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Geometric Quantizations Related to the Laplace Eigenspectra of Compact Riemannian Symmetric Spaces via Borel-Weil-Bott Theory

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GEOMETRIC QUANTIZATIONS RELATED TO THE LAPLACE EIGENSPECTRA OF COMPACT RIEMANNIAN SYMMETRIC SPACES VIA BOREL-WEIL-BOTT THEORY

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Camilo Montoya

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To: Dean Michael Heithaus
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This dissertation, written by Camilo Montoya, and entitled Geometric Quantizations Related to the Laplace Eigenspectra of Compact Riemannian Symmetric Spaces via Borel-Weil-Bott Theory, having been approved in respect to style and intellectual content, is referred to you for judgment.

We have read this dissertation and recommend that it be approved.

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DEDICATION

This dissertation is dedicated to my family Clara and John, with emphasis on the memory and love for my brother David Montoya (5/27/1989 – 2/27/2011) that served as the majority of my motivation in the completion of this work.

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Finally is the entire history and accumulated knowledge of this most beautiful of subjects, the only exact science. The giants whose shoulders we all stand on like Archimedes, Newton, Euler, Gauss, Riemann, Grothendieck, and the plethora of others who helped build the structure to which I am only merely contributing a small piece.

ABSTRACT OF THE DISSERTATION

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Professor Gueo Grantcharov, Major Professor

The purpose of this thesis is to suggest a geometric relation between the Laplace-Beltrami spectra and eigenfunctions on compact Riemannian symmetric spaces and the Borel-Weil theory using ideas from symplectic geometry and geometric quantization. This is done by associating to each compact Riemannian symmetric space, via Marsden-Weinstein reduction, a generalized flag manifold which covers the space parametrizing all of its maximal totally geodesic tori. In the process we notice a direct relation between the Satake diagram of the symmetric space and the painted Dynkin diagram of its associated flag manifold. We consider in detail the examples of the classical simply-connected spaces of rank one and the space SU(3)/SO(3).

We briefly present the necessary background material and also provide detailed study of examples of rank 2 symmetric spaces and possible decomposition of their eigenspaces into irreducible subspaces. In the last part of the thesis, with the aid of harmonic polynomials, we induce Laplace-Beltrami eigenfunctions on the symmetric space from holomorphic sections of the associated line bundle on the generalized flag manifold. We consider a generalization of a method of constructing explicit repre-

sentations of the Laplace-Beltrami eigenfunction using homogeneous harmonic polynomials (under some mild conditions) as the (proper) restrictions in some ambient space, as opposed to the known implicit integral representations of these eigenfunctions [33, 20]. We apply this method to the examples of the simply connected rank one space $\mathbb{H}P^n$ and maximal rank 2 space $SU(3)/SO(3)$, moreover applying the connection to the Borel-Weil theorem we show that our construction produces the explicit representation of all of the eigenfunctions.

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CHAPTER 1

INTRODUCTION

In mathematics the spherical harmonics are generalizations of the sine and cosine functions and are defined as eigenfunctions of the Laplace-Beltrami operator. They have applications to variety of fields such as signal processing, cosmic microwave background radiation and 3D computer graphics. The easiest mathematical representation of the spherical harmonics is through homogeneous polynomials in the ambient Euclidean space. The spheres are the simplest of the mathematical shapes with many symmetries called Riemannian symmetric spaces. They admit generalizations of the spherical harmonics called spherical functions which are special eigenfunctions of the Laplace-Beltrami operator and have various applications. In mathematics the Riemannian symmetric spaces and their spherical functions are studied from analytical algebraic and geometric viewpoints. In this thesis we focus on the algebraic and geometric approaches through representation theory and symplectic geometry.

There are two classical geometric interpretations of the representation theory of the compact Lie groups. On the one side is the Borel-Weil Theorem and its subsequent generalization to the Borel-Weil-Bott theory. In particular, every complex representation of a compact Lie group is realized on the space of holomorphic sections of some line bundle over a flag manifold. On the other side, the harmonic analysis on a Riemannian symmetric space provides irreducible representations of a compact simple Lie group: the natural action of a transitive simple Lie group of isometries G on the common eigenspaces of the (commutative) algebra of the invariant differential operators on the respective compact symmetric space $M = G/K$ is irreducible, and

K is a maximal compact subgroup in G. This algebra contains the Laplace-Beltrami operator and is generated by k generators, where $k = rk(M)$ is the rank of M.

One of the goals of this thesis is to propose a direct geometric relation between the two theories. Some of the main results were reported in [25].

The most explicit illustration of the relation is through geometric quantization of the geodesic flow in the rank one case and we briefly explain it first. We assume for simplicity that such compact Riemannian symmetric space of rank one (CROSS for short) is simply-connected and irreducible, this leaves us with the well-known examples of the spheres, classical projective spaces (complex and quaternionic) and the Cayley plane. These are also the known simply-connected examples in dimension higher than two of Riemannian manifolds all of whose geodesics are closed. On a Riemannian manifold (M, g) all of whose geodesics are closed, there is a natural S^1-action on its tangent bundle TM and the geodesic flow on the cotangent bundle T^*M can be realized as solution to an S^1-invariant Hamiltonian system. For such systems, under mild conditions, there is a moment map and a symplectic reduction process, called also Marsden-Weinstein reduction, defined in more detail in Chaper 6. This reduction produces a reduced space $T^*M//S^1$ that can be identified with the space parametrizing all oriented geodesics and that is equipped with an induced symplectic form. The induced symplectic form depends on a level set of the corresponding moment map $\mu : M \to \mathfrak{g}^*$ which we call the energy level of the geodesic flow. In many examples the cotangent bundle has a "complex polarization" - a complex structure compatible with the symplectic form which becomes Kähler form. A natural question which has its origin in the relation between Kepler's laws and the hydrogen atom is when such manifold could be "quantized". The geometric quantization is not a uniquely defined notion and there are various schemes which implement it. We use the Kähler form and the complex polarization to apply a twisted version of Kostant-Souriau geometric quantization scheme (originally due to [15, 35]) and assign a holomorphic line bundle with first Chern class given by the induced Kähler form with an added extra term. This new term is half of the first Chern class of the canonical bundle of the manifold $T^*M//S^1$. Sometimes this is called *prequantum bundle* and we use this terminology. The quantum condition is the integrality of that corrected form, while the analog of the Hilbert space of quantum observables is the space of holomorphic sections of the prequantum bundle. Our first general result is Theorem 6.1.3 in Chapter 6 in which we show that the quantized energy levels of the geodesic flow on a simply connected rank one symmetric space are, up to a constant, equal to the eigenvalues of the Laplace-Beltrami operator on M, and the corresponding complexified eigenspaces are isomorphic to the spaces of the holomorphic sections of the prequantum bundle over the reduced space which is a generalized flag manifold. Although we provide the representation theory background for a unified proof, we proceed with a case by case proof since it illustrates the explicit nature of the calculations that provides a basis for the next parts of the thesis.

As a preparation for the next Chapters we consider in detail particular examples of higher rank symmetric spaces in Chapter 5. Here we look at the eigenspectra of the Laplace Beltrami operator on $SU(3)/SO(3)$ and explicitly compute the eigenspectra for all compact rank two symmetric spaces and demonstrate the method for such calculations based on results referenced from [8] which provides the necessary machinery of number theory and integral Diophantine equations. The motivation is to continue the compilation of the tables of Laplace-Beltrami operator eigenspectra of rank one compact Riemannian symmetric spaces to higher ranks.

The tabulating the Laplace-Beltrami eigenspectra is well-known and seems to only exist within the literature in rank one case, and can be found in many places within the relevant literature. Their calculation is substantially easier because the spectrums are one dimensional, as a consequence of the root eigenspaces being one dimensional. The table of the eigenspectra of the Laplacian operator Δ_M on the compact rank one symmetric spaces (CROSSes) M is compiled below, as found in [4, 21]. The results in this Chapter allow for the tabulation of the eigenspectra for *rank two compact symmetric spaces*, similar to Table 1 shown below which classifies the Laplace spectra for the CROSSes. A certain avenue of further research after the case $rk(M) = 2$ is complete, is with the continuation of the tabulations for the higher rank symmetric spaces $rk(M) \geq 3$.

Table 1: The Laplace eigenspectrum of compact rank one Riemannian symmetric spaces

CROSS $M = G/K$	**Eigenspectrum Spec**(Δ_M)
$S^n = {}^{SO(n+1)}\!\diagup_{SO(n)}$	$\mathrm{Spec}(\Delta_{S^n}) = \{\lambda_k = k(k+n-1) \mid k \geq 0\}$
$\mathrm{RP}^n = {}^{SO(n+1)}\!\diagup_{SO(n) \times SO(1)}$	$\mathrm{Spec}(\Delta_{\mathbb{R}P^n}) = \{\lambda_k = 2k(2k+n-1) \mid k \geq 0\}$
$\mathbb{C}P^n = {}^{SU(n+1)}\!\diagup_{S(U(n) \times U(1))}$	$\mathrm{Spec}(\Delta_{\mathbb{C}P^n}) = \{\lambda_k = 4k(k+n) \mid k \geq 0\}$
$\mathbb{H}P^n = {}^{Sp(n+1)}\!\diagup_{Sp(n) \times Sp(1)}$	$\mathrm{Spec}(\Delta_{\mathbb{H}P^n}) = \{\lambda_k = 4k(k+2n+1) \mid k \geq 0\}$
$CaP^2 = {}^{F_4}\!\diagup_{Spin(9)}$	$\mathrm{Spec}(\Delta_{CaP^2}) = \{\lambda_k = 4k(k+11) \mid k \geq 0\}$

In the next Chapters we consider the case of general rank. We observe that we can substitute the space parametrizing all geodesics with the space of all maximal totally geodesic flat submanifolds, which are tori in this case. Just as in the rank one case we needed the oriented geodesics, in the higher rank case we need the universal cover of the space parametrizing the maximal flat tori. This space is again a generalized flag manifold and carries a natural "polarization", which could be used for the quantization - a Kähler complex structure. Since our aim is to underline the geometric approach through the Marsden-Weinstein reduction, we also need a Kähler space with a (multi-dimensional) Hamiltonian that, after the symplectic reduction, will become the generalized flag manifold with an appropriate reduced symplectic form, a form which is also integral and Kähler.

This is done in [25] via construction of a Kähler structure on some open subset of the manifold of all tangent spaces of the maximal totally geodesic flat submanifold. Then we prove the main result in Chapter 6 - Theorem 7.2.1, which was announced in [25]. If the symmetric space has a maximal rank $rk(M) = rk(G)$, then the corresponding generalized flag manifold is actually the full flag manifold G/T, where T is a maximal torus in G. From the Borel-Weil theorem follows that every irreducible representation of G appears as a space of holomorphic sections of some line bundle over G/T. This corresponds to the fact that the symmetric spaces of maximal rank provide the largest variety of the irreducible representations of G appearing as subspaces of the eigenspaces of the Laplace-Beltrami operator by the results in [33]. As an example we consider the space $SU(3)/SO(3)$, which is of rank two, and also the simplest example of Riemannian symmetric space of maximal rank. We note that a general correspondence similar to the one in Theorem 7.2.1 has appeared in [20] and [34] Chapter 6, but with a focus on various integral transforms.

In the remaining part of the thesis we indicate a construction, which relates the holomorphic sections of the prequantum bundle to the eigenfunctions of the corresponding eigenvalue on the symmetric space. We use the standard description of the holomorphic sections as holomorphic functions on the total space of the associated principal $\mathbb{C}^*-$bundle with appropriate equivariant condition. Using the basic properties of the Laplace-Beltrami operator under Riemannian submersions, the relation between holomorphic and harmonic functions on a special non-Kähler manifolds, and an extension of the standard relation between harmonic polynomials and eigenfunctions on the spheres, in Theorem 8.0.5, we propose a method of description of the Laplace-Beltrami eigenfunctions out of the holomorphic sections on the associated quantization space. We apply the method, with some modifications, to complex and quaternionic projective spaces as well as the space $SU(3)/SO(3)$. In these examples we describe a spanning set of all eigenfunctions, which consists of algebraic functions. The cases of quaternionic spaces and $SU(3)/SO(3)$ are new and extend the known representations for the spheres and complex projective spaces [43]. We expect that many other symmetric spaces will have similar complete description. Note that the known descriptions of the eigenfunctions so far are based on the integral geometry and various types of Radon transform, which are implicit, as opposed to our method of constructing them from homogeneous harmonic polynomials.

The structure of the thesis is as follows: in the preliminary chapters we collect the necessary background facts about Lie algebras, symmetric spaces, and flag manifolds. Meanwhile we notice a simple connection between the Satake diagram of the symmetric space and the painted Dynkin diagram of the generalized flag manifold. We use it do describe the second cohomology group of the quantization space in terms of the Satake diagram of the initial symmetric space. Then in Chapter 6 we treat the rank one case and in Chapters 5 and 6 we consider the spaces of arbitrary rank. The last two Chapters — 7 and 8 provide the explicit construction of harmonic polynomials from holomorphic sections of the prequantum line bundle over the corresponding generalized flag manifold.

CHAPTER 2

LIE ALGEBRA STRUCTURE THEORY AND DECOMPOSITIONS

2.1 Basic facts about Lie algebras

Definition 2.1.1. A ***Lie algebra*** $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear operation called the ***Lie bracket*** of $\mathfrak{g}$, denoted by $[*, *] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, that satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

The bracket induces a linear transformation for $X, Y \in \mathfrak{g}$, $(\text{ad } X)Y := [X, Y]$. This adjoint transformation is also used to define a canonical symmetric biliniear form B (also known as the Killing form) given by $B(X, Y) := \text{Tr}\left((\text{ad } X) \circ (\text{ad } Y)\right)$.

The general theory of root systems for the classification of (semisimple) Lie algebras is well-known [32], A Lie algebra $\mathfrak{g}$ is said to be ***simple*** if is is nonabelian and has no proper nonzero ideals. A Lie algebra $\mathfrak{g}$ is said to be ***semisimple*** (ss) if it is a direct sum of simple Lie algebras. The stuy of symmetric spaces and their geodesics / maximal tori forces the study of (semisimple) Lie algebras. Here we recall some basic facts about the structure theory of semisimple Lie algebras.

2.2 Cartan decomposition

Definition 2.2.1. An involutive Lie algebra automorphsim $\theta : \mathfrak{g} \to \mathfrak{g}$ is called a ***Cartan involution*** if there exists a bilinear form B_θ defined by $B_\theta(X, Y) := -B(X, \theta Y)$ which is symmetric and positive definite.

Theorem 2.2.2. *The Cartan involution can be used to decompose* $\mathfrak{g}$ *into its* ± 1*-eigenspaces producing what's known as the* **Cartan decomposition** *given by*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}$ *and* $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}$.

The importance of the Cartan decomposition is that it has the property that if $\mathfrak{g}$ is a real Lie algebra and σ a complex conjugation defined on $\mathfrak{g}^{\mathbb{C}}$, then the vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition if there exists a compact real form $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ such that $\sigma(\mathfrak{u}) = \mathfrak{u}$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$, and $\mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{u})$. This fact will prove to be important for classifying the compact Riemannian symmetric spaces.

The Cartan decomposition of a compact Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is trivial because the negative eigenspace $\mathfrak{p}$ is always zero (since $\mathfrak{k} = \mathfrak{g}$ is already compact). But by duality of compact/noncompact real forms of a Lie algebra ([32], Ch. V), we can construct a method to recover the compact symmetric space via the Lie algebras of the corresponding dual noncompact real forms.

The method is as follows. For a compact Riemannian symmetric space G/K with (compact) Lie algebra $\mathfrak{g}$ we consider its complexification $\mathfrak{g}^{\mathbb{C}}$. We then use the classification of its noncompact real forms $\mathfrak{g}_n$ using Satake diagrams (section 3.3) and identify its *unique* maximal compact subalgebra $\mathfrak{k}$ from the Cartan decomposition of $\mathfrak{g}_n = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{k} \subset \mathfrak{g}$ will also be maximal in $\mathfrak{g}$, defining a symmetric pair $(\mathfrak{g}, \mathfrak{k})$, and all distinct $\mathfrak{g}_n$'s will correspond to different $\mathfrak{k}$'s and the pair $(\mathfrak{g}, \mathfrak{k})$ indentify a unique compact Riemannnian symmetric space G/K with that symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

More specifically, if $\mathfrak{g}_n = \mathfrak{k} \oplus \mathfrak{p}$, then its compact dual form will be $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$, since the compact real forms of $\mathfrak{g}^{\mathbb{C}}$ are unique because the complex simple Cartan subalgebra $(\mathfrak{h})^{\mathbb{C}}$ in the root space decomposition from the next section contain unique compact real forms, up to conjugation.

2.3 Root space decomposition

Definition 2.3.1. A ***Cartan subalgebra*** of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is a maximal abelian subalgebra of $\mathfrak{g}$ for which all $H \in \mathfrak{h}$, the endomorphism ad H of $\mathfrak{g}$ is semisimple (and therefore is diagonalizable).

All Cartan subalgebras are conjugate to each other, and though their existence is not a trivial matter, they were proved to always exist in a seminal result known as Lie's theorem on solvable Lie algebras, proved in [32].

Definition 2.3.2. Let α be a linear functional on the vector space $\mathfrak{h}$, $\alpha \in \Delta \subset (\mathfrak{h})^*$ called a root, and denote by $\mathfrak{g}_\alpha$ the ***root space*** consisting of nonzero X's in

$$\mathfrak{g}_\alpha = \Big\{ X \in \mathfrak{g} \;\Big|\; \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h} \Big\}.$$

Then the ***root space decomposition*** of $\mathfrak{g}$ with respect to his

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Since $\mathfrak{h}$ is abelian, we can consider the Cartan subalgebra has $\mathfrak{g}_0$.

CHAPTER 3

ROOT SYSTEM CLASSIFICATION VIA DYNKIN AND SATAKE DIAGRAMS

Dynkin and Satake diagrams are graphs whose main interest is that they are a means of classifying ss Lie algebras over algebraically closed fields (like $\mathbb{C}$). One classifies such Lie algebras via their root system, which can be encoded in special graphs mentioned above.

Definition 3.0.1. Let V be a finite dimensional Euclidean vector space, equipped with the standard Euclidean inner product $(*, *)$. A (reduced) ***root system*** Δ in V is a finite set of nonzero vectors (called roots) that satisfy the following conditions:

(i) The set of roots Δ span V.

(ii) For each root $\alpha \in \Delta$ there exists a reflection s_α along α leaving Δ invariant.

(iii) For any two roots $\alpha, \beta \in \Delta$, the number $\langle \beta, \alpha \rangle := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 2\frac{(\alpha,\beta)}{|\alpha|^2}$ is an integer.

Definition 3.0.2. The root system is said to be ***reduced*** if $\alpha \in \Delta$ implies that $2\alpha \notin \Delta$. We say that two (abstract) root systems Δ in V and Δ in V' are *isomorphic* if there is a vector space isomorphism of V onto V' carrying Δ onto Δ' and preserving the integers in the definition above of root systems.

Definition 3.0.3. Given a root system Δ we can always choose (in several different ways) a subset of ***positive roots*** denoted $\Delta^+ \subset \Delta$, for which every root $\alpha \in \Delta$ either contains $\alpha \in \Delta^+$ or $-\alpha \in \Delta^+$, and if two distinct roots $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta^+$.

An element of Δ^+ is called a ***"simple root"*** if it cannot be written as the sum of two elements of Δ^+. (The set of simple roots Σ is also referred to as a "base" for Δ).

If a set of positive roots $\Delta^+ \subset \Delta$ is chosen, elements of $\Delta^- := -\Delta^+ = \Big\{\alpha \in \Delta \;\Big|\; -\alpha \in \Delta^+\Big\}$ are called "negative roots", and clearly we have

$$\Delta = \Delta^+ \cup \Delta^-.$$

"Such a set of positive roots may be constructed by choosing a hyperplane P not containing any roots and setting Δ^+ to be all the roots lying on a fixed side of P. Furthermore, every set of positive roots arises in this way" [38].

Remark 3.0.4. The importance here in reviewing root systems, is that the simple roots of a semisimple Lie algebra form a root system.

3.1 Lie algebra root systems

We can classify (irreducible) root systems (and therefore simple Lie algebras) using graphs known as *Dynkin diagrams*. Given a root system and a set of simple roots $\Sigma \subset \Delta^+$, we can construct a graph to represent the root system, encoding the lengths and relations of and between all the roots including the simple roots of the system (which form a basis of the underlying vector space of the representation). Hence, Dynkin diagrams classify them in these terms, and in fact, two (simple and hence semisimple) Lie algebras are isomorphic if their corresponding root systems are (the Dynkin diagram of semisimple Lie algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is just the direct sum of the Dynkin diagrams $\mathcal{D}(\mathfrak{g}_1) \oplus \mathcal{D}(\mathfrak{g}_2)$, interpreted as the reducible (disconnected) Dynkin diagram of the two simple Dynkin diagrams). This is clearly the principal objective, as well as the classification of (ss) Lie algebras.

Definition 3.1.1. Let Δ be a root system, $\Delta^+ \subset \Delta$ be the set of positive roots, and $\Sigma \subset \Delta^+$ the set of simple roots. Note that the number of simple roots $|\Sigma|$ is equal to the rank (cf. definition 4.3.1) of the Lie algebra with that particular root system. Given a root system, select a set of positive roots Δ^+ and a set $\Sigma \subset \Delta^+$ of simple roots as in the preceding section. The vertices of the associated *(painted)* ***Dynkin diagram*** $\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}, \Sigma)$ of $\mathfrak{g}$ to correspond to the roots in Σ. Edges are drawn between vectors as follows, according to the angles. (Note that the angle between simple roots is always at least 90 degrees.)

