controlled conditions are summarized in Tables 3.1- 3.3 for $p = 3, 5$ and 10, respectively. In Table 3.1, The MSE values are provided for different sample sizes ($n = 30, 50, 100, 200$) and correlation values ($\rho = 0.90, 0.95, 0.99$). It can be observed that as the sample size increases, the MSE values decreased for all estimators. Additionally, the increase of $\rho$ generally leads to an increase in the MSE values, although some estimators may not follow this pattern. This suggests that the biasing parameters used for logistic regression estimators may not be suitable for logistic regression estimation.

Table 3.2 presents the MSE values for $p = 5$, while Table 3.3 provides the results for $p = 10$. As the number of regressors increases, the MSE values tend to decrease. Among the estimators, AURE and LMRT consistently exhibit the smallest MSE values in our simulation results. However, some estimators show fluctuations in their MSE values, with some initially increasing and then decreasing. Our suggested Stein's estimator demonstrates improved results compared to the previous Stein's estimator.

### 3.2.1 Performance of the estimators as function of $n$

From Tables 3.1-3.3, we observe that as the sample sizes (n) increases, the MSE values decreases for all estimators across different values of $\rho$ and $p$. Among the estimators, AURE4 yields the smallest MSE values followed by LMRT31, LDE42, and AURE1, which also demonstrate better performance. On the other hand, MLE consistently produces the highest MSE values, with LKLE4 and LDE11 also yielding poorer results.

### 3.2.2 Performance of the estimators as function of $p$

Analyzing the Tables 3.1-3.3, we compare the estimated MSE values for different estimators across different values of $p$ (3, 5, and 10) while considering various sample sizes (n) and $\rho$. As the number of regressors ($p$) increases, the estimated MSE generally decreases for various values of $n$ and $\rho$. However, it is worth noting that some estimators deviate from this pattern and show an increase in MSE with the increase in $\rho$. Among the estimators, AURE4 provides the smallest MSE values, while LDE11 and LMRT31 also yield better results. On the other hand, MLE exhibits the highest MSE values, with LKLE2 also demonstrating large MSE values.

### 3.2.3 Performance of the estimators as function of $\rho$

Table 3.1-3.3 also allow us to evaluate the performance of the estimators for different values of $\rho$. Generally, as $\rho$ increases, the MSE values decreases for various sample sizes (30,50, 100, and 200) and number of regressors (3, 5, and 10). However, some estimators do not consistently follow this trend and may not be capable of accurately estimating the biasing parameters in the logistic regression model. Among the estimators, AURE consistently yields better results as $\rho$ increases, while MLE consistently exhibits the worst performance.

### 3.2.4 Relative efficiency of the estimators

The relative efficiency provides a measure of how well an estimator performs compared to MLE. It is calculated as the ratio of the MSE of the MLE to the MSE of the other estimators.

In Table 3.4, the relative efficiency of different estimators compared to MLE is examined for the case where $p = 3$ and $\rho = 0.90$. Similarly, in Table 3.5 and 3.6, the relative efficiency is calculated for different regressor. The purpose is to assess the performance of the best estimators identified in these scenarios and compared them to the MLE.

By considering the performance of the estimators in terms of $n$, $p$, and $\rho$, we can gain insights into the behavior and effectiveness of these estimators. AURE4 consistently performs well across different scenarios, while MLE tends to have higher MSE values. Understanding the performance of these estimators helps in selecting appropriate methods for logistic regression estimation.

# Chapter 4

# Applications

In this chapter, we will discuss two real-life examples to examine the performance of the logistic regression estimators.

## 4.1 Prostate Cancer Data

In this example prostate cancer dataset from Kurtnet et al. (2005) is considered. The response variable, denoted as $y_i$ represents the seminal vesicle invasion, where 1 indicates the presence and 0 indicates absence of seminal vesicle invasion. The dataset includes seven explanatory variables: cancer volume ($x_1$), weight ($x_2$), age ($x_3$), Benign prostate hyperplasia ($x_4$), Capsular penetration ($x_5$), Gleason score ($x_6$ ) and PSA level ($x_7$).

### 4.1.1 Data Checking for Prostate Cancer Data

To assess multicollinearity among the regressors, the eigen values of the matrix $X'\widehat{W}X$ are computed. The resulting eigen values are $\lambda_1 = 69109.31254$, $\lambda_2 = 58.4226$, $\lambda_3 = 35.33693$, $\lambda_4 = 13.643$, $\lambda_5 = 10.925$, $\lambda_6 = 5.075$ and $\lambda_7 = 2.50087$. The condition index number, calculated as $\sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = 166.2351 > 100$, indicating the presence of serious multicollinearity (Liu, 2003) among the regressors. Table 4.2 presents the estimated coefficients and MSE values for different estimators. The coefficients are obtained using equations (5), (9), (14), (19), (24), (29), (34), (40), (45), (50), (55) and (61). The scalar MSE values of these estimators are calculated using their corresponding MSE equations. The estimated parameters values for these estimators are $k_1 = 7$, $k_2 = 9.243284$, $k_3 = 0.014224$, $k_4 = 0.10462$ and $d_1 = 0.0004416$, $d_2 = 0.0025002$. These values represent the specific parameter choices for the estimators used in the analysis.

In this study, we evaluated the theoretical conditions stated in Theorems 2.1 to 2.21 for the prostate cancer dataset. The results of this evaluation are presented in Table 4.1. It can be observed that all the conditions for the prostate cancer data set are less than one, as specified in these theorems.

**Table 4.1:** Validation of the theoretical conditions for prostate cancer data

| Theorems | Conditions | Value |
|---|---|---|
| 2.1 | $b'_{LSRE}[\Lambda^{-1} - c_r^2 A\Lambda^{-1}A']^{-1} b_{LSRE} < 1$ | 0.1359 |
| 2.2 | $b'_{LSRE}[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}]^{-1} b_{LSRE} < 1$ | 0.0174 |
| 2.3 | $b'_{LSRE}\left[\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + (1-d)^2 Q\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}Q'\right]^{-1} b_{LSRE} < 1$ | 0.0509 |
| 2.4 | $b'_{LSRE}[(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_r^2 A\Lambda^{-1}A' + k^4\,\Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2}]^{-1} b_{LSRE} < 1$ | 0.0266 |
| 2.5 | $b'_{LSRE}\left[(I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_r^2 A\Lambda^{-1}A' + (1-d)^4\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$ | 0.0721 |
| 2.6 | $b'_{LSRE}\left[\,\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + (1+d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$ | 0.0509 |
| 2.7 | $b'_{LSRE}\left[\,c_s^2\Lambda^{-1} - c_r^2 A\Lambda^{-1}A' + (c_s - 1)^2\alpha\alpha'\right]^{-1} b_{LSRE} < 1$ | 0.0141 |
| 2.8 | $b'_{LSRE}\left[\,\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + 4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSRE} < 1$ | 0.0268 |
| 2.9 | $b'_{LSRE}[\,(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_r^2 A\Lambda^{-1}A' + k^2 d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}]^{-1} b_{LSRE} < 1$ | 0.0798 |
| 2.10 | $b'_{LSRE}[(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_r^2 A\Lambda^{-1}A' + k^2(1+d)^2(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}]^{-1} b_{LSRE} < 1$ | 0.0170 |
| 2.11 | $b'_{LSKE}\left[\Lambda^{-1} - c_{\mathrm{k}}^2 \mathrm{K}\Lambda^{-1}\mathrm{K}'\right]^{-1} b_{LSKE} < 1$ | 0.2707 |

| | | |
|---|---|---|
| 2.12 | $b'_{LSKE}\left[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.0623 |
| 2.13 | $b'_{LSKE}\left[\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}' + (1-d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.1154 |
| 2.14 | $b'_{LSKE}[(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_k^2K\Lambda^{-1}K' + k^4\,\Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2}]^{-1} b_{LSKE} < 1$ | 0.0645 |
| 2.15 | $b'_{LSRE}\left[(I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_k^2K\Lambda^{-1}K^{'} + (1-d)^4\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$ | 0.1598 |
| 2.16 | $b'_{LSKE}\left[\,\Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_k^2K\Lambda^{-1}K^{'} + (1+d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.1153 |
| 2.17 | $b'_{LSKE}\left[\,c_s^2\Lambda^{-1} - c_k^2K\Lambda^{-1}K^{'} + (c_s-1)^2\alpha\alpha^{'}\right]^{-1} b_{LSKE} < 1$ | 0.0383 |
| 2.18 | $b'_{LSKE}\left[\,\Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2K\Lambda^{-1}K^{'} + 4k^2\Lambda_k^{-1}\alpha\alpha^{'}\Lambda_k^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.0649 |
| 2.19 | $b'_{LSKE}\left[\,(\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_k^2K\Lambda^{-1}K^{'} + k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.1759 |
| 2.20 | $b'_{LSKE}[(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_k^2K\Lambda^{-1}K' + k^2(1+d)^2(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}]^{-1} b_{LSKE} < 1$ | 0.0444 |
| 2.21 | $b'_{LSKE}[c_r^2A\Lambda^{-1}A' - c_k^2K\Lambda^{-1}K' + (c_rA - 1)\alpha\alpha'(c_rA - 1)']^{-1} b_{LSKE} < 1$ | 0.0321 |

Table 4.2 displays the estimated regression coefficients and MSE values for different estimators applied to the prostate cancer dataset. It is evident that the maximum likelihood estimators (MLE) yield the largest MSE values, while some estimators approach this value closely. The LLE2 estimator also produces higher MSE values compared to MLE. On the other hand, the LMRT22 estimator exhibits the smallest MSE values among all the estimators for this dataset. Additionally, LDE21, LDE12, LMRT11, James Stein's

estimator and the proposed Stein estimators LSRE1 and LSRE2 demonstrate good performance, yielding small MSE values.

These results suggest that LMRT22, James Stein' estimator, and the proposed Stein estimators LSRE1 and LSRE2 are more effective in estimating the parameters for the prostate cancer dataset compared MLE and LLE2.

**Table 4.2**: Regression Analysis and MSEs of the logistic regression estimators for the prostate cancer data

| | x1 | x2 | x3 | x4 | x5 | x6 | x7 | MSE |
|---|---|---|---|---|---|---|---|---|
| MLE | -0.00041 | -0.31977 | -0.00263 | 0.01084 | 0.21367 | -0.23406 | 0.14794 | 0.80715 |
| LRE1 | 0.05394 | -0.11081 | -0.02092 | 0.01938 | 0.16602 | -0.15281 | 0.06320 | 0.20379 |
| LRE2 | 0.05612 | -0.09368 | -0.02349 | 0.02027 | 0.15547 | -0.13592 | 0.05713 | 0.17934 |
| LRE3 | -0.00002 | -0.31826 | -0.00271 | 0.01086 | 0.21357 | -0.23392 | 0.14734 | 0.80112 |
| LRE4 | 0.00241 | -0.30905 | -0.00324 | 0.01100 | 0.21292 | -0.23298 | 0.14366 | 0.76478 |
| LLE1 | 0.02051 | -0.24283 | -0.00752 | 0.01257 | 0.20589 | -0.22161 | 0.11663 | 0.52857 |
| LLE2 | -0.03182 | -0.43525 | 0.00472 | 0.00824 | 0.22534 | -0.25273 | 0.19495 | 1.38316 |
| AURE1 | 0.04374 | -0.17504 | -0.01174 | 0.01569 | 0.20186 | -0.21160 | 0.08658 | 0.34984 |
| AURE2 | 0.05075 | -0.15254 | -0.01422 | 0.01768 | 0.19546 | -0.19826 | 0.07738 | 0.29576 |
| AURE3 | -0.00041 | -0.31976 | -0.00263 | 0.01084 | 0.21367 | -0.23406 | 0.14794 | 0.80712 |
| AURE4 | -0.00032 | -0.31935 | -0.00264 | 0.01084 | 0.21368 | -0.23410 | 0.14779 | 0.80567 |
| AULE1 | 0.00490 | -0.29864 | -0.00347 | 0.01074 | 0.21400 | -0.23541 | 0.13976 | 0.73215 |
| AULE2 | 0.01155 | -0.27217 | -0.00453 | 0.01062 | 0.21442 | -0.23710 | 0.12951 | 0.64636 |
| ALLE1 | 0.02053 | -0.24276 | -0.00753 | 0.01258 | 0.20588 | -0.22160 | 0.11660 | 0.52836 |