(i) No edge if the vectors are orthogonal

(ii) An undirected single edge if they make an angle of 120 degrees

(iii) A undirected double edge if they make an angle of 135 degrees

(iv) A directed triple edge if they make an angle of 150 degrees

The term “directed edge” means that double and triple edges are marked with an arrow pointing toward the shorter vector. (Thinking of the arrow as a ”greater than” sign makes it clear which way the arrow is supposed to point.) Thus, these diagrams reduce the problem of classifying root systems to the problem of classifying possible Dynkin diagrams.

3.2 Dynkin diagrams

The classification of Dynkin diagrams corresponding to the classical noncompact complex Lie algebras are listed in the Table 2 below. It is a significant improvement on Cartan’s original classification of simple Lie algebras, which consists of the four infinite families A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$) and D_n ($n \geq 4$), as well as the exceptional algebras, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. Here, “n” denotes the rank of the Lie algebra, as well as the dimension of the vector space, where the Lie algebra root system is represented. The vertices are the simple roots $\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n\}$, and the edges are drawn according to the instructions above. Reference for this can be found in just about any book on Lie theory, including Helgason [32] and Knapp [38] among many. We recall the table of the classification of the classical and exceptional Lie algebras by the Dynkin diagrams of the corresponding root systems.

Table 2

Root system	Lie Algebra $\mathfrak{g}$	Dynkin diagram $\mathcal{D}(\mathfrak{g})$	Rank
A_n	$\mathfrak{sl}(n+1, \mathbb{C})$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \cdots \text{—} \underset{\alpha_{n-1}}{\circ} \text{—} \underset{\alpha_n}{\circ}$	n
B_n	$\mathfrak{so}(2n+1, \mathbb{C})$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \cdots \text{—} \underset{\alpha_{n-1}}{\circ} \Rightarrow \underset{\alpha_n}{\circ}$	n
C_n	$\mathfrak{sp}(2n, \mathbb{C})$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \cdots \text{—} \underset{\alpha_{n-1}}{\circ} \Leftarrow \underset{\alpha_n}{\circ}$	n
D_n	$\mathfrak{so}(2n, \mathbb{C})$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \cdots \text{—} \underset{\alpha_{n-2}}{\overset{\overset{\circ\ \alpha_n}{\vert}}{\circ}} \text{—} \underset{\alpha_{n-1}}{\circ}$	n
E_6	$\mathfrak{e}_6$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \underset{\alpha_3}{\overset{\overset{\circ\ \alpha_6}{\vert}}{\circ}} \text{—} \underset{\alpha_4}{\circ} \text{—} \underset{\alpha_5}{\circ}$	6
E_7	$\mathfrak{e}_7$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \underset{\alpha_3}{\circ} \text{—} \underset{\alpha_4}{\overset{\overset{\circ\ \alpha_7}{\vert}}{\circ}} \text{—} \underset{\alpha_5}{\circ} \text{—} \underset{\alpha_6}{\circ}$	7
E_8	$\mathfrak{e}_8$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \text{—} \underset{\alpha_3}{\circ} \text{—} \underset{\alpha_4}{\circ} \text{—} \underset{\alpha_5}{\overset{\overset{\circ\ \alpha_8}{\vert}}{\circ}} \text{—} \underset{\alpha_6}{\circ} \text{—} \underset{\alpha_7}{\circ}$	8
F_4	$\mathfrak{f}_4$	$\underset{\alpha_1}{\circ} \text{—} \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} \text{—} \underset{\alpha_4}{\circ}$	4
G_2	$\mathfrak{g}_2$	$\underset{\alpha_1}{\circ} \Rrightarrow \underset{\alpha_2}{\circ}$	2

Definition 3.2.1. Associated to each (irreducible) root system, its Dynkin diagram is also encoded in matrices known as ***Cartan matrices***, defined as the square $n \times n$ matrix $\mathfrak{C} = \mathfrak{C}_{\mathfrak{g}} = \mathfrak{C}(\Sigma_{\mathfrak{g}})$ of inner products of simple roots, and $n = \mathrm{rk}(\mathfrak{g}) = |\Sigma|$ is the rank of $\mathfrak{g}$ and the number of simple roots of Δ

$$\mathfrak{C} := \mathfrak{C}(\Sigma_{\mathfrak{g}}) := \big(C_{ij}\big)_{i,j=1,\ldots,n} := \Big(\langle \alpha_i, \alpha_j^{\vee} \rangle\Big)_{i,j=1,\ldots,n},$$

where $\alpha^{\vee}$ is called the coroot corresponding to α given by $\alpha^{\vee} := \frac{2}{\langle \alpha, \alpha \rangle}\alpha$.

These matrices will play a role in the next section regarding the connection to the eigenspectra of the Laplace-Beltrami oprator on the compact RSS's in question. The following table shows the Dynkin diagrams of the classical and exceptional (complex) Lie algebras and their corresponding rank. But to get to our RSS's we will need to enhance our Dynkin diagrams that will shed light on the real forms of our complexified Lie algebras: The Satake diagrams. The Cartan matrices will also make a surprise (but not unexpected) appearance in the quadratic forms used to find the eigenvalues of Δ_M.

As mentioned a moment ago, where Dynkin diagrams may be used to classify the classical complex Lie algebra root systems, a more intricate diagram can be constructed in a manner similar to that of Dynkin diagrams, along with two additional changes to the simple roots Σ by coloring some vertices black and adding some arrows to between a pairs of vertices if needed. These modified diagrams are know as *Satake diagrams*, and the Satake diagrams associated to a Dynkin diagram classify real forms of the complex Lie algebra corresponding to the Dynkin diagram.

Definition 3.2.2. An ***(Riemannian) symmetric (Lie algebra) pair*** (or just ***symmetric pair*** for short) is a pair of lie algebras $(\mathfrak{g}, \mathfrak{k})$ or $(\mathfrak{g}, \mathfrak{k}, \theta)$ with a Cartan involution θ on $\mathfrak{g}$ with the understanding that $\mathfrak{k}$ is the set of fixed points of θ (the $+1$ eigenspace).

The symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is associated to Riemmanian symmetric space G/K (or (G, K)) has been alluded to, but will properly defined in Chapter 4.

3.3 Satake diagrams

Before introducing Satake diagrams, we must first go back to the maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$, and their complexifications for which we can define a so-called "restricted root system" correspoding to $\mathfrak{a}$, which will also be some finite subset of $(\mathfrak{p})^*$. We have that the restriction map $\alpha \mapsto \bar{\alpha}$ from $\mathfrak{h}_{\mathbb{R}}$ to $\mathfrak{a}$, and orthogonal projection map $\alpha \mapsto \frac{1}{2}(\alpha - \theta(\alpha))$ from the dual $(\mathfrak{h}_{\mathbb{R}})^*$ to $(\mathfrak{a})^*$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, coincide (are dual to one another).

Definition 3.3.1. Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair, and Δ a root system with a choice of simple roots $\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and $\mathfrak{a} \subset \mathfrak{p}$ some maximal abelian subspace of $\mathfrak{p}$. The ***Satake diagram*** $\mathcal{S}(\mathfrak{g}, \mathfrak{a})$ of $(\mathfrak{g},\mathfrak{k})$ with respect to $\mathfrak{a}$, that is, to the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a})$ is comprised of three datum:

(i) First is the Dynkin diagram $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$.

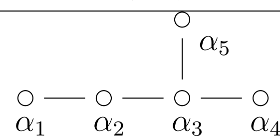
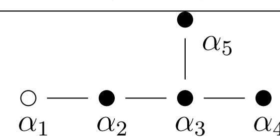
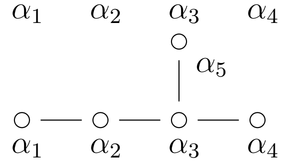
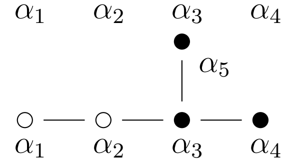
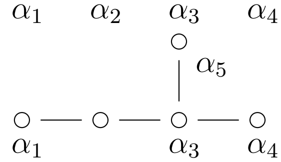
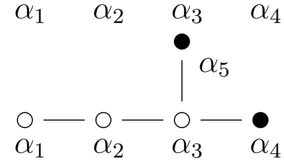
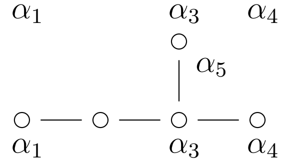
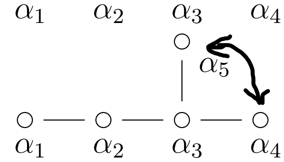
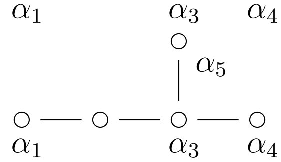
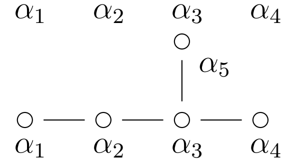
(ii) For elements α in the set Σ we set $\Sigma_0 := \{\alpha \in \Sigma \mid \bar{\alpha} \equiv 0\}$, and color the corresponding nodes in the Dynkin diagram black.

(iii) For distinct elements $\alpha \neq \beta \in \Sigma - \Sigma_0$ that restrict to the same root, that is, that $\bar{\alpha} = \bar{\beta}$, they are joined by a curved arrow.

An illuminating example is shown in Table 3 below, with the specific RSSs and symmetric pairs (specifically defined in Chapter 4), and showing the method of classification in essence by comparing their Dynkin and Satake diagrams.

Example 3.3.2. Let $M_k = \mathrm{Gr}_k(\mathbb{R}^{10})$ be the real Grassmannian manifold of k-planes in $\mathbb{R}^{10}$ (for $k = 1, 2, ..., , 5$). We can draw a table including all the nonisomorphic Grassmanian manifolds M_k as the symmetric space with corresponding symmetric pair, their Dynkin and Satake diagrams.

Table 3

k	M_k	corresp. symmetric pair	$\mathcal{D}(\mathfrak{g}, \mathfrak{k})$	$\mathcal{S}(\mathfrak{g}_\mathfrak{n}, \mathfrak{k}, \mathfrak{a})$
1	$\mathrm{Gr}_1(\mathbb{R}^{10})$	$(\mathfrak{so}(10), \mathfrak{s}(\mathfrak{o}(1) \times \mathfrak{o}(9)))$		
2	$\mathrm{Gr}_2(\mathbb{R}^{10})$	$(\mathfrak{so}(10), \mathfrak{s}(\mathfrak{o}(2) \times \mathfrak{o}(8)))$		
3	$\mathrm{Gr}_3(\mathbb{R}^{10})$	$(\mathfrak{so}(10), \mathfrak{s}(\mathfrak{o}(3) \times \mathfrak{so}(7)))$		
4	$\mathrm{Gr}_4(\mathbb{R}^{10})$	$(\mathfrak{so}(10), \mathfrak{s}(\mathfrak{o}(4) \times \mathfrak{so}(6)))$		
5	$\mathrm{Gr}_5(\mathbb{R}^{10})$	$(\mathfrak{so}(10), \mathfrak{s}(\mathfrak{o}(5) \times \mathfrak{so}(5))),$		

Satake diagrams will allow us to find (and classify) arbitrary rank compact symmetric spaces. A well-known fact (also a theorem in [32]) is that the multiplicity of all the root spaces in the real root space decomposition of $\mathfrak{g}_n$ is 1. An important rank formula (that can be found in [40]) for Satake diagrams which leads directly to the considerations and theorems in Chapter 4.6 is:

$$\mathrm{rk}(\mathfrak{g}^{\mathbb{C}}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{g}_n) + |\Sigma_0| + a$$

where a is is the number of arrows in the Satake diagram and $|\Sigma_0|$ is the number of black vertices.

Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Of course the existence of $\mathfrak{a}$ is guaranteed by dimenionality arguments. For $\alpha \in \mathfrak{a}^*$, we define the restricted root space as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \;\big|\; (\mathrm{ad}\ H)X = \alpha(H)X \text{ for all } X \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, we call α a restricted root of $\mathfrak{g}$, or more precisely, of $(\mathfrak{g}, \mathfrak{k})$, and $\mathfrak{g}_\alpha$ the (real) root space.

The subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ satisfies $\mathfrak{h}_0 \subset \mathfrak{g}_0$, and it is clear that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as a real vector space, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. A real form of $\mathfrak{g}$ that contains $\mathfrak{h}$ is called a ***split-real form*** of $\mathfrak{g}$, where $\mathfrak{h}^{\mathbb{R}} =: \mathfrak{h}_0 = \{H \in \mathfrak{h} \;\big|\; \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}$.

The two main real forms of $\mathfrak{g}$ being considered are normal/split (rank) real forms, and the (essentially unique, up to conjugation) compact real form. For our investigation into maximal rank compact symmetric spaces, we turn our attention to real forms of the split type. Corollary 6.10 in [38] guarantees that any complex semisimple Lie algebra contains a split-real form, but we will go ahead and give a more formal exposition of this statement. Consider the Lie algebra Iwasawa decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that contain a maximal abelian subspace $\mathfrak{a}$ are of the form

$$\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{a},$$

where $\mathfrak{h}^+$ is any Cartan subalgebra of $\mathfrak{m} = \mathfrak{C}_{\mathfrak{k}}(\mathfrak{a}) = \ker(X \mapsto [X, \mathfrak{a}])|_{\mathfrak{k}} = \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0]$. Then take the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ and denote its Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} = (\mathfrak{h}^+)^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$. Thus, we have that

$$\mathfrak{m}^{\mathbb{C}} = (\mathfrak{h}^+)^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in \Delta_0} (\mathfrak{g}^{\mathbb{C}})_\alpha \right) = (\mathfrak{h}^+)^{\mathbb{C}} \oplus \left(\bigoplus_{\substack{\alpha \in \Delta \\ \bar{\alpha} = \alpha|_{\mathfrak{a}} = 0}} (\mathfrak{g}^{\mathbb{C}})_\alpha \right).$$

Hence, the restricted roots are the nonzero restrictions to $\mathfrak{a}$ of the roots, and $\mathfrak{m}$ arises from the roots that restrict to zero on $\mathfrak{a}$.

A corollary to the lemma above is that $\mathfrak{g}$ is split, then $\mathfrak{m} = 0$, $\Delta = \Sigma$, and $(\mathfrak{g}^{\mathbb{C}})_\alpha = (\mathfrak{g}_\alpha)^{\mathbb{C}}$, so dim $\mathfrak{g}_\alpha = 1$ for all $\alpha \in \Sigma$. $\Sigma_0 = 0$ so the corresponding Satake diagram has no black vertices, and since the restricted roots are all the roots, no two distinct roots restrict to the same root (they restrict to themselves), hence the Satake diagram has no arrows either. This observation classifies maximal rank symmetric spaces.

The Satake diagrams are used to classify Riemannian symmetric spaces and the precise relation to painted Dynkin diagrams characterizing their corresponding general flag manifolds will be elucidated more thoroughly in the next chapter.

CHAPTER 4

RIEMANNIAN SYMMETRIC SPACES & GENERALIZED FLAG MANIFOLDS

4.1 Preliminary definitions

Definition 4.1.1. In mathematics, a *smooth manifold* (also known as a differential manifold) is a type of manifold that is locally similar enough to a linear space to allow one to do calculus. More specifically, it is a Hausdorff and second countable topological space M, together with a maximal differentiable atlas on M.

From the classical point of view, the nicest and most well-known smooth manifolds are the Riemannian manifolds, defined below:

Definition 4.1.2. A *Riemannian manifold* M (or (M, g)) is a smooth manifold equipped with a *Riemannian metric* g such that for every $p \in M$, there symmetric positive definite 2-tensor

$$g_p : T_pM \times T_pM \to \mathbb{R}_{\geq 0}.$$

Note that this metric also induces a metric space structure on the manifold M. Next we recall the concept of Lie groups:

Definition 4.1.3. A *Lie group* M is a smooth manifold M equipped with an additional group structure that is compatible with the differentiable structure on the smooth manifold, i.e. M also has a smooth group multiplication as well as inverses.

The most common examples of Lie groups are the matrix Lie groups, such as the special linear group $SL(n, k)$, the special orthogonal group $SO(n, k)$, the orthogonal group $O(n,k)$, and the special unitary group $SU(n)$, and the general linear group $GL(n,k)$, for a field $k=\mathbb{R}$ or $\mathbb{C}$. Also, any closed subgroup of a Lie group is also automatically a Lie group (or Lie (sub)group) as well.

Finally we have the homogeneous spaces, which serve as a precursor to the following section of Riemmanian symmetric spaces, and can be thought of as a coset spaces G/K on which a Lie group G acts transitively preserving the group structure:

Definition 4.1.4. The space $M=G/K$ where G is a Lie group and K some closed subgroup of G is called a *homogeneous space*, which is the coset space with the standard quotient topology, and is a homogeneous space for G with a distinguished point, namely the coset of the identity $o=eK$. Thus a homogeneous space can be thought of as a coset space without a choice of origin, because of transitivity.

4.2 Riemannian symmetric spaces

With the prerequisites from the section above, in this and the following Chapters we consider smooth manifolds M equipped with a Riemannian metric g. Associated to g there is a natural second order differential operator Δ_M on the functions of M, called Laplace-Beltrami operator. It is a generalization of the usual Laplace operator in $\mathbb{R}^n$. In a local coordinate chart $(x^1,x^2,...,x^n)$ of M, where the metric components are g_{ij}, the definition of Δ_M is

$$\Delta_M(f)=\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x_j}\right).$$

A function f on M is called harmonic if $\Delta_M(f)=0$ and by the maximum principle, on a compact M the only harmonic functions are the constants. An eigenfunction f with an eigenvalue λ is a function, for which $\Delta_M(f)=\lambda f$. The set of eigenvalues of Δ_M is known to be an increasing sequence of positive numbers with limit infinity. Moreover the set of eigenfunctions is known to be dense in the set of all smooth functions on M. We focus on a special type of Riemannian manifolds called Riemannian symmetric spaces with the metric coming from canonical Killing form defined.

Definition 4.2.1. A ***Riemannian symmetric space*** (RSS) is a connected real-analytic Riemannian manifold (M, g), such that for every $p \in M$, there is an involutive isometry $s_p \in \mathrm{Isom}_0(M)$ with $s_p(p) = p$, $s_p^2 = \mathrm{id}_M$, and $(ds_p)_p = -\mathrm{id}_{T_pM}$.

There is a well-know connection to the Lie groups, c.f. A. Borel [6]. Letting the connected component of the isometry group be $\mathrm{Isom}_0(M) = G$ and $K := \mathrm{stab}_G(p)$ for $p \in M$, it is known that G is a Lie group acting transitively on M and K is a compact subgroup, so that (M, g) is the coset space G/K with a particular metric g defined in terms of the Killing form B of the Lie algebra of G. A simply connected Riemannian symmetric space of noncompact type is $M = \mathbb{R}^n$ as $G = \mathbb{R}^n \rtimes SO(n)$ and $K = SO(n)$ producing

$$G/K = \mathbb{R}^n \rtimes SO(n)/SO(n) \cong \mathbb{R}^n = M.$$

This is an example of an RSS of noncompact type, since G and therefore M are noncompact. The Riemannian symmetric spaces, which are our primary focus are those of compact type, which include:

1. The spheres: $S^n = SO(n+1)/SO(n)$ and $S^{2n+1} = SU(n+1)/SU(n)$
2. Complex projective spaces: $\mathbb{C}P^n = SU(n+1)/(S(U(1) \times ... \times U(1)))$
3. Quaternionic projective spaces: $\mathbb{H}P^n = Sp(n+1)/(Sp(n) \times Sp(1))$
4. Cayley plane / Octonion projective plane: $CaP^2 = \mathbb{O}P^2 = F_4/Spin(9)$

Definition 4.2.2. (Definition of a CROSS) A *CROSS* is a compact rank one symmetric space. The examples of the simply connected CROSSes are enumerated above. There are also CROSSes which are not simply connected, such as the real projective space $\mathbb{R}P^n$.

When the Lie group G is semisimple, we say that (M, g) is of semisimple type. A Riemannian symmetric space of semisimple type can be represented as a product of irreducible RSSs, for which G is simple. These spaces will be our main object of study.

The process of classifying RSSs could be described briefly and precisely in terms of Lie algebras, and how we can get all of them from the corresponding (painted) Dynkin diagrams and all noncompact real forms from their Satake diagrams and their maximal compact subalgebras. Then by duality we can identify all of the compact Riemannian symmetric spaces. This will be more explicitly revisited in section 4.5. But first we must recall the basics of another Lie group and Lie algebra decomposition, known as the Iwasawa decomposition if a Lie group or Lie algebras.

Definition 4.2.3. (Definition of rank) Let $\mathfrak{a}$ be the maximal abelian subalgebra in $\mathfrak{p}$, and $\mathfrak{m}$ be the Lie algebra of the stabilizer of $\mathfrak{a}$ in $\mathfrak{k}$:

$$\mathfrak{m} = \ker \big(X \mapsto [X, \mathfrak{a}] \big)\big|_{\mathfrak{k}} = \big\{ X \in \mathfrak{k} = \mathrm{Lie}(K) \ \big| \ [X, \mathfrak{a}] = 0 \big\}.$$

The *rank* k of $\mathfrak{a}$ is called the rank of the symmetric space $M = G/K$, which is the maximum dimension of a subspace of the tangent space (to any point) on which the curvature is identically zero. the existence of is guaranteed by dimenionality arguments.

4.3 Iwasawa decomposition

For a noncompact group G, the final decomposition that we need is called the Iwasawa composition, which combines the Cartan decomposition and the root space decomposition of its complexification $\mathfrak{g}^{\mathbb{C}}$. Here we just recall the basic facts, the details can be found in Helgason or Knapp [32, 38].

Here, we combine the root space decomposition of a real Lie algebra with its complexification $(*)^{\mathbb{C}}$ to produce this decomposition which will end up being indispensible for studying noncompact Lie groups and algebras.