|  | x1 | x2 | x3 | x4 | x5 | x6 | x7 | MSE |
|---:|---:|---:|---:|---:|---:|---:|---:|---:|
| ALLE2 | 0.07285 | -0.05034 | -0.01977 | 0.01691 | 0.18643 | -0.19048 | 0.03828 | 0.20017 |
| LSE | -0.00009 | -0.06963 | -0.00057 | 0.00236 | 0.04653 | -0.05097 | 0.03222 | 0.17577 |
| LKLE1 | 0.10830 | 0.09815 | -0.03922 | 0.02792 | 0.11837 | -0.07157 | -0.02155 | 0.29746 |
| LKLE2 | 0.11265 | 0.13241 | -0.04436 | 0.02969 | 0.09726 | -0.03779 | -0.03368 | 0.38254 |
| LKLE3 | 0.00038 | -0.31675 | -0.00280 | 0.01088 | 0.21347 | -0.23378 | 0.14674 | 0.79512 |
| LKLE4 | 0.00524 | -0.29834 | -0.00385 | 0.01115 | 0.21218 | -0.23191 | 0.13938 | 0.72398 |
| LDE11 | -0.00033 | -0.31944 | -0.00264 | 0.01085 | 0.21365 | -0.23403 | 0.14781 | 0.80583 |
| LDE12 | 0.05579 | -0.06059 | -0.02917 | 0.02068 | 0.12761 | -0.09596 | 0.04613 | 0.14709 |
| LDE21 | -0.00030 | -0.31933 | -0.00265 | 0.01085 | 0.21364 | -0.23402 | 0.14777 | 0.80541 |
| LDE22 | 0.05342 | -0.04910 | -0.03134 | 0.01998 | 0.11454 | -0.07975 | 0.04215 | 0.14109 |
| LDE31 | -0.00041 | -0.31977 | -0.00263 | 0.01084 | 0.21367 | -0.23406 | 0.14794 | 0.80715 |
| LDE32 | 0.00057 | -0.31603 | -0.00284 | 0.01089 | 0.21342 | -0.23370 | 0.14645 | 0.79224 |
| LDE41 | -0.00041 | -0.31976 | -0.00263 | 0.01084 | 0.21367 | -0.23406 | 0.14794 | 0.80713 |
| LDE42 | 0.00631 | -0.29446 | -0.00411 | 0.01125 | 0.21176 | -0.23122 | 0.13779 | 0.70891 |
| LMRT11 | 0.05395 | -0.11078 | -0.02093 | 0.01938 | 0.16600 | -0.15279 | 0.06319 | 0.20375 |
| LMRT12 | 0.05277 | -0.04691 | -0.03176 | 0.01976 | 0.11176 | -0.07653 | 0.04134 | 0.14029 |
| LMRT21 | 0.05612 | -0.09365 | -0.02350 | 0.02027 | 0.15545 | -0.13589 | 0.05712 | 0.17930 |
| LMRT22 | 0.04907 | -0.03754 | -0.03358 | 0.01843 | 0.09858 | -0.06228 | 0.03751 | 0.13824 |
| LMRT31 | -0.00002 | -0.31826 | -0.00271 | 0.01086 | 0.21357 | -0.23392 | 0.14734 | 0.80111 |
| LMRT32 | 0.00095 | -0.31456 | -0.00292 | 0.01091 | 0.21332 | -0.23356 | 0.14587 | 0.78642 |
| LMRT41 | 0.00241 | -0.30905 | -0.00324 | 0.01100 | 0.21292 | -0.23298 | 0.14366 | 0.76476 |
| LMRT42 | 0.00871 | -0.28559 | -0.00465 | 0.01143 | 0.21096 | -0.22997 | 0.13419 | 0.67601 |

|  | x1 | x2 | x3 | x4 | x5 | x6 | x7 | MSE |
|---|---|---|---|---|---|---|---|---|
| LSRE1 | 0.02694 | -0.05534 | -0.01045 | 0.00968 | 0.08291 | -0.07631 | 0.03156 | 0.14714 |
| LSRE2 | 0.03143 | -0.05247 | -0.01316 | 0.01135 | 0.08708 | -0.07613 | 0.03200 | 0.14436 |
| LSRE3 | -0.00000 | -0.06960 | -0.00059 | 0.00238 | 0.04671 | -0.05116 | 0.03222 | 0.17560 |
| LSRE4 | 0.00054 | -0.06941 | -0.00073 | 0.00247 | 0.04782 | -0.05232 | 0.03226 | 0.17457 |
| LSKE1 | 0.00490 | 0.00444 | -0.00177 | 0.00126 | 0.00535 | -0.00324 | -0.00097 | 0.21927 |
| LSKE2 | -0.00904 | -0.01062 | 0.00356 | -0.00238 | -0.00780 | 0.00303 | 0.00270 | 0.23328 |
| LSKE3 | 0.00008 | -0.06957 | -0.00061 | 0.00239 | 0.04689 | -0.05135 | 0.03223 | 0.17543 |
| LSKE4 | 0.00121 | -0.06913 | -0.00089 | 0.00258 | 0.04916 | -0.05373 | 0.03230 | 0.17333 |

### 4.1.2 Model Accuracy for Prostate Cancer data

In this Table 4.3, we check the model accuracy for some selected estimators. We can see that they have the almost same accuracy rate (87.63%).

**Table 4.3**: Model accuracy rate for prostate cancer data.

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 75 | 11 |
| 1 | 1 | 10 |

(a) LMRT12

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 74 | 11 |
| 1 | 2 | 10 |

(b) LDE22

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 75 | 11 |
| 1 | 1 | 10 |

(c) LSRE2

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 75 | 11 |
| 1 | 1 | 10 |

(d) LDE12

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 75 | 11 |
| 1 | 1 | 10 |

(e) LSRE1

### 4.2 Cancer Remission Data

The performance of the logistic regression estimators was evaluated using cancer remission data, which were sourced from Lesaffre and Marx (1993) and analyzed by Ozkale et al. (2016). In this data set, the response variable $y_i$ indicates whether a patient experiences complete cancer remission (1) or not (0). There are five explanatory variables: cell index ($x_1$), smear index ($x_2$), infill index ($x_3$), blast index ($x_4$) and temperature ($x_5$). The dataset comprises 27 patients, with nine of them experiencing complete remission.

#### 4.2.1 Data Checking for Cancer Remission Data

The eigen values of the $X'\widehat{W}X$ matrix, which measures the multicollinearity among the regressor variables, were found to be $\lambda_1 = 21.8372$, $\lambda_2 = 0.8011$, $\lambda_3 = 0.3404$, $\lambda_4 = 0.1228$ and $\lambda_5 = 0.0019$. The condition index number, calculated as the square root of the ratio of the largest eigen value to the smallest eigen value, is 107.2066. This value indicates the presence of serious multicollinearity among the regressor variables, suggesting that the variables are highly correlated.

**Table 4.4:** Validation of the theoretical conditions for prostate cancer remission data

| Theorems | Conditions | Value |
|---|---|---|
| 2.2.1 | $b'_{LSRE}[\Lambda^{-1} - c_r^2 A\Lambda^{-1}A']^{-1} b_{LSRE} < 1$ | 0.948 |
| 2.2.2 | $b'_{LSRE}[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}]^{-1} b_{LSRE} < 1$ | 0.686 |

| | | |
|---|---|---|
| 2.2.3 | $b'_{LSRE}\left[\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + (1-d)^2 Q\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}Q'\right]^{-1} b_{LSRE} < 1$ | 1.479 |
| 2.2.4 | $b'_{LSRE}[(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_r^2 A\Lambda^{-1}A' + k^4\ \Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2}]^{-1} b_{LSRE} < 1$ | 1.104 |
| 2.2.5 | $b'_{LSRE}\left[(I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_r^2 A\Lambda^{-1}A' + (1-d)^4\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$ | 1.179 |
| 2.2.6 | $b'_{LSRE}\left[\ \Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_r^2 A\Lambda^{-1}A' + (1+d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$ | 1.488 |
| 2.2.7 | $b'_{LSRE}\left[\ c_s^2\Lambda^{-1} - c_r^2 A\Lambda^{-1}A' + (c_s - 1)^2\alpha\alpha'\right]^{-1} b_{LSRE} < 1$ | 1.123 |
| 2.2.8 | $b'_{LSRE}\left[\ \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_r^2 A\Lambda^{-1}A' + 4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSRE} < 1$ | 0.186 |
| 2.2.9 | $b'_{LSRE}[\ (\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_r^2 A\Lambda^{-1}A' + k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}]^{-1} b_{LSRE} < 1$ | 0.444 |
| 2.2.10 | $b'_{LSRE}[(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_r^2 A\Lambda^{-1}A' + k^2(1+d)^2(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}]^{-1} b_{LSRE} < 1$ | 0.682 |
| 2.2.11 | $b'_{LSKE}\left[\mathrm{\Lambda}^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}'\right]^{-1} b_{LSKE} < 1$ | 0.934 |
| 2.2.12 | $b'_{LSKE}\left[\Lambda_k^{-1}\Lambda\Lambda_k^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}' + k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSKE} < 1$ | 1.011 |
| 2.2.13 | $b'_{LSKE}\left[\Lambda_I^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_I^{-1} - c_{\mathrm{k}}^2\mathrm{K}\Lambda^{-1}\mathrm{K}' + (1-d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSKE} < 1$ | 2.301 |
| 2.2.14 | $b'_{LSKE}[(I - k^2\Lambda_k^{-2})\Lambda^{-1}(I - k^2\Lambda_k^{-2}) - c_k^2 K\Lambda^{-1}K' + k^4\ \Lambda_k^{-2}\alpha\alpha'\Lambda_k^{-2}]^{-1} b_{LSKE} < 1$ | 3.001 |
| 2.2.15 | $b'_{LSRE}\left[(I - (1-d)^2\Lambda_I^{-2})\Lambda^{-1}(I - (1-d)^2\Lambda_I^{-2}) - c_k^2 K\Lambda^{-1}K' + (1-d)^4\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSRE} < 1$ | 1.519 |
| 2.2.16 | $b'_{LSKE}\left[\ \Lambda_I^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)\Lambda_I^{-1} - c_k^2 K\Lambda^{-1}K' + (1+d)^2\Lambda_I^{-1'}\alpha\alpha'\Lambda_I^{-1}\right]^{-1} b_{LSKE} < 1$ | 2.369 |