Theorem 4.3.1. *We let G $(:= Isom_0(M))$ be a real ss Lie group, $\mathfrak{g}$ its Lie algebra with dual $\mathfrak{g}_n$ corresponding to the noncompact group G_n, θ a Cartan involution on $\mathfrak{g}_n$ and $\mathfrak{g}_n = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Then we let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, and $\Delta \subset (\mathfrak{a})^* \subset (\mathfrak{p})^*$ the set of restricted roots with respect to $\mathfrak{a}$, corresponding to eigenvalues of $\mathfrak{a}$ acting on $\mathfrak{g}_n$. We can choose an ordering on the positive roots denoted $\Delta^+ \subset \Delta$. Then the* **Iwasawa decomposition** *of G_n ($\mathfrak{g}_n$ resp.) is*

$$G_n = KAN \quad (\mathfrak{g}_n = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \ \textit{resp.}),$$

where $\mathfrak{k} = Lie(K), \mathfrak{a} = Lie(A)$, and $\mathfrak{n} = Lie(N)$. There is also a complexified version of the Iwasawa decomposition, which we will use shortly.

Example 4.3.2. As a basic example of the Iwasawa decomposition of a noncompact real Lie group is that of the standard decomposition of $SL_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{R})$ as

$$SL_n(\mathbb{R}) = KAN$$

where K is a maximal flat compact subgroup of G, A is (essentially) the maximal abelian subgroup of G which in this case we can take

$$K = SO(n) = \left\{ k \in O(n) \;\middle|\; k^T k = kk^T = \mathrm{Id},\ \det(k) = 1 \right\}$$

to be the special orthogonal matrices, A to be (essentially the maximal abelian) subgroup of diagonal matrices

$$A = \left\{ a = \mathrm{diag}(a_{ii})_{i=1,2,\ldots,n} \;\middle|\; \det(a) = \Pi_{i=1}^{n} a_{ii} = 1 \right\}$$

of determinant 1, and N to be the unipotent subgroup of upper triangular matrices of determinant 1:

$$N = \left\{ n = \begin{pmatrix} 1 & * & * & \ldots & * \\ 0 & 1 & * & \ldots & * \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & * \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \;\middle|\; * \in \mathbb{R} \right\}.$$

Thus, we have that every $g \in G = SL_n(\mathbb{R}) = KAN$ can be expressed as a product in terms of matrix multiplication, $g = kan$.

In the complexified version we also denote $K^{\mathbb{C}}, A^{\mathbb{C}}$ and $N^{\mathbb{C}}$, respectively, and we have that $\mathrm{Lie}(A) = i\mathfrak{a}$. Also of importance is that the complexification $G_n^{\mathbb{C}} = K^{\mathbb{C}}A^{\mathbb{C}}N^{\mathbb{C}}$ is some Zariski open and dense subset of $G^{\mathbb{C}}$.

4.4 Generalized flag manifolds, adjoint actions, and examples

Now for full generality, we must properly introduce generalized flag manifolds, since they provide our parametrization spaces in the quantization procedure.

Definition 4.4.1. A ***generalized flag manifold*** is a homogeneous space of the form $G/K = G/C(S)$, where S is a torus in a compact Lie group G and $C(S)$ is its centralizer. If T is a maximal torus in G, then $C(T) = T$ and G/K is called a ***flag manifold***.

Definition 4.4.2. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and $X \in \mathfrak{g}$. The ***adjoint orbit*** of X is the set $M_X := \mathrm{Ad}(G)X = \{\mathrm{Ad}(g)X \mid g \in G\} \subset \mathfrak{g}$.

Let $K = K_X := \{g \in G \mid \mathrm{Ad}(g)X = X\}$ be the isotropy subgroup of X. Then M_X is diffeomorphic to the homogeneous space G/K. The point X corresponds to the identity coset $o = eK$. These are some of the main examples of generalized flag maniffolds.

Example 4.4.3. Examples of generalized flag manifolds:

i) A ***full*** flag in $\mathbb{C}^n$ is a collection of increasing nested complex subspaces

$$F = (\{0\} = V_0 \subset V_1 \subset .... \subset V_n = \mathbb{C}^n),$$

where each V_i has $\dim_{\mathbb{C}} V_i = i$. Let $\mathbb{F}_n$ be the set of all flags in $\mathbb{C}^n$. The group $SU(n) \ni g$ acts on $\mathbb{F}_n$ by $gF = (gV_0 \subset ... \subset gV_n)$ transitively, so if F^0 is the standard flag given by $V_i^0 = \mathrm{span}_{\mathbb{C}}\{e_1, ..., e_i\}$, then there exists a g such that $gF^0 = F$. Furthermore, the isotropy group of F^0 is consists of all diagonal matrices in $SU(n)$, which is a maximal torus in $SU(n)$. Thus letting $G = SU(n)$, $\mathfrak{g} = \mathfrak{su}(n)$, and $X = \mathrm{diag}(e^{i\lambda_1}, ..., e^{i\lambda_n})$, where λ_i are distinct real numbers with $\sum_i \lambda_i = 0$, then $K = K_X = T^n$ (n-torus), so giving a homogeneous presentation of the flag manifold

$$\mathbb{F}_n = \mathrm{Ad}(SU(n))X = {}^{SU(n)}\!/_{T^n} = {}^{SU(n)}\!/_{S(U(1) \times ... \times U(1))} \quad (n - times).$$

ii) In particular, when ($\lambda_1 = \cdots = \lambda_k = \lambda$) and ($\lambda_{k+1} = \cdots = \lambda_n = \mu$) (with ($\lambda \neq \mu$)), then $K_X = S(U(k) \times U(n-k))$ and

$$\mathrm{Ad}(SU(n))X = {}^{SU(n)}\!\big/_{S(U(k) \times U(n-k))} = Gr_k(\mathbb{C}^n),$$

the Grassmann manifold of k-planes in $\mathbb{C}^n$. Thus, we see that projective spaces $\mathbb{C}P^n$ and Grassmannians $Gr_k(\mathbb{C}^n)$ are special cases of generalized flag manifolds.

iii) We then immediately consider the case of partial flags, where $\mathbb{F}(n_1, ..., n_s)$ is the manifold of all partial flags with $n = \sum_i n_i = n_1 + ... + n_s$ and $N_j = n_1 + ... + n_j$ $\dim_{\mathbb{C}} V_i = n_1 + ... + n_i$. As in the example above, $SU(n)$ acts transitively, and the isotropy subgroup of a fixed point is $S(U(n_1) \times \cdots \times U(n_s))$, the group of matrices of the form $\mathrm{diag}(A_1, \ldots, A_s)$ with $A_i \in U(n_i)$ and $\prod_i \det(A_i) = 1$. Thus, we get the homogeneous presentation of the space of partial flags as

$$\mathbb{F} = \mathbb{F}(n_1, \ldots, n_s) = {}^{SU(n)}\!\big/_{S(U(n_1) \times \cdots \times U(n_s))}.$$

The generalized flag maniflds have important geometric properties. To describe some of them, recal that a complex structure on a manifold M is an endomorphism of its tangent bundle I such that $I^2 = -id$ and I is integrable in the sense that the tensor $N(X,Y) = [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY] = 0$ for every vector fields X, Y on M. A complex structure leads to existence of an atlas of charts on M with values in $\mathbb{C}^n$ and holomorphic transition functions. Main examples of such manifolds are the algebraic submanifolds of $\mathbb{C}P^n$. Every compact algebraic manifold - with algebraic transition functions in appropriate charts, appears as such submanifold by the theorem of Chow. That is why it such manifolds are called projective algebraic. The main property of the generalized flag manifolds is that they admit complex structures which make them projective algebraic manifolds. The theory of projective algebraic manifolds is classical and one of its initial subjects is the relation between holmorphic line bundles on them and their divisors, or codimension 1 subspaces with multiplicities. For information about this theory we refer to Griffths-Harris [27]. We will use the fact that the holomorphic line bundles can be described in terms of their sheaves of holomorphic sections, which have a well defined cohomology. The topology of the line bundles L over a complex manifold M are determined by the first Chern class $c_1(L) \in H^2(M, \mathbb{Z})$.

4.5 Riemannian symmetric spaces and associated generalized flag manifolds

Let $M = G/K$ be a RSS with G-semisimple and compact, so that the Killing form is negative definite on $\mathfrak{g}$, and thus we have a reductive eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to the (Cartan) involution θ on $\mathfrak{g}$. Recall that we can identify the tangent space T_oM with $\mathfrak{p}$, where $o = eK$ is the identity coset in G/K, and we have the bracket relations $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

We denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}^{\mathbb{C}}$ its complexification. $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \subset (\mathfrak{h}^{\mathbb{C}})^*$ is the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$, and the root decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}_\alpha)^{\mathbb{C}},$$

where $\mathfrak{g}_\alpha = \big\{ X \in \mathfrak{g} \;\big|\; \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h} \big\}$ are the root spaces.

Definition 4.5.1. (Definition of rank) Let $\mathfrak{a}$ be the maximal abelian subalgebra in $\mathfrak{p}$, and $\mathfrak{m}$ be the Lie algebra of the stabilizer of $\mathfrak{a}$ in $\mathfrak{k}$:

$$\mathfrak{m} = \ker \big(X \mapsto [X, \mathfrak{a}] \big)\Big|_{\mathfrak{k}} = \big\{ X \in \mathfrak{k} = \mathrm{Lie}(K) \;\big|\; [X, \mathfrak{a}] = 0 \big\}.$$

The *rank* k of $\mathfrak{a}$ is called the rank of the symmetric space $M = G/K$, which is the maximum dimension of a subspace of the tangent space (to any point) on which the curvature is identically zero. the existence of is guaranteed by dimenionality arguments.

As previoiusly mentioned, it is well known that K acts transitively on the set of all maximal abelian subalgebras in $\mathfrak{p}$, and since $\mathfrak{a}$ is maximal, the centralizer $C_{\mathfrak{g}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{g}$ is $\mathfrak{l} = \left\{X \in \mathfrak{g} \;\middle|\; [X, \mathfrak{a}] = 0\right\} = \mathfrak{m} \oplus \mathfrak{a}$.

Remark 4.5.2. If L is the corresponding Lie group with Lie algebra $\mathfrak{l}$, then L contains a maximal torus of G (as a centralizer of an abelian subgroup). Hence, the space G/L is a generalized flag manifold which carries a natural complex structure and a Kähler metric. A key observation in [25] Section 3, "From geometric viewpoint a maximal abelian subalgebra of $\mathfrak{p}$ is tangent to maximal totally geodesic flat torus and every such torus is at a point gK tangent to a left translate of some $\mathfrak{a}$ from $o = eK$ to gK. In particular, the space parametrizing all maximal totally geodesic flat tori can be identified with $G/N_{\mathfrak{a}}$, where $N_{\mathfrak{a}}$ is the normalizer of a fixed $\mathfrak{a}$ in $\mathfrak{g}$. As shown in [33], the centralizer L is a normal subgroup of $N_{\mathfrak{a}}$ and the quotient $N_{\mathfrak{a}}/L$ is a finite group, which is a subgroup of the Weyl group of $\mathfrak{g}$. In these terms the generalized flag manifold G/L is the universal cover of the space parametrizing the set of all such tori. We call the space G/L a **quantization space** of G/K. It is closely related to the horospherical manifold in [34] and [20]."

An important invariant related to the quantization space is the dimension of its second cohomology. We demonstrate here how this dimension could be identified in terms of the data provided by the Satake diagram associated to the noncompact dual Riemannian symmetric space of G/K.

Denote by $\mathfrak{m}_0 = [\mathfrak{m}, \mathfrak{m}]$. Since $\mathfrak{m}$ is compact, $\mathfrak{m}_0$ is the semisimple part of $\mathfrak{m}$. We can consider a maximal torus of $\mathfrak{g}$ which is θ invariant and contains $\mathfrak{a}$. Such torus is known to exists and since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, then this torus is a Cartan subalgebra and has the form $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ with $\mathfrak{t} \subset \mathfrak{k}$. Its complexification $\mathfrak{h}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $\Delta = \Delta^{+} \cup \Delta^{-}$ be a root system with an ordering defining the positive and negative roots of $\mathfrak{g}^{\mathbb{C}}$. There is a set Σ of the so called restricted roots $\Sigma = \Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}) \subset (\mathfrak{a}^{\mathbb{C}})^{*}$ and we can choose a basis $h_1, ..., h_k, h_{k+1}, ..., h_n$ of $(\mathfrak{h}^{\mathbb{C}})^{*}$, of basic roots, such that $h_1, ..., h_k$ (after restricting them via the projection $\mathfrak{h}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}$) are basis for $(\mathfrak{a}^{\mathbb{C}})^{*}$.

We continue to use the same notation $h_1, ..., h_k$ for the restricted roots. After we choose an ordering of the basic roots, or equivalently, a positive Weyl chamber, every element of Δ (respectively, Σ) is an integer linear combinations with all non-negative or all non-positive coefficients of $h_1, ..., h_n$ (respectively, $h_1, ..., h_k$). Similarly we can choose a positive Weyl chamber in the restricted roots. Then $\mathfrak{m}_0^{\mathbb{C}}$ has a root decomposition with root spaces which are among the root spaces $\mathfrak{g}_\alpha$ of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$.

4.6 Relation between Satake and painted Dynkin diagrams

The non-compact real form $\mathfrak{g}_n$ of $\mathfrak{g}^{\mathbb{C}}$, which is dual of $\mathfrak{g}$ have $\mathfrak{k}$ as a maximal compact subalgebra. We want to describe the relation between the Satake diagram of $\mathfrak{g}_n$ and the painted Dynkin diagram associated to the generalized flag manifold G/L (recall L comes from $\mathfrak{a} \oplus \mathfrak{m} = \mathfrak{l} = \mathrm{Lie}(L)$). We first describe the Satake diagram of $\mathfrak{g}_n$. For this we paint the simple roots defining the root spaces of $\mathfrak{m}_0^{\mathbb{C}}$ in black. The Satake diagram of $\mathfrak{g}_n$ is then the Dynkin diagram of $\mathfrak{g}$ with the black and white dots as described, but with an additional arrows between the white roots, when there is an involutive automorphism of $\mathfrak{g}$ such that the difference between the initial and the endpoint of an arrow is a root in $\mathfrak{t}$ (see [40] Sect. 4.4 or [12] Sect. 2.3, for the facts about Satake diagrams).

If we consider a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}}$, which contains $\mathfrak{l}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$ and is minimal (i.e., does not contain another proper parabolic subalgebra), then it defines the complex structure of the generalized flag manifold G/L. Such algebra is not unique, but the different ones are related via elements of the Weyl group and they define different invariant complex structures on G/L. As described in [10], $\mathfrak{q}$ contains a Borel subalgebra and is described by a subset of simple roots of Δ^+ and the complex structure is defined via the ordering. The painted Dynkin diagram of G/L in an analog of the Satake diagram and is defined - see [2, 5] as a diagram with vertices corresponding to the semisimple part of $\mathfrak{l}$ painted in black. Since $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{m}$, the semisimple (ss) part of $\mathfrak{l}$ is precisely $\mathfrak{l}_{ss} = \mathfrak{m}_0$. The description leads to the following observation:

Theorem 4.6.1. *Let $M = G/K$ is an irreducible Riemannian symmetric space and G/L is its quantization space. If $\mathfrak{g}_n$ is the non-compact dual form of $\mathfrak{g}$ with respect to K, then the Satake diagram of $\mathfrak{g}_n$ with the arrows deleted corresponds precisely to the painted Dynkin diagram defining the complex structure of G/L.*

As a consequence of the above theorem, we obtain:

Corollary 4.6.2. The group $H^2(G/L, \mathbb{Z})$ has no torsion and is generated by the elements corresponding to the white vertices of the Satake diagram of $\mathfrak{g}_n$.

Proof. The fact that H^2 has no torsion is well known. According to [7], there is an isomorphism $\mathcal{Z}(\mathfrak{l}^{\mathbb{C}})^* \equiv H^2(G/L, \mathbb{R})$, often called *transgression*, given by $\alpha \mapsto \frac{i}{2\pi} d\alpha$.

It has the property, that the fundamental weights in $\mathcal{Z}(\mathfrak{l}^{\mathbb{C}})^*$ correspond to elements of $H^2(G/L, \mathbb{Z}) \in H^2(G/L, \mathbb{R})$. Then the first k elements $w_1, ..., w_k$ of the basis $w_1, ..., w_n$ of the fundamental weights dual with respect to the Killing form on $\mathfrak{g}$ to $h_1, ...h_n$, define an integral basis of $H^2(G/L, \mathbb{Z})$. It is known [2, 5], that this basis is generated via transgression by simple roots corresponding to the white vertices in the painted Dynkin diagram describing the complex structure of G/L, which are in bijection with the basis of the center of $\mathfrak{l}$. As explained above these are precisely the white vertices in the corresponding Satake diagram. □

4.7 Borel-Weil-Bott Theory

The Borel-Weil-Bott theorem characterizes representations of suitable Lie groups G as space of holomorphic sections of complex line bundles over flag varieties G/B, for B a Borel subgroup. With minor modifications added, this is Kirillov's orbit method, and the construction may be interpreted as sending a symplectic manifold equipped with G-Hamiltonian action to its geometric quantization. It is the generalization of the theorem of Borel–Weil, which we recall here. See [3, 38] for further references.

Theorem 4.7.1. *(Borel-Weil Theorem) If B is the Borel subgroup of the complex semisimple group $G^{\mathbb{C}}$ (which can be considered as the complexification of a compact Lie group G with the maximal torus $T = G \cap B \subset G$), then all unitary irreducible representations can be obtained as the spaces of (anti)-holomorphic line bundles associated to the principal fibration $G \to G/B$ over the generalized flag variety $G^{\mathbb{C}}/B \cong G/T$ with the fiber $\mathbb{C}_\lambda$, which is the 1-dimensional representation corresponding to a dominant integral character λ; and vice-versa, all such spaces of sections are irreducible. The inner product is inherited from the Hermitian structure on the line bundle.*

An integral weight λ determines a G-equivariant holomorphic line bundle L_λ on G/T and the group G acts on its space of global sections, $\Gamma(G/B, L_\lambda)$.

The Borel–Weil theorem states that if λ is a dominant integral weight then this representation is a holomorphic irreducible highest weight representation of G with highest weight λ. Its restriction to K is an irreducible unitary representation of K with highest weight λ, and each irreducible unitary representations of K is obtained in this way for a unique value of λ.

The extension to higher cohomologies instead of spaces of sections, called the Borel–Weil–Bott theorem, which we also recall here and clarify how the classical Borel-Weil theorem above follows as a special case.

Set-up: Let G be a semisimple Lie group over $\mathbb{C}$, and T a fixed maximal torus along with a Borel subgroup B which contains T. Let λ be an integral weight of T; λ defines in a natural way a one-dimensional representation $\mathbb{C}_\lambda$ of B, by pulling back the representation on $T = B/U$, where U is the unipotent radical of B. Since we can think of the projection map $G \to G/B$ as a principal B-bundle, for each $\mathbb{C}_\lambda$ we get an associated fiber bundle $L_{-\lambda}$ on G/B (note the sign). Identifying L_λ with its sheaf of holomorphic sections, we consider the sheaf cohomology groups $H^i(G/B,\, L_\lambda)$. Since G acts on the total space of the bundle L_λ by bundle automorphisms, this action naturally gives a G-module structure on these groups; and the Borel–Weil–Bott theorem gives an explicit description of these groups as G-modules.

First we begin by describing the Weyl group $\mathcal{W}$ action centered at $-\rho$. For any integral weight λ and $w \in \mathcal{W}$, we set $w * \lambda := w(\lambda + \rho) - \rho$, where ρ denotes the half-sum of positive roots of G. It is straightforward to check that this defines a group action, although this action is not linear, unlike the usual Weyl group action. Let ℓ denote the length function on the Weyl group $\mathcal{W}$.

Theorem 4.7.2. *(Borel-Weil-Bott theorem)*

With the set-up from above, given an integral weight λ, one of two cases occur:

i) There is no $w \in \mathcal{W}$ such that $w * \lambda$ is dominant, or equivalently, there exists a nonidentity $w \in \mathcal{W}$ such that $w * \lambda = \lambda$, or

ii) There is a unique $w \in \mathcal{W}$ such that $w * \lambda$ is dominant.

The Borel-Weil-Bott theorem states that in the first case, we have $H^i(G/B, L_\lambda) = 0$ for all i; and in the second case, we have $H^i(G/B, L_\lambda) = 0$ for all $i \neq \ell(w)$, while $H^{\ell(w)}(G/B, L_\lambda)$ is the dual of the irreducible highest-weight representation of G with highest weight $w * \lambda$.

It is worth noting that case (i) above occurs if and only if $(\lambda + \rho)(\beta^\vee) = 0$ for some positive root β. Also, we obtain the classical Borel–Weil theorem as a special case of this theorem by taking λ to be dominant and w to be the identity element $e \in \mathcal{W}$.

4.8 Spherical representations and Cartan-Helgason Theorem

In the previous section we described the relation between a (unitary) representation of a compact Iie group G and the spaces of holomorphic section of line bundles over a generalized flag manifolds. Here we consider the similar question related to the eigenspaces of the Laplace-Beltrami operator on a Rieamnnian symmetric space. First we remind the reader the definition of a finite-dimensional representation of a Lie Group G.

Definition 4.8.1. A representation of a Lie group G is a group action on a finite-dimensional vector space V over the field $\mathbb{C}$, which is a smooth group homomorphism $\rho : G \to \mathrm{GL}(V)$.

We recall the set up of Lie algebra root systems in section 3.2 and root space decompositions, and the further refinement of restricted root systems in 3.3 and the restricted root space decomposition.

Let G be a connected noncompact semisimple Lie group, $G = KAN$ an Iwasawa decomposition, and M the centralizer of A in K, with the usual $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ and $\mathfrak{m}$ the corresponding Lie algebras. Just as before, we extend $\mathfrak{a}$ to a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, and the notation for complexifications is $(*)^{\mathbb{C}}$ as before.