| | | |
|---|---|---|
| 2.2.17 | $b'_{LSKE}\left[\, c_s^2\Lambda^{-1} - c_k^2 K\Lambda^{-1}K' + (c_s - 1)^2\alpha\alpha'\right]^{-1} b_{LSKE} < 1$ | 1.044 |
| 2.2.18 | $b'_{LSKE}\left[\, \Lambda_k^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - KI)\Lambda_k^{-1} - c_k^2 K\Lambda^{-1}K' + 4k^2\Lambda_k^{-1}\alpha\alpha'\Lambda_k^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.309 |
| 2.2.19 | $b'_{LSKE}\left[\, (\Lambda + kdI)^{-1}\Lambda^{-1}(\Lambda + kdI)^{-1} - c_k^2 K\Lambda^{-1}K' + k^2d^2(\Lambda + kdI)^{-1}\alpha\alpha'(\Lambda + kdI)^{-1}\right]^{-1} b_{LSKE} < 1$ | 0.943 |
| 2.2.20 | $b'_{LSKE}[(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1} - c_k^2 K\Lambda^{-1}K' + k^2(1+d)^2(\Lambda + k(1+d)I)^{-1}\alpha\alpha'(\Lambda + k(1+d)I)^{-1}]^{-1} b_{LSKE} < 1$ | 1.028 |
| 2.2.21 | $b'_{LSKE}[c_r^2 A\Lambda^{-1}A' - c_k^2 K\Lambda^{-1}K' + (c_r A - 1)\alpha\alpha'(c_r A - 1)']^{-1} b_{LSKE} < 1$ | -27.94 |

The theoretical conditions for the cancer remission dataset were also examined, and the results showed that most of the evaluating values were less than one, indicating that the conditions were satisfied. However, there were some cases where the estimators did not meet the condition, likely because certain estimators did not perform better than others in terms of smaller mean square error (MSE) criterion.

The estimated regression coefficients and MSE values for the estimators are presented in Table 4.5 for the cancer remission dataset. It is observed that the maximum likelihood estimator (MLE) yielded the largest MSE, while the LKLE estimator resulted in a higher MSE compared to MLE. On the other hand, the Liu estimator (LLE1 and LLE2) produce the smallest MSE values. Furthermore, the proposed estimators also demonstrated better performance compared to other estimators.

**Table 4.5**: Regression Analysis and MSE of the logistic regression estimators for the cancer remission data.

| | x1 | x2 | x3 | x4 | x5 | MSE |
|---:|---:|---:|---:|---:|---:|---:|
| MLE | -0.028 | -1.058 | 1.095 | 0.531 | -2.735 | 538.691 |
| LRE1 | -0.480 | -0.320 | -0.255 | -0.359 | -0.619 | 8.636 |
| LRE2 | -0.440 | -0.298 | -0.245 | -0.364 | -0.551 | 8.884 |
| LRE3 | -0.550 | -0.350 | -0.252 | -0.304 | -0.768 | 8.040 |
| LRE4 | -0.479 | -0.319 | -0.255 | -0.359 | -0.617 | 8.643 |
| LLE1 | -0.629 | -0.376 | -0.163 | -0.096 | -1.106 | 3.141 |
| LLE2 | -0.628 | -0.377 | -0.160 | -0.095 | -1.109 | 3.131 |
| AURE1 | -0.601 | -0.380 | -0.274 | -0.320 | -0.838 | 7.895 |
| AURE2 | -0.569 | -0.371 | -0.285 | -0.373 | -0.757 | 8.249 |
| AURE3 | -0.655 | -0.386 | -0.223 | -0.164 | -1.035 | 7.194 |
| AURE4 | -0.600 | -0.380 | -0.275 | -0.322 | -0.835 | 7.905 |
| AULE1 | -0.677 | -0.388 | -0.018 | 0.158 | -1.510 | 6.916 |
| AULE2 | -0.675 | -0.391 | -0.013 | 0.160 | -1.515 | 6.980 |
| ALLE1 | -0.638 | -0.365 | -0.183 | -0.106 | -1.080 | 7.141 |
| ALLE2 | -0.640 | -0.364 | -0.185 | -0.107 | -1.077 | 7.174 |
| LSE | -0.001 | -0.020 | 0.021 | 0.010 | -0.051 | 9.899 |
| LKLE1 | -0.932 | 0.419 | -1.605 | -1.249 | 1.498 | 570.067 |
| LKLE2 | -0.851 | 0.462 | -1.586 | -1.259 | 1.634 | 571.866 |
| LKLE3 | -1.072 | 0.359 | -1.600 | -1.138 | 1.199 | 565.304 |

|  | x1 | x2 | x3 | x4 | x5 | MSE |
|---:|---:|---:|---:|---:|---:|---:|
| LKLE4 | -0.930 | 0.420 | -1.605 | -1.250 | 1.501 | 570.119 |
| LDE11 | -0.144 | -0.566 | 0.541 | 0.466 | -2.547 | 13.111 |
| LDE12 | -0.193 | -0.541 | 0.504 | 0.452 | -2.481 | 12.158 |
| LDE21 | -0.217 | -0.529 | 0.486 | 0.445 | -2.448 | 11.758 |
| LDE22 | -0.272 | -0.504 | 0.444 | 0.427 | -2.368 | 10.929 |
| LDE31 | -0.035 | -0.632 | 0.626 | 0.493 | -2.685 | 16.853 |
| LDE32 | -0.069 | -0.609 | 0.598 | 0.485 | -2.643 | 15.255 |
| LDE41 | -0.146 | -0.565 | 0.540 | 0.465 | -2.544 | 13.073 |
| LDE42 | -0.195 | -0.540 | 0.503 | 0.451 | -2.478 | 12.124 |
| LMRT11 | -0.479 | -0.319 | -0.255 | -0.359 | -0.617 | 8.643 |
| LMRT12 | -0.479 | -0.319 | -0.255 | -0.359 | -0.617 | 8.644 |
| LMRT21 | -0.439 | -0.298 | -0.245 | -0.364 | -0.549 | 8.889 |
| LMRT22 | -0.439 | -0.297 | -0.245 | -0.364 | -0.549 | 8.891 |
| LMRT31 | -0.549 | -0.349 | -0.253 | -0.305 | -0.766 | 8.048 |
| LMRT32 | -0.549 | -0.349 | -0.253 | -0.305 | -0.766 | 8.050 |
| LMRT41 | -0.478 | -0.319 | -0.255 | -0.360 | -0.615 | 8.650 |
| LMRT42 | -0.478 | -0.318 | -0.255 | -0.360 | -0.615 | 8.651 |
| LSRE1 | -0.771 | -0.513 | -0.410 | -0.577 | -0.994 | 8.182 |
| LSRE2 | -0.730 | -0.495 | -0.407 | -0.603 | -0.914 | 8.471 |
| LSRE3 | -0.850 | -0.540 | -0.390 | -0.469 | -1.187 | 7.521 |
| LSRE4 | -0.770 | -0.513 | -0.410 | -0.578 | -0.992 | 8.190 |
| LSKE1 | 0.012 | -0.005 | 0.021 | 0.016 | -0.020 | 9.963 |

|  | x1 | x2 | x3 | x4 | x5 | MSE |
|---|---|---|---|---|---|---|
| LSKE2 | 0.012 | -0.006 | 0.022 | 0.017 | -0.023 | 9.952 |
| LSKE3 | 0.012 | -0.004 | 0.018 | 0.013 | -0.014 | 9.988 |
| LSKE4 | 0.012 | -0.005 | 0.021 | 0.016 | -0.020 | 9.963 |

### 4.1.2 Model Accuracy for Prostate Cancer data

In this Table 4.6, we check the model accuracy for some selected estimators. We can see that they have the almost same accuracy rate (70.37%).

**Table 4.6**: Model accuracy rate for cancer remission data.

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 15 | 5 |
| 1 | 3 | 4 |

(a) LLE1

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 15 | 5 |
| 1 | 3 | 4 |

(b) LLE2

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 15 | 5 |
| 1 | 3 | 4 |

(c) AULE1

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 16 | 5 |
| 1 | 2 | 4 |

(d) AULE2

|  | Actual |  |
|---|---|---|
| Predicted | 0 | 1 |
| 0 | 15 | 5 |
| 1 | 3 | 4 |

(e) ALLE1

# Chapter 5

# Some Concerning Remarks

In this thesis, we proposed two new Stein's estimators for logistic regression model: one for logistic ridge regression and another one for logistic K-L estimator. We also considered several other one and two parameter estimators for the logistic regression model, including, maximum likelihood estimator, ridge regression estimator, Liu estimator, almost unbiased ridge and Liu estimators, adjusted Liu estimator, James stein's estimator, Kibria and Lukman estimator, Dorugade estimator and Modified ridge estimator. We derived expressions for the bias, covariance, and mean squared error (MSE) of these estimators, which depend on the biasing parameters $k$ and $d$. These estimators were compared theoretically based on the criterion of smaller mean square error (MSE). A monte Carlo simulation study was conducted to assess the performances of these logistic regression estimators. The simulation study evaluated the performance of the estimators for different values of $k$ and $d$, as well as for varying sample sizes, number of regressors, and correlation among the regressors. The study involved 2000 replications, and the average MSE values were computed.

The simulation results showed that increasing the correlations among the explanatory variables generally led to an increase in MSE values, although there were some exceptions. Among the estimators, AURE4 performed better than other estimators. The performance of the estimators also improved with increasing sample sizes, as the MSE values decreased. Furthermore, as the number of regressors increased, the estimated MSE values increased as well. These findings indicated that our proposed estimators perform well, particularly in comparison to the MLE, which consistently yielded the worst results.

Prostate cancer data has been used to illustrate the analysis of this thesis. This data is about the presence and absence of the seminal vesical invasion. The number of patients in this study was 97 and among them 21 of them had the presence of seminal vesicle invasion. There are seven explanatory variables in this dataset. It appears from the condition index is that there are multicollinearity problems among these regressors. The analysis of the prostate cancer data revealed that LMRT22 exhibited the smallest MSE values, while MLE yielded the worst but LLE2 estimator gives the MSE value more than the MLE. However, it is important to note that in the simulation study, which involved 2000 replications and averaging of MSE values, some proposed estimators may have produced higher MSE values than MLE. In the analysis of the prostate cancer data, only a single estimate was obtained, which could explain why the MSE values of LLE2 estimators exceeded MLE.

The cancer remission data was also analyzed to study the performance of logistic regression estimators. The dataset consisted of information on 27 patients, with nine of them experiencing complete cancer remission. The goal was to predict whether a patient would experience complete remission based on several variables. The condition index was calculated and confirmed the presence of multicollinearity in the dataset. This indicates that some of the explanatory variables are highly correlated with each other. The performance of various logistic regression estimators was evaluated using mean squared error (MSE) as the criterion. From the analysis, we observed that MLE gave the largest MSE values among the estimators. The LKLE estimator also resulted in higher MSE values compared to MLE. On the other hand, the LLE estimator showed the smallest MSE values among all the estimators evaluated.

We have considered the Logistic regression model in this thesis. But in the future, this thesis can be extended to consider different regression models, such as Poisson regression, Beta regression, Gamma regression, Negative binomial regression, LASSO regression, and others. It will also be interesting to see the performance of the estimators when data contains both outlier and extreme values.

## References

1. Abonazel, M. R., & Farghali, R. A. (2019). Liu-type multinomial logistic estimator. *Sankhya B*, *81*(2), 203-225.

2. Amin, M., Akram, M. N., Kibria, B. G., Alshanbari, H. M., Fatima, N., & Elhassanein, A. (2023). On the Estimation of the Binary Response Model. *Axioms*, *12*(2), 175.

3. Asar, Y. (2017). Some new methods to solve multicollinearity in logistic regression. *Communications in Statistics-Simulation and Computation*, *46*(4), 2576-2586.