Since $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$ is diagonalizable on $G^{\mathbb{C}}$, the root system $\Sigma(\mathfrak{g}, \mathfrak{h})$ defines a restricted root system

$$\Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}) := \Big\{ \alpha|_{\mathfrak{a}^{\mathbb{C}}} \;\Big|\; \alpha \in \Sigma(\mathfrak{g}, \mathfrak{h}),\ \alpha|_{\mathfrak{a}^{\mathbb{C}}} \neq 0 \Big\},$$

and the corresponding positive root system $\Delta^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$ and simple roots $\Sigma^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$.

The essential difference between root systems and restricted root systems is that one can have several roots with the same restriction to $\mathfrak{a}^{\mathbb{C}}$, so the restricted root spaces $\mathfrak{g}^{\mathbb{C}}_{\alpha}$ can have dimension greater than 1. The weights (like the roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$) are real valued on the space $\mathfrak{h}_{\mathbb{R}} = i(\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{a}$, where $\mathfrak{h}_{\mathfrak{k}} := \mathfrak{h} \cap \mathfrak{k}$.

The choice of $\mathfrak{n}$ corresponds to an ordering of the dual space $\mathfrak{a}^*$, and denote by Σ^+ the corresponding set of positive restricted roots. Choosing a compatible ordering on the dual of $\mathfrak{h}_{\mathbb{R}}$ produces an ordering on the set of all weights. The Killing form $B(\cdot, \cdot)$ on $\mathfrak{g}^{\mathbb{C}}$ induces an inner product on the algebras $\mathfrak{a}$, $\mathfrak{h}_{\mathbb{R}}$, and their duals.

Next we define $\widehat{G}$ - the set of inequivalent unitary representations, that contains as subset the **spherical representation** of a symmetric pair (G, K) coming from the symmetric space G/K. These spherical representations are representation of G which have a fixed vector by K. We can now state the Cartan-Helgason theorem which characterizes (G, K)-spherical representations by the highest weight:

Theorem 4.8.2. *(Cartan-Helgason Theorem, Thm 11.4B [45]) Fix a positive restricted root system $\Sigma^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$ and a compatible positive system $\Sigma^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Let $[\rho] \in \widehat{G}$ and $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$*

(i) The representation ρ has a nonzero K-fixed vector (i.e. is (G, K)-spherical) if and only if $\rho(M)$ fixes the highest weight vector of ρ.

(ii) The linear functional λ on $\mathfrak{h}_{\mathbb{R}}$ is the highest weight of an irreducible (G, K)-spherical representation of G if and only if $\lambda(\mathfrak{h}_{\mathbb{R}}) = 0$, and $\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+$ for all $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{h})$.

In other words, extending linear functionals from $\mathfrak{a}^{\mathbb{C}}$ to $\mathfrak{h}^{C}$ by zero on $\mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}}$, the (G, K)-spherical representations of G are parameterized, up to equivalence, in terms of their highest weights, by $\left\{ (\alpha|_{\mathfrak{a}^{\mathbb{C}}})^* \;\middle|\; \frac{\langle \lambda, \gamma \rangle}{\langle \gamma, \gamma} \in \mathbb{Z}^+, \text{ for all } \gamma \in \Delta^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}) \right\}$.

It is known that the algebra of invariant differential operators on a Riemannian symmetric space is commutative, and in particular every eigenspace of the Laplace-Beltrami operator is a direct sum of of common eigenspaces. Such common eigenspaces are known to be precisely irreducible spherical representaion for the pair (G, K) when G is simple simply-connected and compact.

CHAPTER 5

EIGENSPECTRA OF THE LAPLACE-BELTRAMI OPERATOR

Recall the formula for the eigenvalues of the Laplacian operator Δ_M from D. Gurarie's "*Symmetries and Laplacians*" [30]. We restate Theorem 2 in section 7.7:

Theorem 5.0.1. *(Eigenspectra of Δ_M)*

Let $M = G/K$ and Δ_M be Laplacian operator of M, $\mathfrak{g} = Lie(G) = \mathfrak{k} \oplus \mathfrak{p}$ and Σ a (usually canonical) choice of simple roots $\Sigma = \{\alpha_1, ...\alpha_r\}$ where the rank of $M = G/K$ is $k = dim_{\mathfrak{a} \subset_{max} \mathfrak{p}}(\mathfrak{a}) = rk(M)$. Then

(i) The eigenvalues $\{\lambda_\beta\}$ of Δ_M on M are labeled by the restricted weight lattice [given by the subset] $\{\beta = \sum k_j \alpha_j \mid k_j \geq 0\}$ of the fundamental Weyl chamber $\mathcal{C}$, the β^{th} eigenvalue is equal to

$$\lambda_\beta = ||\beta - \rho||^2 - ||\rho||^2,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha = \frac{1}{2}\sum_{\alpha_j \in \Sigma} m_j \alpha_j$ is the half sum of all positive roots (taken with their multiplicities in the second equality).

(ii) The multiplicity of λ_β is equal to the degree of an irreducible representation π^β of the group G.

In the case of A_n, the dominant weights are given by $\beta = \sum_{i=1}^{n-1} c_i \alpha_i$ for c_i integers, defined more precisely in the next definition.

Definition 5.0.2. A weight β is called ***dominant*** if $\langle \beta, \alpha_i \rangle \geq 0$ for all simple roots α_i, meaning β is in the closure of the fundamental Weyl chamber $\mathcal{C}$.

Example 5.0.3. Eigenvalues of the Laplace operator on $M = SU(n)/SO(n)$ for $n \geq 3$.

We note that M has root system $A_{n-1} = \mathfrak{sl}(n, \mathbb{C})$. Making use of the definition, we proceed to calculate in defining inequalities of the fundamental Weyl chamber for this particular root system, primarily using the definition 5.0.2 above.

For this we compute $\langle \beta, \alpha_i \rangle \geq 0$ for all i. Useful to note that

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

which are precisely the entries in the Cartan matrix of the algebra A_{n-1}.

For $i = 1$, we have

$$\langle \beta, \alpha_1 \rangle = \langle \sum_{k=1}^{n-1} c_k \alpha_k, \alpha_1 \rangle = \sum_{k=1}^{n-1} c_k \langle \alpha_k, \alpha_1 \rangle = 2c_1 - c_2 \geq 0,$$

for $1 < i < n-2$, we deduce

$$\langle \beta, \alpha_i \rangle = \langle \sum_{k=1}^{n-1} c_k \alpha_k, \alpha_i \rangle = \sum_{k=1}^{n-1} c_k \langle \alpha_k, \alpha_i \rangle = -c_{i-1} + 2c_i - c_{i+1} \geq 0,$$

and finally for $i = n-2$, we get

$$\langle \beta, \alpha_{n-1} \rangle = \langle \sum_{k=1}^{n-1} c_k \alpha_k, \alpha_{n-1} \rangle = \sum_{k=1}^{n-1} c_k \langle \alpha_k, \alpha_{n-1} \rangle = -c_{n-2} + 2c_{n-1} \geq 0$$

We begin by finding the defining inequalities of its fundamental Weyl chamber, which is as easy as

$$\sum_{k=1}^{n-1} c_k \langle \alpha_k, \alpha_1 \rangle = 2c_1 - c_2 \geq 0$$

in the case $i = 1$. For $1 < i < n-2$, we deduce

$$\langle \beta, \alpha_i \rangle = \langle \sum_{k=1}^{n-1} c_k \alpha_k, \alpha_i \rangle = \sum_{k=1}^{n-1} c_k \langle \alpha_k, \alpha_i \rangle = -c_{i=-1} + 2c_i - c_{i+1} \geq 0,$$

And finally, for $i = n-2$, we get

$$\langle \beta, \alpha_{n-1} \rangle = \langle \sum_{k=1}^{n-1} c_k \alpha_k, \alpha_{n-1} \rangle = \sum_{k=1}^{n-1} c_k \langle \alpha_k, \alpha_{n-1} \rangle = -c_{n-2} + 2c_{n-1} \geq 0.$$

Combining these inequalities gives us the defining inequalities in our chosen basis of the simple roots for the fundamental Weyl chamber

$$\mathcal{C} : \begin{cases} 2c_1 \geq c_2 \\ 2c_i \geq c_{i-1} + c_{i+1} \text{ for } i = 2, 3..., n-2 \\ 2c_{n-1} \geq c_{n-2} \end{cases}$$

So computing it for

$$\beta = \sum_{i=1}^{n-1} c_i \alpha_i = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + ... + c_i \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} + c_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 - c_1 \\ \vdots \\ c_i - c_{i-1} \\ \vdots \\ -c_{n-1} \end{pmatrix}.$$

To considerably simplify matters a bit further, we make the substitution $x_i = c_i - 1$ for $i = 1, 2, ..., n-1$, so $c_i - c_{i-1} = c_i - 1 - (c_{i-1} - 1) = x_i - x_{i-1}$ and we can rewrite

$$\beta - \rho = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ \vdots \\ x_i - x_{i-1} \\ \vdots \\ -x_{n-1} \end{pmatrix},$$

and applying the theorem at the beginning of this section, we have

$$\begin{aligned}
\|\beta-\rho\|^2 &:= x_1^2+(x_2-x_1)^2+...+(x_{n-2}-x_{n-1})^2+(-x_{n-1})^2 \\
&= x_1^2+x_2^2-2x_1x_2+...+x_{n-1}^2-2x_{n-2}x_{n-1}+x_{n-2}^2+x_{n-1}^2 \\
&= 2\left(\sum_{i=1}^{n-1}x_i^2-\sum_{i=1}^{n-2}x_ix_{i+1}\right) \\
&= \begin{pmatrix} x_1, x_2, ..., x_{n-1}\end{pmatrix}\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ & & & ... & & \\ 0 & & -1 & 2 & -1 & 0 \\ 0 & ... & 0 & -1 & 2 & -1 \\ 0 & ... & ... & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1}\end{pmatrix} \\
&= x^T * \mathfrak{C} * x
\end{aligned}$$

This calculation gives us a convenient and useful expression for the norm. In particular, we see that the eigenspaces the Laplace-Beltrami Δ_M corresponding to eigenvalue $\lambda_\beta = \|\beta-\rho\|^2 - \|\rho\|^2$ split into irreducible representation subspaces when the Diophantine equation $\sum_{i=1}^{n-1}x_i^2 - \sum_{i=1}^{n-2}x_ix_{i+1} = Q$ where $Q = \lambda_\beta + \|\rho\|^2$ does not depend on $x_i > 1$, has more than one integer solution in the fundamental Weyl chamber.

We can carry out the same procedure for the remaining classical and exceptional noncompact (ss) Lie algebras. The resulting Diophantine equation in standard coordinates for the other Lie algebras, but under a linear change of coordinates it can be put into a similarly elegant form like the one above for $A_{n-1} = \mathfrak{sl}(n, \mathbb{C})$.

Using the well-known and well documented data for simple Lie algebras, found in a plethora of resources including ([38], Appendix C), we can carry out the same calculations to find the Diophantine equation and Weyl chamber for all the algebras, to later apply them to corresponding RSS's, and carry out a similar procedure using the corresponding the root system and (fundamental) Weyl chamber (equivalently, the ordering we choose for simple roots), and can use it to find the eigenspectrum for higher rank compact RSS's, which consequently is related with the geometric quantization of its geodesic flow. The Laplace eigenspectra of CROSSes are well known, see Gurarie [30] for example, yet the same is not true for the compact RSS of higher rank than 1. So we will dedicate a section here to use number theoretic techniques to describe the spectrum of rank two systems, and then beyond.

5.1 Laplace eigenspectra of rank 2 Riemannian symmetric spaces

The rank 2 root systems are $A_1 \times A_1, A_2, B_2, C_2, D_2$, and the the root system of the exceptional Lie group G_2, but only 4 irreducible systems upto isomorphism since $A_1 \times A_1 \cong D_2$ and $B_2 \cong C_2$. For example, the rank 2 compact symmetric space $M = SU(3)/SO(3)$ corresponds a the split real form of $\mathfrak{su}(3)^{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$ which has root system A_2.

5.2 Lie algebras with root system A_2

Proposition 5.2.1. *Eigenspectrum of $M = SU(3)/SO(3)$ using the A_2 root system.*

Setting $n = 3$ in example 5.0.3, and for simplicity calling $x_1 = x$ and $x_2 = y$, Theorem 5.0.1 *and example* 5.0.3. *enables to find the eigenvalues of Δ_M by finding dominant weights β in the fundamental Weyl chamber $\mathcal{C}$ as the integer (or lattice* *points) that satisfy the eigenvalue formula*

$$\lambda_\beta = ||\beta - \rho||^2 - ||\rho||^2,$$

which we can reformulate as the solutions to the integral Diophantine equation (since $||\rho||^2$ *is always constant), natural numbers* $n \in \mathbb{N}_{>0}$ *such that*

$$C(A_2, n) := \Big\{ n = \lambda_\beta + ||\rho||^2 := ||\beta - \rho||^2 \;\Big|\; x^2 - xy + y^2 = n \Big\}.$$

Proof. For $M = SU(3)/SO(3)$ we look at the few small individual cases separately, like the case $Q = 1$. But what will turn out to be a reccuring theme, perfect squares such as $1, 4, 9, ..., n^2, ...$ always have $Q = n^2$ itself as the trivial solution (n, n) meaning if $x = n$ and $y = n$, since clearly

$$(x^2 - xy + y^2)|_{(x,y)=(n,n)} = n^2 - n^2 + n^2 = n^2$$

is always satisfied, so is essentially a trivial solution.

The nontrivial solutions require the number theory concerning integer solutions to integral quadratic forms of a finite number of variables.

For primes $p = 2$ and $p = 3$, these can be worked out by hand. So suppose p is an (odd) prime greater than 3. We are interested in the solutions to the equation

$$\frac{1}{2}||\beta - \rho||^2 = x^2 - xy + y^2 = p.$$

The first step is to see which primes this Diophantine quadratic form corresponding to A_2 has (nontrivial) solutions in $\mathcal{C}$. To this end, we begin by multiplying by 4 and performing an affine transformation:

$$\begin{aligned} 4(x^2 - xy + y^2) = 4x^2 - 4xy + 4y^2 &= (2x)^2 - 2(2x)(y) + y^2 + 3y^2 \\ &= (2x - y)^2 + 3y^2, \end{aligned}$$

hence, we have

$$4(x^2 - xy + y^2) = (2x - y)^2 + 3y^2 = 4p$$
$$\Rightarrow X^2 + 3Y^2 = 4p,$$

after changing variables to $X = 2x - y$ and $Y = y$.

We now let $Z = \frac{X}{Y}$ and solve the corresponding modular congruence after modding out by p and use quadratic residues to determine if some solutions can be lifted from the modular equations to the Diophantine ones.

$$Z^2 + 3 \equiv 0 \pmod{p} \quad \text{or} \quad Z^2 \equiv -3 \pmod{p}.$$

Lemma 5.2.2. *$Z^2 \equiv -3 \pmod{p}$ has (nontrivial) solutions if and only if $p \equiv 1 \pmod{6}$.*

Proof. The proof is straightforward and relatively simple using formulas for *Legendre symbols* (and its properties) and *the law of quadratic reciprocity* (QR). For the reader's convenience we will recall the facts and formulas used in the following proof:

(i) $Z^2 \equiv -3 \pmod{p}$ has (non-trivial) solutions $\iff$ the Legendre symbol $\left(\frac{-3}{p}\right) = 1$.

(ii) Legendre symbol's multiplicative properties: for $a, b \in \mathbb{Z}$ we have $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

(iii) Particular formulas for $a = -1$ and $b = 3$:

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

$$\left(\frac{3}{p}\right) = (-1)^{\lfloor \frac{p+1}{6} \rfloor} = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12} \end{cases}$$

(iv) The Law of Quadratic Reciprocity (QR): $\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$

So it follows, beginning from (i) above, that

$$\begin{aligned} Z^2 \equiv -3 \pmod{p} \text{ has (non-trivial) solutions} &\iff \text{the Legendre symbol } \left(\frac{-3}{p}\right) = 1 \\ &\iff \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = 1 \\ &\iff (-1)^{\frac{p-1}{2}}\left(\frac{3}{p}\right) = 1 \\ &\iff (-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}\frac{3-1}{2}}\left(\frac{p}{3}\right) = 1 \\ &\iff \left(\frac{p}{3}\right) = 1. \end{aligned}$$

Hence p must have remainder 1 modulo 3, i.e. $p = 3k+1$. However, since we are only considering *odd* primes, $k \in 2\mathbb{Z}$ must be even because for odd k, $k = 2k_1 + 1$, then $3k+1 = 3(2k_1+1)+1 = 6k_1+3+1 = 6k_1+4 = 2(3k_1+2) \in 2\mathbb{Z}$, and hence cannot be a prime number. Thus $p = 3(2k_1)+1 = 6k_1+1$. This completes the proof. $\square$

Recalling the well known theorems found in [8] to carry out the remaining characterization of the Laplace eigenspectra of $SU(3)/SO(3)$ corresponding to A_2 and *all* other rank two compact RSSs, via the same approach.

Theorem 5.2.3 (Chevalley's Theorem)**.** *If $F(x_1, ..., x_n)$ is a form of degree less than n, then the congruence*

$$F(x_1, ..., x_n) \equiv 0 \ (mod\ p)$$

has nonzero solutions.

Theorem 5.2.4 (Warning's Theorem)**.** *The number of solutions to the congruence*

$$F(x_1, ..., x_n) \equiv 0 \ (mod\ p)$$

is divisible by p, provided that the provided that the degree of F is less than n.

We apply immediate consequences of the theorems above to the case of (binary) quadratic forms, that is, forms of the type $f(x, y) = ax^2 + 2bxy + cy^2$. In this case the discriminant of f is $d(f) = ac - b^2$.

Corollary 5.2.5. The congruence

$$f(x, y) \equiv 0 \ (\text{mod } p)$$

has a nonzero solution if and only if p divides $-d(f)$ or $-d(f)$ is a quadratic residue modulo p.

Furthermore, the ***number of nonzero solutions*** of the congruence $f(x, y) \equiv 0 \ (\text{mod } p)$ where $f(x, y)$ is a quadratic form with discriminant $d(f)$ not divisible by p, provided $p \neq 2$, is

$$N(p) := \text{number of nonzero solutions to } f(x, y) := C(A_2) \equiv 0 \ (\text{mod } p).$$

To continue the case of the system A_2, we take $a = 2, b = -1, c = 2$ to get the solutions of the binary quadratic Diophantine equation (set equal to some number $n = p$) $C(A_2) = f(x, y) = \frac{1}{2}||\beta - \rho||^2 = x^2 - xy + y^2 \equiv n$, $(n \in \mathbb{N}_{>0})$ coming from what we call from here the corresponding *Cartan quadratic form* of the root system (A_2 in this case) obtained from Gurarie's formula for the eigenvalues, and solve $f(x, y) = x^2 - xy + y^2 \equiv 0$ (mod p). Note that $d(f) = (2)(2) - (-1)^2 = 3$ is not divisible by $p > 5$, which includes all $6k + 1 = 3(2k_1) + 1$ primes (for $k, k_1 \geq 1$), and in Lemma 6.2 we obtain the primes p for which this has nonzero solutions, namely all p's of the form $p = 6k + 1 > 5$. The only other case, where we have at least one nonzero solution (except as pointed out $n = k^2$ for $k \geq 1$), is for the numbers $n = 3^k p$'s that are divisible by 3 (and hence all powers of 3, $3^k, k \geq 1$), which are also of the same form: $p^2 = (6k+1)^2 = 36k^2+12k+1 = 6k(6k+2)+1$ and $n^2 = (3(2k_1)+1)^2 = 6k_2+1$, and the same for $n = 3^k p$.

We require a few more straightforward observations. The first is the simple observation of noticing the symmetry: if (x, y) is a solution, then so is $(-x, -y), (y, x)$, and $(-y, -x)$, for obvious reasons. Hence solutions come in symmetric pairs except in the exceptional case in the next observation. We will see that we can disregard negative solutions because we are interested only in solutions in the fundamental Weyl chamber, which we will elaborate more very soon.

As foreshadowed earlier, since we always have the trivial solution $1^2 - (1)(1) + 1^2 = 1$, that is, $(x, y) = (1, 1)$ and $x^2 - xy + y^2 = n^2$ is equivalent to $\left(\frac{x}{n}\right)^2 - \left(\frac{x}{n}\right)\left(\frac{y}{n}\right) + \left(\frac{y}{n}\right)^2 = 1$, we always have a trivial solution which is the solution $(1, 1)$ scaled by n: $n * (1, 1) = (n, n)$.

The following observation has to do with powers of the 1 modulo 6 primes and $n = 3p$, is powers of them, p^k. The case of n-perfect squares is determined above, and powers of them here. Because of the trivial solution when $n = N^2 n_1$ has a factor of a perfect square, the pairity of k changes the situation because if k is even, then we get an extra trivial solution because of the previous observation. The final observation is that we must take the solutions that lie in the fundamental Weyl chamber $\mathcal{C} = \mathcal{C}(A_2)$, in this particular case defined by the open cone in the first quadrant defined by $0 < x < 2y < 4x$. So we define our counting function for the number of nonzero solutions, and it follows that

$$
\begin{aligned}
N(p) = \left|\left\{(x.y) \in \mathbb{N}\times\mathbb{N} \;\middle|\; x^2 - xy + y^2 \equiv 0 \ (\mathrm{mod}\ p)\right\}\right| &= (p-1)\left(1 + \cancelto{1}{\left(\frac{-d(f)}{p}\right)}\right) \\
&= 2(p-1)
\end{aligned}
$$

if p is 1 modulo 6. Next we restrict our solutions to the Weyl chamber $\mathcal{C}$, which is equivalent to modding out by the Weyl group $\mathcal{W} = \mathcal{W}(A_2) \cong S_3$, the symmetric group on 3 letters, of cardinality $|S_3| = 3! = 6$, which not coincidentally the number if distinct Weyl chambers, because the Weyl group simply permutes the fundamental chamber a $|\mathcal{W}|$ number of times.