4. Asar, Y., & Genç, A. (2016). New shrinkage parameters for the Liu-type logistic estimators. *Communications in Statistics-Simulation and Computation*, *45*(3), 1094-1103.

5. Asar, Y., & Genç, A. (2017). Two-parameter ridge estimator in the binary logistic regression. *Communications in Statistics-Simulation and Computation*, *46*(9), 7088-7099.

6. Awwad, F. A., Odeniyi, K. A., Dawoud, I., Algamal, Z. Y., Abonazel, M. R., Kibria, B. G., & Eldin, E. T. (2022). New two-parameter estimators for the logistic regression model with multicollinearity. *WSEAS Trans. Math*, *21*, 403-414.

7. Chang, X. (2015, November). On the almost unbiased ridge and Liu estimator in the logistic regression model. In *2015 International Conference on Social Science, Education Management and Sports Education* (pp. 1658-1660). Atlantis Press.

8. Dorugade, A. V. (2014). A modified two-parameter estimator in linear regression. *Statistics in Transition. New Series*, *15*(1), 23-36.

9. Farebrother, R. W. (1976). Further results on the mean square error of ridge regression. *Journal of the Royal Statistical Society. Series B (Methodological)*, 248-250.

10. Farghali, R. A., Qasim, M., Kibria, B. M. G., & Abonazel, M. R. (2021). Generalized two-parameter estimators in the multinomial logit regression model: methods, simulation, and application. *Communications in Statistics-Simulation and Computation*, 1-16.

11. Hoerl, A. E., & Kennard, R. W. (1970). Ridge regression: Biased estimation for nonorthogonal problems. *Technometrics*, *12*(1), 55-67.

12. Hoerl, A. E., Kannard, R. W., & Baldwin, K. F. (1975). Ridge regression: some simulations. *Communications in Statistics-Theory and Methods*, *4*(2), 105-123.

13. Inan, D., & Erdogan, B. E. (2013). Liu-type logistic estimator. *Communications in Statistics-Simulation and Computation*, *42*(7), 1578-1586.

14. Kejian, L. (1993). A new class of blased estimate in linear regression. *Communications in Statistics-Theory and Methods*, *22*(2), 393-402.

15. Kibria, B. M. G. (2022). More than hundred (100) estimators for estimating the shrinkage parameter in a linear and generalized linear ridge regression models. *Journal of Econometrics and Statistics*, *2*(2), 233-52.

16. Kibria, B. M. G. (2003). Performance of some new ridge regression estimators. *Communications in Statistics-Simulation and Computation*, *32*(2), 419-435.

17. Kibria, B. M. G., & Lukman, A. F. (2020). A new ridge-type estimator for the linear regression model: Simulations and applications. *Scientifica*, *2020*.

18. Kibria, B. M. G., Månsson, K., & Shukur, G. (2012). Performance of some logistic ridge regression estimators. *Computational economics*, *40*, 401-414.

19. Kutner, M., Nachtsheim, C. J., Neter, J., & Li, W. (2005). Applied linear statistical models. McGraw-Hill. *New York*, *7*.

20. Lesaffre, E., & Marx, B. D. (1993). Collinearity in generalized linear regression. *Communications in Statistics-Theory and Methods*, *22*(7), 1933-1952.

21. Lukman, A. F., Emmanuel, A., Clement, O. A., & Ayinde, K. (2020). A modified ridge-type logistic estimator. *Iranian Journal of Science and Technology, Transactions A: Science*, *44*, 437-443.

22. Lukman, A. F., Kibria, B. M. G., Nziku, C. K., Amin, M., Adewuyi, E. T., & Farghali, R. (2023). KL Estimator: Dealing with Multicollinearity in the Logistic Regression Model. *Mathematics*, *11*(2), 340.

23. Lukman, A. F., & Olatunji, A. (2018). Newly proposed estimator for ridge parameter: an application to the Nigerian economy. *Pakistan Journal of Statistics*, *34*(2), 91-98.

24. Månsson, K., Kibria, B. G. M., & Shukur, G. (2011). On Liu estimators for the logit regression model. *Economic Modelling*, *29*(4), 1483-1488.

25. Nagarajah, V., & Wijekoon, P. (2015). Stochastic restricted maximum likelihood estimator in logistic regression model. *Open Journal of Statistics*, *5*(07), 837.

26. Özkale, M. R., & Arıcan, E. (2016). A new biased estimator in logistic regression model. *Statistics*, *50*(2), 233-253.

27. Saleh, A. K. Md. E., & Kibria, B. M. G. (1993). Performance of some new preliminary test ridge regression estimators and their properties. *Communications in Statistics-Theory and Methods*, *22*(10), 2747-2764.

28. Schaefer, R. L., Roi, L. D., & Wolfe, R. A. (1984). A ridge logistic estimator. *Communications in Statistics-Theory and Methods*, *13*(1), 99-113.

29. Şiray, G. Ü., Toker, S., & Kaçiranlar, S. (2015). On the restricted Liu estimator in the logistic regression model. *Communications in Statistics-Simulation and Computation*, *44*(1), 217-232.

30. Trenkler, G., & Toutenburg, H. (1990). Mean squared error matrix comparisons between biased estimators—An overview of recent results. *Statistical papers*, *31*(1), 165-179.

31. Urgan, N. N., & Tez, M. (2008). Liu estimator in logistic regression when the data are collinear. In *20th euro mini conference* (pp. 323-327).

32. Varathan, N., & Wijekoon, P. (2018). Optimal generalized logistic estimator. *Communications in Statistics-Theory and Methods*, *47*(2), 463-474.

33. Wu, J. (2016). Modified restricted Liu estimator in logistic regression model. *Computational Statistics*, *31*, 1557-1567.

34. Wu, J., & Asar, Y. (2017). More on the restricted Liu estimator in the logistic regression model. *Communications in Statistics-Simulation and Computation*, *46*(5), 3680-3689.

35. Yang, H., & Chang, X. (2010). A new two-parameter estimator in linear regression. *Communications in Statistics—Theory and Methods*, *39*(6), 923-934.

# Appendix

In this appendix, we provided R-codes that were used in this thesis.

```
set.seed(4)
n=30

p=3

w=matrix(rnorm(n*p),ncol=p)

mean(w)

sd(w)

x=sqrt(1-gamma^2)*w+gamma*w[,ncol(w)]

X=t(x)%*%x

eigen(X)

v=eigen(X)$values

q=eigen(X)$vectors

det(t(q)%*%q)

b=matrix(q[,1])

t(b)%*%b

I=diag(rep(1,p))

pi=exp((x)%*%b)/(1+exp((x)%*%b))

a= function(x)
```

```
 {

   m=matrix(rep(0,n))

   for ( i in 1:n )

   {

     m[i,]=rbinom(1,1,x[i,])

   }

   return(m)

 }

 y=a(pi)

z=log(pi)+(y-pi)/(pi*(1-pi))

g=diag(c(pi*(1-pi)))

k1=p

k2=p^(1+(1/p))

k3=prod(1/qi)^(1/p)

k4=max(1/qi)

d1=min((alpha^2)/((1/(eigen(t(x)%*%g%*%x)$values))+alpha^2))

d2=min((eigen(t(x)%*%g%*%x)$values)*(1+alpha^2)/(1+(alpha^2)*(eigen(t(x)%*%g%
*%x)$values)))

mle=solve(t(x)%*%g%*%x)%*%(t(x)%*%g%*%z) #maximumum likelihood estimator
```

```
lre=solve(t(x)%*%g%*%x+k*I)%*%(t(x)%*%g)%*%x%*%mle # logistic ridge
estimator

lle=solve((t(x)%*%g)%*%x+I)%*%(((t(x)%*%g)%*%x+d*I))%*%mle #logistic liu
estimator

aure = (I-
k^2*solve((t(x)%*%g)%*%x+k*I)%*%solve((t(x)%*%g)%*%x+k*I))%*%mle #almost
unbiased ridge estimator

aule=  (I-(1-d)^2*solve((t(x)%*%g)%*%x+I)%*%solve((t(x)%*%g)%*%x+I))%*%mle
#almost unbiased liu estimator

alle= solve((t(x)%*%g)%*%x+I)%*%(((t(x)%*%g)%*%x-d*I))%*%mle # adjusted
logistic liu estimator

c=(t(beta)%*%beta)/(t(beta)%*%beta+sum(1/(eigen((t(x)%*%g)%*%x)$values)))
lse=c(c)*mle #logistic james stein estimator
```

```
lkle=solve((t(x)%*%g)%*%x+k*I)%*%((t(x)%*%g)%*%x-k*I)%*%mle # logistic KL estimator

lde=solve((t(x)%*%g)%*%x+k*d*I)%*%((t(x)%*%g)%*%x)%*%mle # logistic D estimator

lmrt=solve((t(x)%*%g)%*%x+k*(1+d)*I)%*%((t(x)%*%g)%*%x)%*%mle #logistic modified ridge estimator

A=solve((t(x)%*%g)%*%x+k*I)%*%(t(x)%*%g)%*%x
c1=t(mle)%*%A%*%mle/(t(mle)%*%t(A)%*%A%*%mle+sum(eigen(A%*%solve((t(x)%*%g)%*%x)%*%t(A))$values))
lsre=c(c1)*A%*%mle # logistic stein estimator for ridge

K=solve((t(x)%*%g)%*%x+k*I)%*%((t(x)%*%g)%*%x-k*I)
c2=t(mle)%*%K%*%mle/(t(mle)%*%t(K)%*%K%*%mle+sum(eigen(K%*%solve((t(x)%*%g)%*%x)%*%t(K))$values))
lske=c(c2)*K%*%mle #logistic stein estimator for KL estimator.
```

```
# simulation


x <- replicate(2000, {

 n=100

 gamma=0.9

 p=10

 z=matrix(rnorm(n*p),ncol=p)

 x=sqrt(1-gamma^2)*z+gamma*z[,ncol(z)]

 X=t(x)%*%x

 v=eigen(X)$values

 del=diag(v)

 q=eigen(X)$vectors

 b=matrix(q[,1])

 pi=exp(x%*%b)/(1+exp(x%*%b))

 a= function(x)

 {

  m=matrix(rep(0,n))

  for ( i in 1:n )
```

```
  {
   m[i,]=rbinom(1,1,x[i,])
  }
  return(m)
}
y=a(pi)
z=log(pi)+(y-pi)/(pi*(1-pi))
g=diag(c(pi*(1-pi)))
beta=solve((t(x)%*%g)%*%x)%*%(t(x)%*%g)%*%z
mx=max(eigen(t(x)%*%g%*%x)$values)
alpha=(eigen(t(x)%*%g%*%x)$vectors)%*%beta
qi=mx/((n-p)+mx*alpha^2)
k1=p
k2=p^(1+(1/p))
k3=prod(1/qi)^(1/p)
k4=max(1/qi)
d1=min((alpha^2)/((1/(eigen(t(x)%*%g%*%x)$values))+alpha^2))
```

```
d2=min((eigen(t(x)%*%g%*%x)$values)*(1+alpha^2)/(1+(alpha^2)*(eigen(t(x)%*%g%
*%x)$values)))
  k=k4


  I=diag(rep(1,p))
  K=solve((t(x)%*%g)%*%x+k*I)%*%((t(x)%*%g)%*%x-k*I)



c2=t(beta)%*%K%*%beta/(t(beta)%*%t(K)%*%K%*%beta+sum(eigen(K%*%solve((t(
x)%*%g)%*%x)%*%t(K))$values))


  jk=c(c2)*K%*%beta


  mse=t(jk-b)%*%(jk-b)
  mse
})
mean(x)
```