So we can summarize this by attaching an index of the root system to the counting function to remind us that we are only counting solutions in the fundamental Weyl chamber.

$$
N_{A_2}(p) := \left|\left\{(x.y) \in \mathbb{N}\times\mathbb{N} \;\middle|\; x^2 - xy + y^2 = p,\ 0 < x < 2y < 4x\right\}\right|.
$$

Proposition 5.2.6. *The number of nontrivial solutions of inside the fundamental Weyl chamber $\mathcal{C}$ of the root system A_2 (related to the eigenspectrum of the Laplace operator on the corresponding symmetric space), via the Diophantine equation induced by the Cartan quadratic form $C(A_2)$ set equal to a prime $p > 5$ of the form $p = 6k+1$ is*

$$
N_{A_2}(p) = 2k^2.
$$

Proof. We noted above that for such 1 mod 6 primes greater than 5, $N(p) = 2(p-1)$. Moreover, the Weyl group of A_n is isomorphic to the symmetric group S_{n+1} on $n+1$ letters (of order $(n+1)!$), and remembering we will get the symmetric solution to

$(x, y) \sim (y, x)$:

$$\mathcal{W}(A_2) \cong S_3 \quad \Longrightarrow \quad |S_3| = 3! = 6 = |\mathcal{W}(A_2)|.$$

Hence, for $p = 6k + 1 > 5$ we have now elementary calculation

$$N_{A_2}(p) = k\frac{N(p = 6k+1)}{|\mathcal{W}(A_2)|} = k\frac{2(p-1)}{|6|} = k\frac{2(6k+1-1)}{6} = k\frac{(6k)}{3} = 2k^2.$$

That concludes the proof of the proposition. □

The multiplicative nature of the solutions $N_{A_2}(p_1p_2) = N_{A_2}(p_1)N_{A_2}(p_2)$, shown by induction on powers of a prime factor $n = p^l, p = 6k+1 > 5, l > 1$ is $N_{A_2}(p^l) = l + 1$ from which it generalizes for products $n = \prod_{i=1}^{r} p_i^{l_i}$:

Corollary 5.2.7. (i) For all $l_i \geq 0$, the number of solutions for $n = \prod_{i=1}^{r} p_i^{l_i}$ is

$$N_{A_2}(n) = \prod_{i=1}^{r} N_{A_2}(p_i^{l_i}) = \prod_{i=1}^{r}(l_i + 1) = (l_1 + 1)(l_2 + 1)...(l_r + 1)$$

where again when $p_i = 6k + 1 > 5$ is prime.

(ii) The final case is when some n from case 2. is multiplied by a factor of a power of 3, i.e. $n' = 3^l n > 3$, $(l \geq 1)$ has the same number of solutions as n, that is, $N_{A_2}(n') = N_{A_2}(n)$.

(iii) Finally, if any such n discussed so far contains (or more precisely is multiplied by another prime number that is not $p \neq 6k + 1$ like 5, 11, 17, 23, 29, not $31 = 6 * 5 + 1$, etc.., then there is NO (nontrivial) solutions.

This corollary completes the necaessary number theory required to completely describe and characterize the spectrum of A_2, corresponding to the Laplacian operator Δ_M eigenspectrum on the corresponding symmetric space $M = SU(3)/SO(3)$. In this case, the half sum of the positive roots $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ is $\rho = \frac{1}{2}(2\alpha_1 + 2\alpha_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ has norm $||\rho||^2 = 2$, so the eigenvaues are given by dominant weights λ_β such that $n = \lambda_\beta + ||\rho||^2 = \lambda_\beta + 2 = ||\beta - \rho||^2 = C(A_2, n)$ (and $\lambda_\beta = ||\beta - \rho||^2 - ||\rho||^2$) has nonzero integral solutions and the eigenspectrum of $\frac{1}{2}\Delta_M$ for the compact RSS $M = SU(3)/SO(3)$.

Putting together all the considerations above in Corollary 5.2.7, the eigenspectrum of the (semi) Laplacian operator on the compact Riemannian symmetric space $M = SU(3)/SO(3)$ of rank 2, corresponding to the root system A_2, is *characterized* ***explicitly*** below:

Theorem 5.2.8. *The eigenspectrum of the (semi) Laplacian operator $\frac{1}{2}\Delta_M$ of $M = SU(3)/SO(3)$ as above, the compact dual RSS corresponding to the split real form of A_2 is:*

$$\begin{aligned} \mathrm{Spec}\left(\frac{1}{2}\Delta_{SU(3)/SO(3)}\right) &= \left\{\lambda_\beta = n - 2 \;\middle|\; n \text{ is any number of the type in Corollary 5.27 above}\right\} \\ &= \left\{ \begin{cases} \lambda_\beta = n - 2 \\ \lambda_\beta = n' - 2 \end{cases} \;\middle|\; \begin{cases} n = \prod_{i=1}^{r} p_i^{l_i},\ l_i \geq 0, \\ n' = 3^l n > 3,\ l \geq 1, \end{cases} \quad 5 < p_i = 6k + 1\text{-prime} \right\}. \end{aligned}$$

A few more elaborate but similarly computed examples are the remaining rank 2 systems, the algebra $B_2 = \mathfrak{so}(2(2) + 1, \mathbb{C}) \cong C_2$, $D_2 = \mathfrak{so}(2(2), \mathbb{C}) \cong A_1 \times A_1$, and the exceptional rank 2 Lie algebra with root system G_2.

5.3 Lie algebras with root system B_2

To continue the computations of Laplace eigenspectra of compact RSSs of rank 2 using the rank 2 root systems, we cite the root system data found in many locations including [38] Appendix C, pg. 510. For D_2. using the space with standard basis e_1 and e_2, $\mathbb{R}^2 = \text{span}_{\mathbb{R}}(e_1, e_2) = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, the root system B_2 has the set of of simple roots $\Sigma = \{\alpha_1, \alpha_2\}$ with $\alpha_1 = e_1 - e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\alpha_2 = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then as before the eigenvalues λ_β of $\frac{1}{2}\Delta_M$ are given by dominant weights in the fundamental Weyl chamber $\mathcal{C} = \mathcal{C}(D_2) \ni \beta = x\alpha_1 + y\alpha_2 = \begin{pmatrix} x \\ y - x \end{pmatrix}$ such that $\lambda_\beta = ||\beta - \rho||^2 - ||\rho||^2$, where the half sum of positive roots is $\rho = \frac{3}{2}e_1 + \frac{1}{2}e_2 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$. Hence, by the similar analysis and changing variables $\begin{cases} 2u = 2x - 3 \\ 2v = 2y - 4 \end{cases}$ to write

$$
\begin{aligned}
||\beta - \rho)||^2 = ||\frac{1}{2}(2\beta - 2\rho)||^2 = \frac{1}{4}\left|\left|\begin{pmatrix} 2x - 3 \\ 2y - 2x - 1 \end{pmatrix}\right|\right|^2 &= \frac{1}{4}\left([2x-3]^2 + [2y - 2x - 1]^2\right) \\
&= \frac{1}{4}\left([2x-3]^2 + ([2y-4] - [2x-3])^2\right) \\
&= \frac{1}{4}\left([2u]^2 + [2v - 2u]^2\right) = u^2 + (v-u)^2.
\end{aligned}
$$

Setting equal to some prime number $p > 3$ and diving out by a variable, say u, letting $z = v/u - 1$, and modding out by p,

$$z^2 + 1 \equiv 0 \pmod{p},$$

which has solutions for $p \equiv 1 \pmod{4}$.

The final eigenspectra for RSSs of Lie algebras with root system G_2 are computed in the same way, which completes the necessary work to compile the table for rank two spaces.

CHAPTER 6

SYMPLECTIC REDUCTION AND GEOMETRIC QUANTIZATIONS

First let us recall some basics of Marsden-Weinstein (or symplectic) reduction. Let (M, ω) be a symplectic manifold and suppose G acts on M leaving ω invariant. This action is called a Hamiltonian if there is a smooth equivariant map $\mu : M \to \mathfrak{g}^*$, s.t. for all $X \in \mathfrak{g}$,

$$d\mu(X) = i_X\omega,$$

and the function μ_X is defined by

$$\mu_X(p) = \langle \mu(p), X \rangle, \qquad X \in \mathfrak{g},\ p \in M.$$

The moment map is assumed to be equivariant with respect to the co-adjoint action Ad^* of $\mathfrak{g}^*$, that is, $\langle \mathrm{Ad}^*(g)\xi, Y \rangle = \langle \xi, \mathrm{Ad}(g^{-1})X \rangle$, for all $g \in G$, $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$; and the corresponding infinitesimal action ad^*, $\langle \mathrm{ad}^*(X)\xi, Y \rangle = \langle \xi, -[X, Y] \rangle$.

An alternative definition of the moment map can be given in terms of Hamiltonian vector fields and Poisson brackets. Refer to [31] for such a formulation.

Definition 6.0.1. A point $p \in M$ is called a ***regular point*** if the tangent map $d\mu_p : T_pM \to \mathfrak{g}^*$ is surjective. The set of regular points of μ is denoted by M_{reg}. We will write μ_{reg} for the restriction of μ to M_{reg}.

An element $\xi \in \mathfrak{g}^*$ is called a ***regular value*** of μ if all points in the inverse image $\mu^1(\xi)$ are regular. By the implicit function theorem, the subset $\mu^{-1}_{reg}(\xi) \subset M$ is a smooth submanifold, for all $\xi \in \mathfrak{g}^*$. Because the map μ is equivariant, the submanifold $\mu^{-1}_{reg}(\xi)$ is invariant under the action of the stabilizer group G_ξ.

Definition 6.0.2. Let (M, ω) be a connected symplectic manifold, equipped with a Hamiltonian action of G. Let $\xi \in \mathfrak{g}^*$ be a regular value of the moment map μ. When such a moment map μ exists, the action is called Hamiltonian and the space $N = N_\xi := \mu^{-1}(\xi)/G_\xi$ is called the ***Marsden-Weinstein (symplectic) reduction***, and ξ is a fixed element of the adjoint action of G on $\mathfrak{g}^*$. N will be denoted by $M//G$, and inherits a natural symplectic form ω_{red}

such that $i^*\omega = \pi^*\omega_{red}$, where $i : \mu^{-1}(\xi) \hookrightarrow M$ and $\pi : \mu^{-1}(\xi) \to N_\xi = \mu^{-1}(\xi)/G_\xi$ are the natural inclusions and projections.

Suppose that the stabilizer G_ξ of ξ acts properly and freely on the submanifold $\mu^{-1}(\xi)$ of M, i.e. assume the G-action is transitive and the moment map μ is proper, so that the orbit space (or reduced space) $M_\xi = \mu^{-1}(\xi)/G_\xi$ is a smooth manifold, and $M_\xi = \mu^{-1}(\xi)/G_\xi = \mu^{-1}(G\cdot\xi)/G$, where $G\cdot\xi$ is the co-adjoint orbit through ξ. Then there is a unique symplectic form ω_ξ on M_ξ such that $\pi^*\omega_\xi = i^*\omega$.

Remark 6.0.3. If a Lie group G of isometries acts on a manifold M, this action induces the Hamiltonian action on T^*M and the reduced space $T^*M//G$ is a reduced Hamiltonian system. *Whenever $T^*M//G = N$ for some Riemannian manifold N, then the solutions of the new system comprise of the geodesic flow on N.*

6.1 The geodesic flow of symmetric spaces of rank one

We first recall Marsden-Weinstein (or symplectic) reduction and then the modified Konstant - Souriau geometric quantization scheme (twisted half of the canonical bundle - see Czyz [15] and Hess [35]). In [39] the authors related the energy spectrum of the quantized geodesic flow on a sphere with the eigenvalues of the Laplace-Beltrami operator.

In what follows we first notice a generalization of this result from a representation theory viewpoint to all compact rank-one Riemannian symmetric spaces (CROSSes). Then we provide a similar detailed computations for two examples - $\mathbb{C}P^n$ and $\mathbb{H}P^n$. The exposition follows [25] and is used later to determine a generating set of eigenfunctions defined by harmonic polynomials in an ambient space.

For more details on the Marsden-Weinstein (or symplectic) reduction we refer the reader, for example, to [1], here we provide a partial review of the results we need later. If (M,ω) is a symplectic manifold and H is a function on M, then the vector field X_H defined as $dH(Y) = \omega(X_H, Y)$ is called Hamiltonian vector field. We will call H a Hamiltonian function and the triple (M,ω,H) - a Hamiltonian system. If G is a group of symplectomorphisms then under mild conditions there is a map $\mu : M \to \mathfrak{g}^*$, defined by

$$d\mu(X) = i_X\omega,$$

where $\mathfrak{g}$ is the Lie algebra of G and $X \in \mathfrak{g}$ is identified with the induced vector field on M. When such μ exists, the action is called *Hamiltonian* and the space $N = \mu^{-1}(c)/G$ is called the Marsden-Weinstein reduction or the symplectic reduction, where c is a fixed element of the adjoint action of G on $\mathfrak{g}^*$. We denote N by $M//G$. The space $M//G$ inherits a natural symplectic form ω_{red} such that $i^*(\omega) = \pi^*(\omega_{red})$, where $i : \mu^{-1}(c) \to M$ is the inclusion and $\pi : \mu^{-1}(c) \to N = \mu^{-1}(c)/G$ is the natural projection. The following results will be used repeatedly in the paper (c.f. [1] for the proof).

Proposition 6.1.1. *If (N,ω_{red}) is the symplectic reduction of (M,ω) under the action of a Lie group G and H is a G-invariant function on M, then there is a unique function H_{red} on N such that $\pi^*(H_{red}) = i^*(H)$. Moreover, the flow of the vector field X_H preserves $\mu^{-1}(c)$ and projects on N to the flow of the vector fields $X_{H_{red}}$. Moreover, if we have a second Hamiltonian action of a Lie group G_1 on M which commutes with the action of G, then the level sets of its moment map μ_1 are $G-$invariant and $\mu_1|_{\mu^{-1}(c)} = \pi^*(\overline{\mu_1})$ where $\overline{\mu_1}$ is the moment map associated to the action of G_1 on N.*

Proposition 6.1.2. *If G is a compact group of isometries acting freely on the Riemannian manifold (M,g) and $N = M/G$ is the orbit space, then for the canonical symplectic forms Ω_M, Ω_N on T^*M, T^*N, respectively, we have $T^*N = T^*M//G$ with Ω_N being the reduced form from Ω_M.*

The geodesic flow on a Riemannian manifold is represented as a Hamiltonian flow on its cotangent bundle. The cotangent bundle of each Riemannian manifold (M,g) has a canonical symplectic form given in local coordinates as $\Omega = \sum dx_i \wedge dy_i$ where $(x_1,...,x_n)$ are local coordinates of M and $(x_1,...,x_n,y_1,...,y_n)$ are the associated local coordinates of T^*M.

Then the function $H(x,v) = \frac{1}{2}g(v,v)$ for $x \in M$ and $v \in T_x^*M$ has a Hamiltonian vector field X_H and its flow lines project on M to give the geodesics. In particular, $i_{X_H}\Omega = dH$. If all the geodesics of M are closed, then they define an S^1-action on T^*M with orbits $(c(t), g(c'(t))$ for a geodesic $c(t)$ and the dual 1-form $g(c'(t))$ of its tangent vector $c'(t)$. This means that the moment map $\mu(x,v)$ at $(x,v) \in T^*M, v \in T_x^*M$ for this S^1-action is precisely $\mu = H$. When we fix the level c of the moment map, the points in the reduced space $H^{-1}(c)/S^1$ represent (oriented) geodesics on M with tangent vectors of length c. This shows that $\mathrm{Geod}(M) = T^*M//S^1$ as sets where $\mathrm{Geod}(M)$ is the set of oriented geodesics on M. We note that the reduced form Ω_c from the canonical form on T^*M depends on the choice of the level set $\mu^{-1}(c)$ for the moment map of the action μ (which is called the *energy of the geodesic flow*).

Now recall some facts about the quantization scheme of Konstant and Souriau with the amends of Czyz and Hess [15, 35]. Let X be a compact Kähler manifold with Kähler form λ. We say that the holomorphic line bundle L is a *quantum line bundle* if its first Chern class satisfies

$$c_1(L) = \frac{1}{2\pi}[\lambda] - \frac{1}{2}c_1(X).$$

Thus, X will be quantizable if and only if $\frac{1}{2\pi}[\lambda] - \frac{1}{2}c_1(X) \in H^2(X,\mathbb{Z})$. The corresponding quantum Hilbert space is the (finite dimensional) linear space $H^0(X, \mathcal{O}(L))$. We want to apply the scheme to the space of geodesics of a Riemannian manifold all of whose geodesics are closed.

Main examples of such manifolds are the CROSSes. Recall that for a compact irreducible Riemannian symmetric space G/K with simple Lie group G we associated a quantization space G/L, which covers the space parametrizing all maximal totally geodesic flat tori in G/K. The space G/L is a generalized flag manifold, so smooth projective variety and from the description of its second cohomology we know that its Picard group contains the center of $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{m}$. In particular it contains the fundamental weights $w_1, ..., w_k$ which correspond to the restricted roots for $\mathfrak{a}$. Now denote by $\mathcal{L} = \mathcal{L}_{i_1,...,i_k}$ the holomorphic line bundle on

G/L determined by $w = i_1 w_1 + ... + i_k w_k$, where $i_j \geq 0$. By Bott vanishing the higher cohomology of $\mathcal{L}$ are zero. The space $H^0(G/L, \mathcal{O}(\mathcal{L}))$ is a (unitary) representation of G with highest weight w. The Borel-Weil theorem shows that the representation is irreducible if w is dominant, and corresponds to the (unique) irreducible representation with highest weight w [42].

On the other side, the general theory for the Laplace spectrum on symmetric spaces ([11, 43]) tells us that the eigenvalues are given by $\lambda = ||\rho((\mathfrak{a}^{\mathbb{C}})^*) + w||^2 - ||\rho((\mathfrak{a}^{\mathbb{C}})^*)||^2$ where w is as before and $\rho((\mathfrak{a}^{\mathbb{C}})^*)$ is the half sum of positive restricted roots of $\mathfrak{a}^{\mathbb{C}}$. When the center of $\mathfrak{l}$ is $\mathfrak{a}$, ρ represents one half of the first Chern class of G/L, so $\rho((\mathfrak{a}^{\mathbb{C}})^*) + w$ is the first Chern class of $\mathcal{L} \otimes K^{\frac{1}{2}}$.

In this Section we focus on the case of Riemannian symmetric spaces of rank one, since the correspondence in this case is most studied and related to the classical quantization of the geodesic flow. In the next sections we'll generalize the scheme to the symmetric spaces of higher rank. When the rank of M is one, the space of the restricted roots Σ is 1-dimensional as is the Weyl chamber in it. The set of fundamental weights in it is (see [32]):

$$\Lambda^+ = \left\{ \lambda \in \mathfrak{a}^{\mathbb{C}} \;\middle|\; \frac{\langle \lambda, \psi \rangle}{\langle \psi, \psi \rangle} \in \mathbb{Z}^+, \text{ for all } \psi \in \Sigma \right\}$$

and in the rank one case is generated by a single element θ. The considerations above give the following result, which we will generalize in the next section.

Theorem 6.1.3. *let $M = G/K$ be an irreducible simply-connected compact Riemannian symmetric space of rank one (CROSS). Then up to re-scaling of the metric on M the following are true:*

i) Under the transgression the reduced symplectic form Ω_c on Geod(M) $= G/L = T^*M//S^1$ corresponds to $\pi\sqrt{2c}\theta$ and with the choice of the positive Weyl chamber and complex structure as above, $c_1(G/L)$ corresponds to $N_M\theta$ for a positive integer N_M.

ii) *The quantum condition on $(Geod(M), \Omega_c)$ (i.e. $[\Omega_c] \in H^2(G/L,))$ provides the following energy spectrum: $c_k = 1/2(N_M + 2k)^2$.*

iii) *The spectrum of the (semi)-Laplacian $\frac{1}{2}\Delta_M$ on M is given by $\lambda_k = ||k\theta + \rho(\mathfrak{a}^{\mathbb{C}})||^2 - ||\rho(\mathfrak{a}^{\mathbb{C}})||^2$ and $c_k = ||k\theta + \rho(\mathfrak{a}^{\mathbb{C}})||^2$ where $\rho(\mathfrak{a}^{\mathbb{C}})$ is the half-sum of the positive restricted roots of $\mathfrak{a}^{\mathbb{C}}$.*

iv) *The multiplicities of c_k and λ_k coincide with the dimension of the (finite - dimensional) representation $L(k\theta)$ of $\mathfrak{g}$ with highest weight $k\theta$ relative to $(\mathfrak{h}, \Delta)$. Moreover the representation $L(k\theta)$ is isomorphic to both the (complex) eigenspace $\mathcal{L}^2(M)^{\lambda_k}$ of Δ_M corresponding to λ_k and the quantization space $H^0(Geod(M), \mathcal{O}(\mathcal{L}_k))$.*

Proof. The spaces in the Theorem are classified and are $S^n, \mathbb{C}P^n, \mathbb{H}P^n, CaP^2$. In the two examples below we give a proof in case of n and n. The case of S^n is considered in [39]. In [39] the space of oriented geodesics of $M = S^n$ is explicitly identified with the complex quadric in n via the Marsden-Weinstein reduction. It was noted that the energy levels of the moment map that satisfy a quantization condition coincide, up to an additive constant, with the eigenvalues of the Laplace-Beltrami operator and the their multiplicity are the same as the (complex) dimension of the holomorphic sections of the corresponding quantum bundle $L(k\theta)$ (see also [41] for related results).

Finally, consider the case $CaP^2 = F_4/Spin(9)$. Since M has rank one, then the reduction identifies the level set of the moment map with a spheric bundle over M. From [18] Proposition 3.3 follows that it is diffeomorphic to $F_4/Spin(7)$. This gives an identification of the quantization space with $F_4/Spin(7) \times S^1$. Its painted Dynkin diagram from [5], Table 4 and Corollary 4.6.2 follows that $H^2(F_4/Spin(7) \times S^1, \mathbb{Z}) = \mathbb{Z}$, so the reduced form Ω_c is proportional to the generator. Since the generator corresponds to θ under the transgression, and the proportionality constant depends on c, and (i) follows. Then (ii) follows by the quantization condition, (iii) and (iv) by combining the results from [11] and Borel-Weil Theorem. $\square$

We note that the simply-connected requirement could be lifted and similar statement could be stated for $\mathbb{R}P^n$.

For the remainder of the paper, we use the notation $(\cdot,\cdot)$ for the standard $SO(n)$-invariant bilinear form on $\mathbb{C}^n$, i.e., $(a,b) = \sum_{i=1}^n a_i b_i$ for $a, b \in \mathbb{C}^n$.

6.2 The sphere S^n

We will quickly outline the scheme for the n-sphere, with reference to [39, 41]

$$S^n = \Big\{x \in \mathbb{R}^{n+1} \;\Big|\; (x,x) = \sum x_k^2 = 1\Big\}.$$

Using the round metric we identify the tangent and cotangent bundles with the set

$$T^*S^n = \Big\{(x,y) \in \mathbb{R}^{2n+2} \;\Big|\; (x,x) = 1,\ (x,y) = 0\Big\}.$$

The canonical symplectic form on T^*S^n is given by the restriction of $\sum dx_i \wedge dy_i$. Moreover we can define a compatible (also called an adapted) complex structure via

$$z_i = y_i + \sqrt{-1}|y|x_i$$

so that the canonical symplectic form becomes Kähler. The set $T^*S^n - \mathbf{0}$ which is the set of all non-zero cotangent vectors is the complex quadric

$$\Big\{z \in \mathbb{C}^n \;\Big|\; (z,z) = 0,\ z \neq 0\Big\}.$$

The geodesic flow is defined as the flow of the Hamiltonian vector field X_H for the function $H(x,y) = \frac{1}{2}||y||^2$. Since all geodesics in S^n are closed, they define a S^1-action on T^*S^n with orbits - the orbits of X_H. In particular H is also a moment map for this action and the energy level sets $H^{-1}(c)$ are S^1-invariant. The quotient $H^{-1}(c)/S^1$, which is called symplectic reduction, can be identified with the projective quadric

$$Q = \Big\{[z] \in \mathbb{C}P^n \;\Big|\; (z,z) = 0\Big\}.$$

Eigenvalues of the Laplace-Beltrami operator are known to be $\lambda_k = k(n+k-1)$. Also the corresponding (complex valued) eigenfunctions are restrictions of harmonic homogeneous polynomials of degree k in $\mathbb{R}^{n+1}$. A simple way to describe them is as the span of the set $p_a(x) = (a,x)^k$ where $a \in \mathbb{C}^n$ with $(a,a) = 0$.

The reduction construction provides a Kähler form Ω_c. Then the following is true:

i) The form Ω_c defines a first Chern class of a holomorphic line bundle L_k with $c_1(L_k) = c[\Omega] - \frac{1}{2}c_1(Q)$ over Q when

$$c = c_k = \frac{1}{2}\left(k + \frac{n-1}{2}\right)^2 = \lambda_k + \frac{1}{2}\left(\frac{n-1}{2}\right)^2.$$

ii) The space of holomorphic sections of L_k is identified as the span of functions $P_a(z) = (a,z)^k$ on $\mathbb{C}^{n+1}$.

Our goal is to generalize this picture to other manifolds - the remaining compact Riemannian symmetric spaces (CROSSes).

Remark 6.2.1. The re-scaling factor mentioned in the Theorem could be different for the different spaces. It is known that the eigenvalues of the Laplace-Beltrami operator for S^n and n are $k(n+k-1)$ and $4k(n+k)$ in the round metric on S^n and the Fubini-Studi metric on n respectively. So the two metrics are rescaled differently, one by a factor 4 times the other.

We continue with the explicit calculations of the two classical simply-connected projective spaces.

6.3 Complex projective space

We'll use and extend here the results of [16]. Throughout this subsection for complex vectors $z, w \in \mathbb{C}^n$ we denote by $\langle z, w\rangle = \mathrm{Re} \sum z_i \overline{w}_i$ their hermitian scalar product and by $(z,w) = \sum z_k w_k$ the complex scalar product so $||z|| = \sqrt{\langle z, z\rangle}$. For a point $[z] = [z_0, z_1, ..., z_n]$ in the complex projective space $\mathbb{C}P^n$, we identify the holomorphic cotangent space

$$T^*_{[u]}\mathbb{C}P^n \cong \Big\{([u], v) \in \{[u]\} \times \mathbb{C}^{n+1} \;\Big|\; (u, \overline{v}) = 0\Big\}$$

where we used the Fubini-Study metric to identify the tangent and cotangent bundles. To achieve a global description of the cotangent bundle, we use the Hopf map $\pi : S^{2n+1} \to \mathbb{C}P^n$ which is induced by the standard action of S^1 on S^{2n+1}. This map is defined by $u \mapsto [u]$, where $u \in \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ with $||u|| = 1$. After identifying the tangent and cotangent bundles of the sphere via the canonical metric, we can identify the cotangent bundle as

$$T^*S^{2n+1} = \Big\{(u, v) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \;\Big|\; ||u|| = 1,\ \langle u, v\rangle = 0\Big\}$$

Then the S^1-action ρ for the Hopf projection π extends to T^*S^{2n+1} as

$$\rho(e^{i\theta})(u, v) = (e^{i\theta}u, e^{i\theta}v)$$

. This action preserves the canonical symplectic form on T^*S^{2n+1}, which is given by $i^*\mathrm{Re}\,(du \wedge d\overline{v})$. The moment map for the action ρ can be used to show the following theorem. This theorem is first proven in [16], but for reader's convenience a short proof is presented. We consider the cotangent bundle with its zero section deleted $T^*_0\mathbb{C}P^n$ (and $T^*_0S^{2n+1}$) in order to avoid the singularity issues since they are irrelevant in the paper.

Lemma 6.3.1. *The space $T^*_0\mathbb{C}P^n$ is diffeomorphic to both X_C and $\widetilde{X_C}$ where*

$$X_C \cong \Big\{[u, v] \;\Big|\; ||u|| = 1,\ (u, \overline{v}) = 0, v \neq 0\Big\}$$

with $[u, v]$ representing the class of (u, v) under $(u, v) \sim (e^{i\theta}u, e^{i\theta}v)$ and

$$\widetilde{X_C} \cong \Big\{[[u, v]] \;\Big|\; \langle u, u\rangle = \langle v, v\rangle \neq 0,\ (u, v) = 0\Big\}$$

with $[[u, v]]$ defined by the relation $(u, v) \sim (e^{i\theta}u, e^{-i\theta}v)$. Moreover $T^*_0\mathbb{C}P^n$ is biholomorphic to $\widetilde{X_C}$ when it is identified with $T^*_0S^{2n+1}//S^1$ and the reduced complex structure.

Proof. It is well-known that under the action ρ, $T^*\mathbb{C}P^n = T^*S^{2n+1}//S^1$. The moment map Φ associated to the action ρ is simply $\Phi(u,v) = \mathrm{Im}\langle u,v\rangle$. Hence, $T_0^*S^{2n+1}//S^1 = \Phi^{-1}(\mu)/S^1$, for a generic $\mu \in \mathbb{R} = iu(1)$, is identified with X_C which gives the diffeomorphism $T_0^*\mathbb{C}P^n \cong X_C$. The diffeomorphism between X_C and $\tilde{X_C}$ is given by the formulas:

$$\tilde{u}_k = \frac{1}{\sqrt{2}}(||v||u_k + iv_k),$$

$$\tilde{v}_k = \frac{1}{\sqrt{2}}(\overline{v}_k - i||v||\overline{u}_k).$$

The biholomorphism follows from the fact that the reduction is Kähler, when we consider the canonical form on T^*S^{2n+1} as a Kähler form for the complex structure induced from the embedding in $\mathbb{C}^{2n+2}$ as in [41] for example. $\square$

In the particular case of T^*S^{2n+1} we obtain the following.

Proposition 6.3.2. *The canonical symplectic form Ω_C on $T^*\mathbb{C}P^n \cong X_C$ is*

$$\Omega_C = \frac{1}{2}(du \wedge d\overline{v} + d\overline{u} \wedge dv)$$

and the Hamiltonian system $\mathcal{H}_{\mathbb{C}P^n} = (X_c, \Omega_C, H_C = \frac{||v||^2}{2})$ induces the geodesic flow on $\mathbb{C}P^n$. The system is equivalent to $(\widetilde{X}_C, \widetilde{\Omega}_C, \widetilde{H}_C)$ in view of the diffeomorphism in Lemma 6.3.1.

Since the orbits of $\mathcal{H}_{\mathbb{C}P^n}$ correspond precisely to the geodesics of $\mathbb{C}P^n$, we first identify the space parametrizing the geodesics. For this we first consider the geodesic flow on the sphere S^{2n+1}. Since all of the geodesics on the sphere are closed, the flow of X_H in the cotangent space has also only closed trajectories. They define an S^1-action which is given by $(u,v) \mapsto (e^{i\theta}u, e^{-i\theta}v)$. This action commutes with the action inducing the Hopf projection and is Hamiltonian. So it defines an action on $T^*\mathbb{C}P^n$ which has orbits - the flow lines of the Hamiltonian vector field defining the geodesics on $\mathbb{C}P^n$. We can identify a geodesic $c(t)$ in $\mathbb{C}P^n$ with the line $(c(t), c'(t))$ in $T\mathbb{C}P^n \cong T^*\mathbb{C}P^n$ when t is a parameter such that

c' has constant norm. From here we see that the space parametrizing the geodesics can be identified with the Marsden-Weisntein quotient. Let $N_c = \widetilde{H}_C^{-1}(c)/S^1$ be the reduced space. To identify N_c with a flag manifold, we use the Hamiltonian system $(\widetilde{X}_C, \widetilde{\Omega}_C, \widetilde{H_C})$. Let

$$\begin{aligned}\mathbb{F} &= \Big\{([z],[w]) \in \mathbb{C}P^n \times \mathbb{C}P^n \;\Big|\; (z,w) = 0\Big\} \\ &= \Big\{([z],[w]) \in \mathbb{C}P^n \times \mathbb{C}P^n \;\Big|\; \langle z,z\rangle = \langle w,w\rangle,\ (z,w) = 0\Big\}.\end{aligned}$$

One can see that $\mathbb{F}$ is biholomorphic to the (1,2)-flag in $\mathbb{C}^{n+1}$ with homogeneous representation $\mathbb{F} = U(n+1)/U(1)\times U(1)\times U(n-1)$. Denote by p_1 and p_2 the two projections on the corresponding factors of $\mathbb{C}P^n \times \mathbb{C}P^n$. Let α be the generator (the Fubini-Study form) of $H^2(\mathbb{C}P^n, \mathbb{Z})$. Then $\omega_1 = p_1^*\alpha$ and $\omega_2 = p_2^*\alpha$ are generators of $H^2(\mathbb{F}, \mathbb{Z})$. With this notation we have the following.

Proposition 6.3.3. *If $c \neq 0$ then the reduced manifold N_c is biholomorphic to the flag $\mathbb{F}$ and the reduced Kähler form is $\widetilde{\omega}_c = \pi\sqrt{2c}(\omega_1 + \omega_2)$.*

Proof. The S^1-action of the geodesic flow on $T^*\mathbb{C}P^n$ is induced from the one on T^*S^{2n+1}. Hence, this action

$$\lambda(z,w) = (\lambda z, \lambda w),$$

for $(z,w) \in \widetilde{H}_C^{-1}(c)$. For the sphere S_R^{2n+1} of radius R the Hopf projection fits in the diagram $\mathbb{C}^{n+1} \xleftarrow{i} S_R^{2n+1} \xrightarrow{h} \mathbb{C}P^n$ with $h^*\alpha = \frac{1}{\pi R^2} i^*\Omega$ (see [39]). If $\tilde{\pi}_c$ is the projection $\widetilde{H}_C^{-1}(c) \to N_c = \mathbb{F}$ then we have the following commutative diagram:

$$\begin{tikzcd} \widetilde{H}_C^{-1}(c) \arrow[r, "\tilde{\pi}_c"] \arrow[d] & N_c \arrow[d, "\tilde{i}_c"] \\ S^{2n+1} \times S^{2n+1} \arrow[r, "h\times h"] & \mathbb{C}P^n \times \mathbb{C}P^n \end{tikzcd}$$

where the vertical arrows correspond to the natural embeddings. Therefore,

$$\begin{aligned}\tilde{\pi}_c^*(\sqrt{2c}\pi(\omega_1+\omega_2)) &= \pi\sqrt{2c}\tfrac{\sqrt{-1}}{2\pi}\left(\tfrac{dz\wedge d\overline{z}}{||z||^2} + \tfrac{dw\wedge d\overline{w}}{||w||^2}\right) \\ &= \tfrac{1}{\sqrt{2c}}\tfrac{\sqrt{-1}}{2}(dz\wedge d\overline{z} + dw\wedge d\overline{w}) \\ &= \tfrac{1}{2}(du\wedge d\overline{v} + d\overline{u}\wedge dv) \\ &= \tilde{i}_c^*(\tilde{\Omega}_c).\end{aligned}$$

$\square$

In the above calculation we used that $\widetilde{H}_C(z,w) = c$, so $||z||^2 = ||w||^2 = 2c$. We want to use the modified Kostant - Souriau scheme to "quantize" the geodesic flow of $\mathbb{C}P^n$.

Proposition 6.3.4. *We have* $c_1(\mathbb{F}) = n(\omega_1 + \omega_2)$.

Proof. We apply the adjunction formula for a hypersurface of degree (1,1) in $\mathbb{C}P^n \times \mathbb{C}P^n$ to obtain

$$\begin{aligned} c_1(\mathbb{F}) &= -(c_1(K_{\mathbb{C}P^n \times \mathbb{C}P^n}|_{\mathbb{F}}) + c_1([\mathbb{F}]|_{\mathbb{F}})) \\ &= c_1(\mathbb{C}P^n \times \mathbb{C}P^n)|_{\mathbb{F}} - c_1([\mathbb{F}]|_{\mathbb{F}}) \\ &= (n+1)(\omega_1 + \omega_2) - (\omega_1 + \omega_2) \\ &= n(\omega_1 + \omega_2). \end{aligned}$$

$\square$

Theorem 6.3.5. *The energy spectrum of the geodesic flow on* $\mathbb{C}P^n$ *is*

$$E_k = \frac{1}{2}(n+2k)^2, k \in \mathbb{N},$$

with corresponding multiplicities $m_k = \binom{n+k}{k}^2 - \binom{n+k-1}{k-1}^2$.

Proof. For the exact cohomology sequence:

$$H^1(\mathbb{F}, \mathcal{O}) \to H^1(\mathbb{F}, \mathcal{O}^*) \to H^2(\mathbb{F}, \mathbb{Z}) \to H^2(\mathbb{F}, \mathcal{O})$$

and the identities $H^(\mathbb{F}, \mathcal{O}) = H^2(\mathbb{F}, \mathcal{O}) = 0$ follows that:

$$c_1 : H^1(\mathbb{F}, \mathcal{O}^*) \xrightarrow{\cong} H^2(\mathbb{F}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore, every holomorphic line bundle L on $\mathbb{F}$ is equivalent to $L_{k_1,k_2} = k_1\pi_1^*(H) + k_2\pi_2^*(H)$, where H is the hyperplane section on $\mathbb{C}P^n$.

The quantum condition on c is

$$\frac{1}{2\pi}[\omega_c] - \frac{1}{2}c_1(\mathbb{F}) = c_1(L_{k_1,k_2}),$$

which implies

$$\frac{\sqrt{2c}}{2} - \frac{n}{2} = k,$$

where $k = k_1 = k_2$ is a positive integer. In particular

$$c = \frac{1}{2}(2k+n)^2.$$

To count the multiplicities (i.e. $\dim H^0(\mathbb{F}, \mathcal{O}(L))$ we consider the exact sequence of sheaves:

$$0 \to \mathcal{O}_{\mathbb{C}P^n \times \mathbb{C}P^n}(L_{k-1,k-1} \otimes L_{1,1}) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}P^n \times \mathbb{C}P^n}(L_{k,k}) \xrightarrow{r} \mathcal{O}|_{\mathbb{F}}(L_{k,k}) \to 0, \tag{6.3.1}$$

where α is the multiplication of sections of $L_{k,k}$ by the polynomial $\sum_0^n z_i w_i$ which defines $\mathbb{F}$ in $\mathbb{C}P^n \times \mathbb{C}P^n$ and r is the restriction. The corresponding exact cohomology sequence gives:

$$0 \to H^0(\mathbb{C}P^n \times \mathbb{C}P^n, \mathcal{O}(L_{k-1,k-1})) \to H^0(\mathbb{C}P^n \times \mathbb{C}P^n, \mathcal{O}(L_{k,k}))$$

$$\to H^0(\mathbb{F}, \mathcal{O}(L_{k,k})) \to H^1(\mathbb{C}P^n \times \mathbb{C}P^n, \mathcal{O}(L_{k-1,k-1})) = 0$$

where the last term is zero by the Kodaira vanishing theorem. Thus, we have

$$\begin{aligned}
m_k &= \dim(H^0(\mathbb{F}, \mathcal{O}(L_{k,k})) \\
&= \dim(H^0(\mathbb{C}P^n \times \mathbb{C}P^n, \mathcal{O}(L_{k,k})) - \dim(H^0(\mathbb{C}P^n \times \mathbb{C}P^n, \mathcal{O}(L_{k-1,k-1}))) \\
&= \tbinom{n+k}{k}^2 - \tbinom{n+k-1}{k-1}^2.
\end{aligned}$$

□

6.4 Quaternionic projective space

We first note that the results in this subsection were independently obtained in [36] and some of them appear in [16]. For readers convenience, in this section we use a slightly different notations to distinguish between real complex and quaternionic scalar products.

In particular, we use $\langle x,y\rangle_{\mathbb{R}}$, $\langle x,y\rangle_{\mathbb{C}}$, and $\langle x,y\rangle_{\mathbb{H}}$ for $\overline{x}.y=\sum\overline{x_i}y_i$, when x_i,y_i are in $\mathbb{R},\mathbb{C}$, and $\mathbb{H}$ respectively. The corresponding norms arising from their real parts are denoted by $||.||_{\mathbb{R}},||.||_{\mathbb{C}},||.||_{\mathbb{H}}$ respectively. The geodesic flow on $\mathbb{H}P^n$ can be described in a similar way as the one for $\mathbb{C}P^n$ but with the aid of the quaternionic Hopf map. For that we use three equivalent representations of T^*S^{4n+3}:

$$\begin{aligned}T^*S^{4n+3} &= \Big\{(x,y)\in\mathbb{R}^{4n+3}\times\mathbb{R}^{4n+3}\ \big|\ ||x||_{\mathbb{R}}=1,\ \langle x,y\rangle_{\mathbb{R}}=0\Big\}\\ &= \Big\{(u,v)\in\mathbb{C}^{2n+2}\times\mathbb{C}^{2n+2}\ \big|\ ||u||_{\mathbb{C}}=1,\ \mathrm{Re}\langle u,v\rangle_{\mathbb{C}}=0\Big\}\\ &= \Big\{(p,q)\in\mathbb{H}^{n+1}\times\mathbb{H}^{n+1}\ \big|\ ||p||_{\mathbb{H}}=1,\ \langle p,q\rangle_{\mathbb{R}}=0\Big\}\end{aligned}$$

where $p_k:=u_{2k}+u_{2k+1}j, q_k:=v_{2k}+v_{2k+1}j$ and $\langle p,q\rangle_{\mathbb{R}}=\mathrm{Re}\langle p,q\rangle_{\mathbb{H}}=\mathrm{Re}\sum\overline{p}_kq_k$. The quaternionic Hopf map in this case is $\chi:S^{4n+3}\to\mathbb{H}P^n,\ \ p\mapsto[p]$ where $[p]=[p_0,p_1,...,p_n]$ is the class of p for the relation $p\sim\sigma p,\sigma\in Sp(1)$. The next lemma is again from [16].

Lemma 6.4.1. *The cotangent space $T^*\mathbb{H}P^n$ is diffeomorphic to both X_H and $\widetilde{X}_H$ defined as follows:*

$$X_H:=\Big\{\lfloor p,q\rfloor\in\mathbb{H}^{n+1}\times\mathbb{H}^{n+1}\ \big|\ ||p||_{\mathbb{H}}=1,\ \langle p,q\rangle_{\mathbb{H}}=0\Big\},$$
$$\widetilde{X}_H:=\Big\{\lfloor z,w\rfloor\in\mathbb{C}^{2n+2}\times\mathbb{C}^{2n+2}\ \big|\ ||z||_{\mathbb{C}}=||w||_{\mathbb{C}},\langle z,w\rangle_{\mathbb{C}}=0,\ I(z,w)=0\Big\}$$

where $I(z,w)=z_0w_1-z_1w_0+...+z_{2n}w_{2n+1}-z_{2n+1}w_{2n}$ and $\lfloor p,q\rfloor$ and $\lfloor z,w\rfloor$ denote the equivalence classes of (p,q) and (z,w) under $(p,q)\sim(\sigma p,\sigma q)$ and $(z,w)\sim(z,w)g$ for $\sigma\in Sp(1)$ and $g\in SU(2)\cong Sp(1)$.

Proof. Consider the action of $SU(2)$ on S^{4n+3} defined by

$$\Psi_g(p,q):=(p,q)g,g\in SU(2). \tag{6.4.1}$$

This action has a moment map $G:T^*S^{4n+3}\to\mathfrak{su}(2)^*$, given by the formulas

$$G(p,q)=(A(p,q),B(p,q),C(p,q)),$$

where

$$\langle p,q\rangle_{=}\mathrm{Re}(\langle p,q\rangle_{\mathbb{H}})+A(p,q)i+B(p,q)j+C(p,q)k,$$

and the imaginary quaternions are identified with $\mathfrak{su}(2)^*$. Hence, $T^*S^{4n+3}//SU(2) = X_H \cong T^*\mathbb{H}P^n$.

To prove that X_H and $\widetilde{X}_H$ are diffeomorphic, consider the map $t_H : X_H \to \tilde{X}_H, (z,w) = t_H(p,q)$, where

$$\begin{aligned} z_{2k} &:= \frac{1}{\sqrt{2}}(||v||_{\mathbb{C}} u_{2k} + \sqrt{-1} v_{2k}), \\ z_{2k+1} &:= \frac{1}{\sqrt{2}}(-||v||_{\mathbb{C}} \overline{u}_{2k+1} - \sqrt{-1}\overline{v}_{2k+1}), \\ w_{2k} &:= \frac{1}{\sqrt{2}}(v_{2k+1} - \sqrt{-1}||v||_{\mathbb{C}} u_{2k+1}), \\ w_{2k+1} &:= \frac{1}{\sqrt{2}}(\overline{v}_{2k+1} - \sqrt{-1}||v||_{\mathbb{C}} \overline{u}_{2k+1}). \end{aligned}$$

□

The action Ψ defined in (6.4.1) commutes with the geodesic flow of S^{4n+3}. Recall the diffeomorphism $t_H : X_H \to \widetilde{X}_H$ defined at the end of the last proof. Like in the previous subsection, we have the following.

Proposition 6.4.2. *Let $\Omega_H = \Omega_{T^*\mathbb{H}P^n}$ be the canonical symplectic form on $T^*\mathbb{H}P^n$. Then*

$$\Omega_H = \frac{1}{2}(du \wedge d\overline{v} + d\overline{u} \wedge dv).$$

Moreover, the geodesic flow of $\mathbb{H}P^n$ is the flow of the equivalent Hamiltonian systems

$$(X_H, \Omega_H, G_H) \cong (\widetilde{X}_H, \widetilde{\Omega}_H, \widetilde{G}_H),$$

where $G_H = \dfrac{||q||^2_{\mathbb{H}}}{2} = \dfrac{||v||^2_{\mathbb{C}}}{2}$, $\widetilde{\Omega}_H = t_H^*\Omega_H$ and $\widetilde{G}_H = t_H^*(G_H)$.

Next we compute the energy spectrum of the geodesic flow on $\mathbb{H}P^n$ in a similar way as in the case of $\mathbb{C}P^n$. We consider again the reduced space $\mathbb{O}_c = T^*\mathbb{H}P^n//S^1 = \tilde{G}^{-1}(c)/S^1$ with the induced symplectic form ω_c obtained from $\tilde{i}_c^*\Omega_H = \tilde{\pi}_c^*\omega_c$, where $\tilde{i}_c : \widetilde{G}^{-1}(c) \to T^*\mathbb{H}P^n$ and $\tilde{\pi}_c : \widetilde{G}^{-1}(c) \to \mathbb{O}_c$. Denote by $\mathbb{F}_{is}$ the isotropic Grassmann manifold

$$\begin{aligned} \mathbb{F}_{is} &= \Big\{\Lambda \in Gr_2(\mathbb{C}^{2n+2}) \ \Big|\ I|_\Lambda = 0\Big\} \\ &= \Big\{[[z,w]] \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \ \Big|\ ||z||_{\mathbb{C}} = ||w||_{\mathbb{C}} = 1, \langle z, w\rangle_{\mathbb{C}} = I(z,w) = 0\Big\}, \end{aligned}$$

where $[[z,w]]$ is representative of (z,w) for $(z,w) \cong (\lambda z, \lambda w)g,\ \lambda \in S^1, g \in SU(2)$ or equivalently $(z,w) \cong (z,w)g, g \in U(2)$. Alternatively, $\mathbb{F}_{is}$ is a hyperplane in $Gr_2(\mathbb{C}^{2n+2})$:

$$\mathbb{F}_{is} \cong \Big\{(\lambda_{ij}) \in Gr_2(\mathbb{C}^{2n+2}) \ \Big|\ \lambda_{01} + \lambda_{23} + ... + \lambda_{2n+1,2n+2} = 0\Big\},$$

where (λ_{ij}) are the Plücker coordinates on $Gr_2(\mathbb{C}^{2n+2})$, as well as a homogeneous space: $\mathbb{F}_{is} \cong Sp(n+1)/U(2)Sp(n-1)$.

Proposition 6.4.3. *If $c \neq 0$ then the reduced space $\mathbb{O}_c$ is isomorphic to $\mathbb{F}_{is}$ equipped with the Kähler form $\widetilde{\omega}_c = \pi\sqrt{2c}\omega$, where ω is the restriction of the canonical Kähler form on $Gr_2(\mathbb{C}^{2n+2})$ which generates $H^2(Gr_2(\mathbb{C}^{2n+2}), \mathbb{Z})$.*

Proof. The S^1 action of the geodesic flow on $\widetilde{G}_H^{-1}(c) \subset T^*n \cong \widetilde{X}_H$ is:

$$\lambda\lfloor z, w\rfloor = \lfloor \lambda z, \lambda w\rfloor.$$

which commutes with the action of $Sp(1) \cong SU(2)$ defining the quaternionic Hopf fibration. Now from $\widetilde{G}_H(z,w) = c$ we have $||z||^2_{\mathbb{C}} = ||w||^2_{\mathbb{C}} = 2c$. If $\lambda_{ij} = z_i w_j - z_j w_i$ are the Plücker coordinates on $Gr_2(\mathbb{C}^{2n+2})$, then

$$\begin{aligned} \tilde{\pi}_c^*(\pi\sqrt{2c}\omega) &= \pi\sqrt{2c}\tfrac{\sqrt{-1}}{2\pi}\tfrac{d\lambda_{ij}\wedge d\bar{\lambda}_{ij}}{\sum_{i,j}||\lambda_{ij}||^2} \\ &= \tfrac{1}{\sqrt{2c}}\tfrac{\sqrt{-1}}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w}) \\ &= \tilde{i}_c^*(\widetilde{\Omega}_H). \end{aligned}$$

□

Proposition 6.4.4. *We have $c_1(\mathbb{F}_{is}) = (2n+1)\omega$.*

Proof. We note that $c_1(Gr_2(\mathbb{C}^{2n+2})|_{\mathbb{F}_{is}} = (2n+2)\omega$ and then proceed with the adjunction formula as in Proposition 2.4 using the fact that $\mathbb{F}_{is}$ is a hypersurface in $Gr_2(\mathbb{C}^{2n+2})$. □

Theorem 6.4.5. *The energy spectrum of the geodesic flow on $\mathbb{H}P^n$ is*

$$E_k = \frac{1}{2}(2n+1+2k)^2, k \in \mathbb{N}$$

with corresponding multiplicities:

$$m_k = \frac{2n+2k+1}{(k+1)(2n+1)}\binom{2n+k}{k}\binom{2n+k-1}{k}.$$

Proof. We only sketch the proof since it is similar to the n case. We have $c_1 : H^1(\mathbb{F}_{is}, \mathcal{O}^*) \to H^2(\mathbb{F}_{is}, \mathbb{Z}) = \mathbb{Z}$. Therefore all holomorphic line bundles on $\mathbb{F}_{is}$ which arise from the quantization are $L_k := S^{\otimes k}$, where $S = \iota^*([H])$ and ι is the inclusion $\iota : \mathbb{F}_{is} \to Gr_2(\mathbb{C}^{2N+2})$. Hence,

$$\frac{\sqrt{2c}}{2} - \frac{2n+1}{2} = k.$$

The dimension can be calculated via the Weyl dimension formula (see for example [11]). $\square$

This Theorem finishes the case by case proof of Theorem 6.1.3.

Remark 6.4.6. Existence of a symplectic form on the cotangent bundle doesn't have a direct analog to use in the case of higher-rank symmetric spaces. However not all simple and simply connected Lie group acts transitively on a CROSS. And the representation theory suggests that the correspondence could be extended to the higher dimensional case. In the next Sections we present one possible extension of the correspondence to the higher rank spaces based on geometry of toric bundles over flag manifolds.

CHAPTER 7

GENERALIZATION TO HIGHER RANK

7.1 Symplectic geometry of complex torus bundles

Let $T^n = S^1 \times S^1 \times ... \times S^1$ be the (real) n-dimensional torus. Then its tangent bundle is trivial and there is a well-known identification $T(T^n) \equiv T(S^1) \times T(S^1) \times ... \times T(S^1) \equiv (\mathbb{C}^*)^n \equiv (T^n)^{\mathbb{C}}$ which is the complex n-dimensional torus. In particular it is an open and dense subset of $\mathbb{C}^n$ and has an induced complex structure and Kähler metric. If $z_k = r_k e^{i\theta_k}$ are the coordinates in $(\mathbb{C}^*)^n$ then the Kähler form can be written as $\omega = \sum d(r_k^2) \wedge d\theta_k = d(\sum r_k^2 d\theta_k)$. One can see that the T^n action on $(\mathbb{C}^*)^n$ $(z_1, ..., z_n) \to (e^{i\alpha_1} z_1, ..., e^{i\alpha_n} z_n)$ is Hamiltonian with moment map $\mu(z_1, ..., z_n) = (r_1^2, ..., r_n^2) = (|z_1|^2, ..., |z_n|^2)$. Now we want to extend it to torus bundles:

Theorem 7.1.1. *Let $\pi : P \to M$ be a principal T^n-bundle over a Kähler manifold M with characteristic classes of type $(1,1)$. Let $P^{\mathbb{C}} = P \times_{T^n} (\mathbb{C}^*)^n$ be the associated complex torus bundle with the standard right action of T^n on $(\mathbb{C}^*)^n$. then $P^{\mathbb{C}}$ is open and dense subset of the vertical tangent bundle $\mathcal{V}$ of P and carries a natural complex structure and compatible symplectic (pseudo-Kähler) form ω. Moreover the T^n action on $P^{\mathbb{C}}$ is Hamiltonian and the Marsden-Weinstein reduction $P^{\mathbb{C}}//T^n$ is diffeomoerphic to M for a generic level set of the corresponding moment map.*

Proof. A principal torus bundle is determined, up to an isomorphism, by its characteristic classes on the base. Then we have a closed and integral $(1,1)$-forms on M, $\omega_1, ..., \omega_n$ and a connection 1-forms $\theta_1, .., \theta_n$ on P, such that $d\theta_k = \pi^*(\omega_k)$. The projection map $z : P \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ defines functions $r_k^2 = |z_k|^2$, which are T^n-invariant and descend to $P^{\mathbb{C}}$. Now the forms θ_k also descend to connection 1-forms on $P^{\mathbb{C}}$ and we can define an almost complex structure on $P^{\mathbb{C}}$ as $I(dr_k^2) = \theta_k$ and on the horizontal co-vectors is just a pull-back of the complex structure on the base. It defines the standard complex structure on the fibres $(\mathbb{C}^*)^n$. Its integrability follows from the fact that ω_k are of type $(1,1)$ (see [23]). The symplectic form is $\omega = \sum d(r_k^2\theta_k) + \pi^*(\omega_M)$, where ω_M is a Kähler form which is positive enough to ensure that ω is non-degenerate in the horizontal directions for almost all x_i. Now it is clear that for a basis of vertical vector fields X_k which are defined by the T^n action and satisfy $\theta_i(X_j) = \delta_i^j$, the moment map is $\mu = (r_1^2, ..., r_n^2)$ as a $\mathbb{R}^n$-valued function on $P^{\mathbb{C}}$. So it is clear that for $c \in \mathbb{R}^n$ where all coordinates are positive, $\mu^{-1}(c) \equiv P$ where we identify P with the set of points in $P^{\mathbb{C}}$ with $r_k = 1$ for all k. Then it is clear that $P^{\mathbb{C}}//T^n \equiv M$. □

We can see that the reduced symplectic form depends on the level c and is integral whenever c satisfies some integrality condition - which will provide the quantum condition for the correspondence in the higher rank symmetric spaces.

We identify the reduced symplectic form on $P^{\mathbb{C}}//T^n$ in the following way:

Corollary 7.1.2. In the notations of the Theorem 7.1.1 and its proof, the symplectic form on P is given by $\omega = d(\sum x_i^2\theta_i) + \pi^*\omega_M$, and the reduced symplectic form on $P^{\mathbb{C}}//T^n = \mu^{-1}(c_1, ..., c_n) \equiv M$ for a generic choice of $(c_1, ..., c_n)$ is $\tilde{\omega}_M = \sum c_i^2 d\theta_i + \omega_M$.

7.2 Symmetric spaces of general rank

Now we apply the Corollary and the Theorem of the previous Section to the space parametrizing the maximal totally geodesic tori of a Riemannian symmetric space. Let as before $M = G/K$ be a symmetric space with G compact and semisimple. Every maximal totally geodesic tori is tangent to a translated maximal commutative subspace of $\mathfrak{m}$. Denote again by $\mathfrak{a}$ one such fixed subspace. Also L is the connected subgroup of G with Lie algebra $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We can also write $L = MA$ where M and A are the corresponding Lie groups (in the Iwasawa decomposition, see [20], [22]).

Then G/L is a generalized flag manifold parametrizing the maximal totally geodesic tori in M. As such it caries a natural complex structure, which depends on the choice of a

Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and a partial order in it which determines a positive Weyl chamber and it defines a positive Weyl chamber in $(\mathfrak{a}^{\mathbb{C}})^*$. The later is dual to the cone of restricted dominant weights. Then as a complex manifold G/L is equivalent to $G^{\mathbb{C}}/M^{\mathbb{C}}A^{\mathbb{C}}N^{\mathbb{C}}$ and has a principle $A^{\mathbb{C}}$-bundle $G^{\mathbb{C}}/M^{\mathbb{C}}N^{\mathbb{C}} \to G^{\mathbb{C}}/M^{\mathbb{C}}A^{\mathbb{C}}N^{\mathbb{C}}$ with total space - the *horospherical manifold* Θ. Since $A^{\mathbb{C}}$ is the complexification of the real torus $T^r = A$ and can be identified with the cotangent bundle T^*T^r, then Θ can be identified with the total space of the vertical (co)tangent bundle of the principal bundle $G/M \to G/L$ with fiber A. In case the rank of M is $r = 1$, this is just T^*M. Since the characteristic classes of the bundle $G/M \to G/L$ are determined via transgression by the simple roots in $\mathfrak{a}^*$, we can apply the constructions of the previous section.

Theorem 7.2.1. *Let $M = G/K$ be a compact Riemannian symmetric space of rank k with G semisimple and let $\theta_1, ..., \theta_k$ be the basis of fundamental weights that is dual to the simple restricted roots of $\mathfrak{a}^{\mathbb{C}}$. Let Θ be the associated horospherical manifold and $\Theta \to G/L$ be the corresponding principal $(\mathbb{C}^*)^k$-bundle, where G/L is the quantization space of G/K. Let $\omega_M = \frac{1}{2\pi} d\rho$ be the 2-form on G/L representing $\frac{1}{2}c_1(G/L)$, so ρ is the half sum of the positive roots in $\mathfrak{g}^{\mathbb{C}}$ vanishing on $\mathfrak{l}^{\mathbb{C}}$ (as in [2, 5]). Then there exists a symplectic form ω on Θ with the following properties:*

i) There are positive numbers n_i such that the reduced form $\tilde{\omega}$ on $\Theta//T^k$ corresponding to ω via the Marsden-Weinstein reduction is $\tilde{\omega} = \sum_{i=1}^{k} n_i d\theta_i + \omega_M$, on G/L.

ii) When the 2-form $d\alpha$ for $\alpha = n_1\theta_1 + n_2\theta_2 + ... + n_k\theta_k$ is integral (up to a factor of 2π) and determines a dominant weight, then the corresponding quantum bundle $\mathcal{L}$ defined by $\tilde{\omega} \in c_1(\mathcal{L})$ has the property that its space of holomorphic sections $H^0(G/L, \mathcal{O}(\mathcal{L}))$ is an irreducible unitary representation of G with highest weight α.

iii) The complexified eigenspaces of the Laplace-Beltrami operator Δ_M corresponding to the eigenvalue $\lambda_\alpha = ||\alpha + \rho_{\mathfrak{a}}||^2 - ||\rho_{\mathfrak{a}}||^2$ on M have dimension equal to the sum over *all α with $||\alpha + \rho_{\mathfrak{a}}||^2 = \lambda_\alpha + ||\rho||^2$ of the dimensions of $H^0(G/L, \mathcal{O}(\mathcal{L}))$ defined in ii). All eigenvalues of Δ_M are equal to λ_α for some α.*

Proof. Since the first Chern class $c_1(G/L) > 0$ is positive for generalized flag manifolds, Ω_M is positive definite and Kähler.The form ω is closed and since θ_i correspond to a basis of the positive Weyl chamber in $\mathfrak{a}^{\mathbb{C}}$, we see that the ω is also Kähler as a sum of a positive and non-negative form.

The reduced form coincides with of $\tilde{\omega}_M$ by Corollary 7.1.2, which proves *i*).

The quantum line bundle is well defined since for generalized flag manifolds the Picard group is isomorphic to $H^2(G/L, \mathbb{Z})$. Then *ii*) follows from Borel-Weil Theorem. Finally *iii*) is valid in view of the fact that the eigenspaces of the Laplace-Beltrami operator are sums of irreducible G-modules.

$\square$

CHAPTER 8

LAPLACE EIGENFUNCTIONS AND HOLOMORPHIC SECTIONS

In this Chapter we describe a procedure to obtain an explicit algebraic expression of the eigenfunctions of the Laplace-Beltrami operator on compact symmetric spaces through harmonic polynomials.

We shortly describe the idea of the construction first. Consider the space of holomorphic sections of a line bundle in the Borel-Weil Theorem. It is identified with holomorphic functions f on a principal $\mathbb{C}^*$-bundle P over the (generalized) flag manifold $\mathbb{F} = G/L$ such that $f(xa) = \chi(a)f(x)$, where χ is the character of the representation in $H^0(\mathbb{F}, L_\chi)$ for the associated with P line bundle L_χ from the Borel-Weil Theorem. The structure group of P could be reduced to S^1 so P has a structure of a cone $P \cong \mathbb{R}^+ \times S$, for S - the total space of an S^1-bundle over $\mathbb{F}$. Note that P is different from Θ and sometimes can be represented as its quotient. The S^1 action on S is induced from the $\mathbb{C}^*$-action on P such that for $a = re^{i\theta} \in \mathbb{C}^*$ we have the action $R_a(x,t) = (e^{i\theta}x, rt)$. We note that S has a Sasakian metric g_S (see [9]) and there is a cone metric g_P on P such that $g_P = dr^2 + r^2 g_S$. Then g_P is the Kähler cone metric - as in Boyer-Galicki approach to Sasakian geometry [9]. In particular, every holomorphic function on P is also harmonic.

Now the relation between the Laplace-Beltrami operators on the cone P and the base S is

$$\Delta_P(u) = \frac{\partial^2 u}{\partial r^2} + n\frac{1}{r}\frac{\partial u}{\partial r} + r^{-2}\Delta_S(u)$$

where $u = u(r,x)$ and $\Delta_S(u)$ is calculated when S is embedded in P as $r = constant$. If the function u is corresponding to a holomorphic section, then the equivariance condition above gives for $x = e^{i\theta}y$

$$u(x,r) = u(e^{i\theta}y, r.1) = r^k e^{ik\theta} u(e^{-i\theta}x, 1),$$

where $k = \chi(re^{i\theta})$. Then from the formulas we obtain $\frac{\partial u}{\partial r} = \frac{k}{r}u$ and

$$\Delta_S u = \lambda u \tag{8.0.1}$$

when $u(x) = u(x, 1)$ and λ depends on k. In particular u determines an eigenfunction of the Laplace-Beltrami operator on S. Now, in many cases, we can pull-back the function to Θ and if this pull-back is K-invariant, then it will define a function on G/K. This function is an eigenfunction if the projection is a Riemannian submersion with totally geodesic fibers.

To make this strategy work we have to resolve two problems. First we need to see when a pull-back to Θ is possible. Second, the metrics on $\mathbb{F}$ which will lead to such projection are not Kähler - they arise from the bi-invariant metric on G. So we need a modification of this idea for non-Kähler metrics. We start with the second problem.

Recall that a Hermitian metric g on a complex manifold M with a fundamental form ω is called balanced, if $d\omega^{n-1} = 0$ where n is the complex dimension of M. A result in [28] shows that a holomorphic function on a balanced manifold is again harmonic. To use this property we need a few lemmas.

Lemma 8.0.1. *Suppose that M is a compact complex manifold of dimension n with a balanced metric g_M which has fundamental form ω_M, i.e. $d\omega_M^{n-1} = 0$ Let $\pi : P \cong \mathbb{R}^+ \times S \to M$ be a principal $\mathbb{C}^*$-bundle with $U(1)$-connection 1-form θ on S and a cone metric of the form $g = dr^2 + r^2(\theta^2 + \pi^*(g_M))$. Let $d\theta = \pi^*(\omega)$, where ω is a form of type (1,1), be the curvature of S (and P). With respect to a natural complex structure I on P compatible with g_P, such g_P is balanced iff*

$$(\omega - \omega_M) \wedge \omega_M^{n-1} = 0.$$

Proof. The complex structure on P is given by $I(dr) = r\theta, I(\theta) = d\log r$ and the pull-back of the complex structure on M on the horizontal spaces $\ker(\theta) \cap \ker(dr)$. The fact that it is integrable follows from the condition that ω is (1,1) (see [23]). Suppose that the complex dimension of M is n, so $\dim_{\mathbb{C}} P = n + 1$. Then the fundamental Kähler form of g_P is given by $\omega_P = rdr \wedge \theta + r^2\pi^*(\omega_M)$. For convenience we write ω_M for $\pi^*(\omega_M)$ where it is not confusing. Now $\omega_P^n = nr^{2n+1} dr \wedge \theta \wedge \omega_M^{n-1} + \omega_M^n$ and

$$d\omega_P^n = nr^{2n+1} dr \wedge (\omega_M - \omega) \wedge \omega_M^{n-1}$$

since $d\omega_M^{n-1} = 0$. $\square$

Therefore, we deduce

Lemma 8.0.2. *Let $P \cong \mathbb{R}^+ \times S$ be a principal $\mathbb{C}^*$-bundle over a generalized flag manifold $M = G/L$ where G is a compact simply-connected and semisimple Lie group and L is a centralizer of a torus. Assume that S has a connection 1-form θ which is G-invariant and its curvature ω has a cohomology class $[\omega]$ such that $[\omega]/2\pi \in H^2(M, \mathbb{Z})$ and is not an integer multiple of another class. Then there is a projection $\pi_1 : G \to S$ which is a factor-bundle. With respect to the natural complex structure, P admits a balanced cone metric $g_P = dr^2 + r^2 g_S$ with an induced g_S metric on S, such that the projection π is a Riemannian submersion with totally geodesic fibers when G is equipped with its biinvariant metric, after possible rescaling.*

Proof. The fact that there is such a projection follows from [24]. For every invariant g_M, the Hodge-dual $*d(\omega_M)^{n-1}$ of $d\omega_M^{n-1}$ is an invariant 1-form and the Euler characteristic of M is positive, so g_M is balanced. From Lemma 8.0.1 we see that both ω_M^n and $(\omega \wedge \omega_M^{n-1})$ are proportional to the invariant volume form on M, which means that up to a rescaling of the metric on M we could make them equal, so g_P is balanced. $\square$

Lemma 8.0.3. *Every holomorphic function on P is harmonic with respect to the metric g_P from Lemma 8.0.2.*

Proof. The result follows for example, from [28]. $\square$

Note that most of the results in the Lemmas above are valid for non-integrable almost complex structures. However, we are focusing on the integrable case, since we need the conditions for the Borel-Weil Theorem to be satisfied in order to provide the relation to the Laplace-Beltrami eigenfunctions on G/K.

Lemma 8.0.4. *A harmonic function F on P which satisfies $f(x,r) = r^k f(x,1)$ induces an eigenfunction of the Laplace-Beltrami operator on S.*

Proof. It follows from (8.0.1) and the calculations there.

Now we consider the problem of existence of a pull-back of a function to Θ. Denote by L_{ss} the subgroup of G with Lie algebra $\mathfrak{l}_{ss} = [\mathfrak{l}, \mathfrak{l}] = [\mathfrak{m}.\mathfrak{m}]$ which is the semisimple part of $\mathfrak{l}$. Consider G/L_{ss} as a T^k-principal bundle over the flag manifold G/L. It has characteristic classes given by $\gamma_i = \frac{1}{2\pi} d\omega_i$ where $\omega_i, i = 1, 2, ..., k$, are the fundamental weights. Any principal S^1-bundle can be characterized topologically by its first Chern class, which is a positive integer combination of these. Let S be determined by $c_1(S) = \sum n_i \gamma_i$, where n_i are positive integers. According to Lemma 3 in [24], if $\gcd(n_1, ..., n_k) = 1$, then we can find a basis of generators $\beta_1 = c_1(S),\ \beta_2, ..., \beta_k$ of $H^2(\mathbb{F}, Z)$ and they will define an equivalent principal bundle to $G \to \mathbb{F} = G/T^k$. In particular, there is a principal T^{k-1}-bundle $G/L_{ss} \to S$ and we can use the construction above. If this condition does not hold, then $c_1(S) = m\beta$, for some β and m positive integer, which satisfies it. Now we can replace S with another bundle $\overline{S}$ with characteristic class β. By a standard argument (for example comparing the Euler classes - see e.g. [14] Ex 3.26.) $S = \overline{S}/\mathbb{Z}_m$ as a finite cover, so we have the projections $G/L_{ss} \to \overline{S} \to S$. Now G/L_{ss} have two fibrations - over S and over the symmetric space G/K. When we induce the metrics on G/K, G/L_{ss}, and S from the biinvariant metric on G, both fibrations are Riemannian submersions with totally geodesic fibers. For a such a Riemannian submersion $\pi : M \to N$ the relation between the Laplace-Beltrami operators on M and N is:

$$\Delta^M(f \circ \pi) = (\Delta^N f) \circ \pi \tag{8.0.2}$$

for any smooth function f on N - see [44]

□

The above considerations lead to:

Theorem 8.0.5. *Suppose that G/K is a compact Riemannian symmetric space and $\mathbb{F} = G/L$ is the associated generalized flag manifold - the quantization space. Let $f \in H^0(\mathbb{F}, L_\chi)$ be a holomorphic section of a positive line bundle L_χ over a flag manifold $\mathbb{F}$ which is considered as function on the corresponding principle bundle $P = S \times \mathbb{R}^+$ with $f(za) = \chi(a)f(z)$ for $a \in \mathbb{C}^*$. Let $\pi : G/L_{ss} \to S$ and $\pi_1 : G/L_{ss} \to G/K$ be the natural projections. If the pull-back of f to $G/L_{ss} \times \mathbb{R}^+$ via π is K-invariant, then f satisfies the conditions of Lemma 8.0.4 and the function $u(x) = f(x, 1)$ on S defines an eigenfunction $\overline{u}$ of the Laplace-Beltrami operator on the Riemannian symmetric space G/K with $\pi_1^*(\overline{u}) = \pi^*(u)$.*

Proof. From the Lemmas above, u is an egienfunction on S. By the property (8.0.2) $\pi^*(u)$ is an eigenfunction on G, and by the K-invariance it is a pull-back of a function on G/K. Then again by (8.0.2) $\overline{u}$ is an eigenfunction on G/K. □

Remark 8.0.6. Using the Cartan embedding $i : G/K \to G$, we see that we need functions on G/K depending on the parameters defining the image of G/K. We are going to use this in the examples below. Moreover, the generalized flag manifold $\mathbb{F}$ has more than one invariant complex structures - see [7] for example. Each of them defines a set of eigenfunctions described in the Theorem. In the examples below this process in fact generates all of the eigenfunctions. It is likely that this happens for most of the irreducible compact symmetric spaces.

We mention briefly the relation of the construction to the so-called spherical representations. A representation π of group G in a vector space V (with respect to the Riemannian symmetric space G/K) is called spherical, if V contains a vector, fixed by all operators in $\pi(K)$. Any unitary spherical representation of G with a unit vector $\mathbf{e}$ fixed by $\pi(K)$, the function $G \ni dx \to \langle \mathbf{e}, \pi(x)\mathbf{e} \rangle$ is positive-definite and spherical ([33], Therorem 3.4). A function on a Lie group G is called positive definite, if for every $x_1, .., x_n \in G$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$ we have

$$\sum_{i,j} \phi(x_i^{-1} x_j) \alpha_i \overline{\alpha_j} \geq 0.$$

Also ϕ is called *spherical* if it is K-bi-invariant (left and right) and also a common eigenfunction of all left-invariant operators on G, which are also right $K-$invariant. The Cartan-Helgason Theorem [33] characterizes the irreducible spherical representations as the ones for which the highest weight $\lambda : \mathfrak{h} \to \mathbb{C}$ satisfies

$$\lambda(i(\mathfrak{h} \cap \mathfrak{k}) = 0$$

and

$$\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+, \forall \alpha \in \Sigma^+.$$

Note that the irreducible representations for compact G are characterized by the second condition, with $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+$ instead of $\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+$. In particular, when G/K is a Riemannian symmetric space of maximal rank, the first condition is trivial, so the spherical representations form"half" of all irreducible representations - we call them *even*.

In fact, the relation between the Borel-Weil theorem and the Laplace-Beltrami eigenfunctions can potentially reveal more information. The spaces of holomorphic sections in the Borel-Weil theorem are irreducible representations and the eigenspaces on the symmetric space G/K are not unless it is a CROSS. The irreducible spherical representations are characterized as the common eigenspaces of the invariant differential operators on G/K. So the eigenspaces are sums of spherical representations and we expect that the correspondence in the Theorem could be extended to the irreducible spherical representations. Another approach to that (see [34, 20]) is through the integral geometry and variations of Radon transform. But such approach provides only expressions of the spherical functions in terms of integral formulas, which are not explicit in general.

8.1 Harmonic polynomials and eigenfunctions

The linear action of a (Lie) group on a vector spaces V naturally extends to an action on the symmetric algebra and is naturally accompanied with "invariant polynomials and corresponding harmonic polynomials". The crucial connection between these invariants of joint differential operators and their eigenfunctions will only be briefly outlined here; a more thorough exposition can be found in Helgason, [33], Chapter III.

Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and $G := \mathrm{GL}(V)$ of linear transformations of the vector space V. The action of G on V also induces an action of G on the polynomial ring of V, where the $G-$*invariants* (fixed points) form a subring.

As usual, let V^* denote the dual space of V, $\mathcal{S}(V)$ and $\mathcal{S}(V^*)$ the standard *real* symmetric algebras. $\mathcal{S}(V^*)$ then consists of the polynomials on V^*. The respective complexifications will be denoted at $\mathcal{S}^{\mathbb{C}}(V) := \mathbb{C} \otimes \mathcal{S}(V)$ and $\mathcal{S}^{\mathbb{C}}(V^*) := \mathbb{C} \otimes \mathcal{S}(V^*)$.

Definition 8.1.1. For $X \in V$, the ("directional derivative type") differential operator is the map $\partial(X) : \mathcal{S}(V) \to \mathbb{R}$ is defined as

$$(\partial(X)f)(Y) := \left(\frac{d}{dt} f(Y + tX)\right)\Bigg|_{t=0} \qquad \text{for } Y \in V, f \in C^{\infty}(V).$$

Thus the mapping $X \mapsto \partial(X)$ extends to an isomorphism of the symmetric algebra $\mathcal{S}(V)$ (and the complex symmetric alegbra $\mathcal{S}^{\mathbb{C}}(V)$ respectively). $\mathrm{GL}(V)$ acts on both V and V^* by the remarkably beautiful equation: letting $v \in V$ and $v^* \in V^*$, for $g \in G$ the equation is

$$(g \cdot v^*)(v) = v^*(g^{-1} \cdot v).$$

This extends to a map L from $\mathcal{S}(V)$ to the algebra of differential operators on V which is an isomorphism. Take a positive bilinear form B on V and define with it the isomorphism $B : V \to V^*$. The space $\mathcal{S}(V^*)$ has a bilinear form $\langle\langle , \rangle\rangle$ defined as

$$\langle\langle p, q \rangle\rangle = (\partial(P)q)(0)$$

where P is the image of p under the isomorphism $L \circ B$. This coincides with he usual extension of B to $\mathcal{S}(V)$, so is a positive scalar product. We have the following property for polynomials p, q, r and their corresponding differential operators P, Q, R:

$$\langle\langle p, qr \rangle\rangle = \langle\langle \partial(Q)p, r \rangle\rangle \tag{8.1.1}$$

Proposition 8.1.2. *([33] pg. 347) For $p, q, r \in \mathcal{S}(V^*)$ then we have*

$$\langle\langle p, qr\rangle\rangle = (\partial(QR)p)(0) = (\partial(R)\partial(Q)p)(0) = \langle\langle \partial(Q)p, r\rangle\rangle.$$

Definition 8.1.3. Let $I(V^*)$ is the ideal in $\mathcal{S}(V^*)$ generated by the invariant polynomials, and $I^+(V^*) \subset I(V^*)$ are the subset of polynomials without constant term. For the action of G denote by $H(V^*)$ the set of $G-$*harmonic* polynomials h, i.e. $\partial(J)(h) = 0$ for every invariant differential operator $J = L \circ B(j)$, $j \in I^+(V^*)$. Assuming that G is compact by [33] Ch. 3, Theorem 1.1:

$$\mathcal{S}(V^*) = I(V^*)H(V^*),$$

and from the proof we see that

$$\mathcal{S}^k(V^*) = (I^+(V^*)\mathcal{S}(V^*))^k \oplus H^k(V^*)$$

is an orthogonal decomposition with respect to $\langle\langle , \rangle\rangle$. We are going to use a particular case, when $I^+(V)$ is generated by one homogeneous polynomial p of degree l. Then the multiplication by p gives an embedding $P : \mathcal{S}^k(V^*) \to \mathcal{S}^{k+l}(V^*)$ such that we have an identification of the quotient space

$$\mathcal{S}^{k+l}(V^*)/P(\mathcal{S}^k(V^*)) = H^{k+l}(V^*)$$

with the harmonic polynomials, which in this case are coicide with $\ker(\partial(P))$.

The classical example we want to generalize is that of the sphere S^n. It is known that the spherical harmonics (Laplace-Beltrami eigenfunctions), are restrictions of the harmonic polynomials on $\mathbb{R}^{n+1}$. A similar description is known for $\mathbb{C}P^n$. Before we present it we recall briefly the facts we need from the theory of invariant harmonic polynomials.

Let V be a real or complex vector space and G a group of linear transformations of V. Then G acts on the ring of polynomials identified as the symmetric algebra $\mathcal{S}(V^*)$. If $f \in \mathcal{S}(V^*)$ is a polynomial and $X \in V$, then the directional derivative $\partial(X)$ acts on f and extends to a map L from $\mathcal{S}$ to the algebra of differential operators on V which is an isomorphism. Take a positive bilinear form B on V and define with it the isomorphism $B : V \to V^*$. The space $\mathcal{S}(V^*)$ has a bilinear form $\langle\langle , \rangle\rangle$ defined as

$$\langle\langle p, q \rangle\rangle = (\partial(P)q)(0)$$

where P is the image of p under the isomorphism $L \circ B$. This coincides with he usual extension of B to $\mathcal{S}(V)$, hence, is a positive scalar product. We have the following property for polynomials p, q, r and their corresponding differential operators P, Q, R:

$$\langle\langle p, qr \rangle\rangle = \langle\langle \partial(Q)p, r \rangle\rangle \tag{8.1.2}$$

so the multiplication by q is adjoint to the operator $\partial(Q)$.

Definition 8.1.4. For nonnegative integers $p, q \geq 0$ and any $a, b \in \mathbb{C}^{n+1}$ satisfying $(a, b) = 0$, define the (polynomial) function

$$h(z) := h(z, \bar{z}) := h^{p,q}_{a,b}(z, \bar{z}) := (a, z)^p (b, \bar{z})^q \tag{8.1.3}$$

These so-called harmonic polynomials, have a clear $SU(n + 1)$ action, which can be verified without much effort that for $s \in SU(n + 1)$ we have

$$sh^{p,q}_{a,b} = h^{p,q}_{(s^t)^{-1}a, sb}.$$

This allows us to decompose the space $H(V)$ (for $V = \mathbb{C}^n$) of harmonic polynomials as a sum of $SU(n + 1)$-modules spanned by the h's in definition (8.1.4). More precisely, with emphasis on the second claim

Theorem 8.1.5. *(Th. 14.4 in [43]) As an $SU(n + 1)$-module the space*

$$H(\mathbb{C}^{n+1}) = \bigoplus_{p,q \geq 0} H^{p,q}(\mathbb{C}^{n+1})$$

i.e., every harmonic polynomial of type (p, q) is a linear combination of the h's:

$$H^{p,q}(\mathbb{C}^{n+1}) = span_{\mathbb{C}} \Big\{ h^{p,q}_{a,b} \;\Big|\; a, b \in \mathbb{C},\; (a, b) = 0 \Big\}.$$

Next we prove that the h's are indeed harmonic, i.e. are in the kernel of the Laplace-Beltrami operator.

Proposition 8.1.6. *If $a, b \in \mathbb{C}^{n+1}$ satisfy $(a, b) = 0$, p and q are nonnegative integers, and h is defined as above, then $\Delta h(z, \bar{z}) = pq(a, z)^{p-1}(b, \bar{z})^{q-1}$. Moreover, h is harmonic, i.e., $h(z, \bar{z}) = 0$.*

Proof. For $h(z, \bar{z})$ as defined again as,

$$h := h^{p,q}_{a,b} := h^{p,q}_{a,b}(z, \bar{z}) = (a, z)^p (b, \bar{z})^q.$$

Computing the first partial of z_k of h we obtain

$$\begin{aligned} \frac{\partial}{\partial z_k} h(z, \bar{z}) := \frac{\partial}{\partial z_k}(a, z)^p (b, \bar{z})^q = (b, \bar{z})^q \frac{\partial}{\partial z_k}\Big[(a, z)^p\Big] \\ = p(b, \bar{z})^q (a, z)^{p-1} \frac{\partial}{\partial z_k}(a, z) = p(b, \bar{z})^q (a, z)^{p-1}(a_k) \end{aligned}$$

and same for $\bar{z}_k$:

$$\frac{\partial}{\partial \bar{z}_k}(a, z)^p (b, \bar{z}_k)^q = (a, z)^p \frac{\partial}{\partial \bar{z}_k}\Big[(b, \bar{z})^q\Big] = q(a, z)^p (b, \bar{z})^{q-1}(b_k)$$

and of course we will employ the well-known of equality $4\Delta = \partial\bar{\partial}$ the theory of functions of one and several complex variables All that remains is summing over k taken in the order to compute the Laplacian operator to immediately get, if $(a, b) = 0$ then

$$4\Delta h(z, \bar{z}) = \sum_{k=1}^{n+1} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} h(z, \bar{z}) = pq(a, z)^{p-1}(b, \bar{z})^{q-1}\cancelto{0}{(a, b)} = 0. \tag{8.1.4}$$

□

8.2 Complex projective space $\mathbb{C}P^n$

In this section we drescribe the generaliztion of the quantization procedure to produce the the geodesic flow of the CROSSes $M = G/K$:

$$M = \mathbb{C}P^n = {}^{SU(n+1)}\!\big/_{S(U(n) \times U(1))},$$

and higher ranks as well, analogous to the symplectic reduction of the Hamiltonian flow. In this case, they compute and show the biholomorphism to

$$\begin{aligned}\operatorname{Geod}(\mathbb{C}P^n) = {}^{G}\!/_{L} &= {}^{SU(n+1)}\!/_{S(U(1)\times U(1)\times U(n-1))} \\ &\cong \Big\{([z],[w])) \in \mathbb{C}P^n \times \mathbb{C}P^n \ \Big|\ (z,w)=0\Big\}\end{aligned}$$

We denote the last set by X. In the rank one case, G/L can be interpreted as the (oriented) parametrization space of the geodesic flow of the symmetric space. In higher rank spaces, they no longer parametrize geodesics but totally geodesic maximal abelian subgroups, at the group level, and then at the algebra level all Cartan subalgebras, that again are all conjugate to each other. We show that the space spanned by the functions $h(z)$ is the orthogonal complement to this quadric in $\mathbb{C}P^n$.

Letting $p = h(z,w) = h^{p,q}_{a,b}(z,w)$, $q(z,w) = (z,w) = \sum_k z_k w_k$, and r remaining arbitrary in the ideal generated by $q(z,w)$, we calculate immediately employing equation 8.2.1 below:

$$\langle\langle h, qr\rangle\rangle = \langle\langle \partial(Q)h, r\rangle\rangle := ((\partial(\partial(Q)h)r)(0). \tag{8.2.1}$$

The quantization space is the generalized flag manifold $\mathbb{F} = SU(n+1)/S(U(1)\times U(1)\times U(n-1))$ and the horospherical manifold corresponding to $\mathbb{C}P^n$ is a bundle over the flag $SU(n+1)/S(U(1)\times U(1)\times U(n-1))$, which is embedded as a quadric (1,1)-hypersurface in $\mathbb{C}P^n \times \mathbb{C}P^n$ (cf. Section 3.1). This bundle should correspond to $L_{p,q}$ for some non-negative integers p, q.

Alternatively, or by the duality property of formula (8.1.2), the space of sections of $L_{k,k}, k > 1$ can be identified with the polynomials $p(z,w)$ for which$\sum \frac{\partial^2}{\partial z_i \partial w_i}(p) = 0$.

Then considering

$$H^0(\mathbb{F}, \mathcal{O}(L_{p,q})) = \mathcal{S}^{p,q} / \left((z,w)\mathcal{S}^{p-1,q-1}\right),$$

where $\mathcal{S}^{p,q}$ denotes the space of polynomials in $z, w \in \mathbb{C}^{n+1}$ of homogeneous degree (p,q) (in particular, $(z,w) \in \mathcal{S}^{1,1}$). Alternatively,

$$\mathcal{S}^{p,q} = \left\{F(z,w) \in S^*(\mathbb{C}^{2n+2}) \ \middle|\ F(\alpha z, \beta w) = \alpha^p \beta^q F(z,w), \text{ for all } a, b \in \mathbb{C}\right\}.$$

Since, the space $\mathcal{S}^{p,q}$ is spanned by the polynomials $F = f(z)g(w)$, $\deg(f) = p$, $\deg(g) = q$,, then $H^0(\mathbb{F}, \mathcal{O}(L_{k,k}))$ is spanned by (the restriction to $\mathbb{F}$ of) the polynomials $p(z,w) = (a,z)^k(b,w)^k$ for all a, b with $(a,b) = 0$.

On the other side, by Corollary after Theorem 14.4 in [43], the functions $(a,z)^k(b,\overline{z})^k$ span the harmonic polynomials on $\mathbb{C}^{n+1}$ which induce *the eigenfunctions for the kth eigenvalue $\lambda_k = 4k(n+k)$ of the Laplace-Beltrami operator on $\mathbb{C}P^n$ relative to the Fubini-Study metric* (cf. Theorem 6.1.3). Therefore, the λ_k-eigenspace equals the span of the restriction given by $w = \overline{z}$ of all $p(z,w) = (a,z)^k(b,w)^k$ with $(a,b) = 0$.

To finish the proof, we need to show that this inner product defined on this particular space vanishes (i.e., is orthogonal with respect to the $\langle\langle,\rangle\rangle$ inner product) equation (8.1.1) in section 8.1, or equivalently, equation (8.1.2) is equal to zero, which is calculated to be so in equation (8.1.4) so after combining it with $\partial(Q)h \equiv 0$ and we obtain equation (8.2.3). But now we have everything we need to calculate the orthogonal complement of $X \subset \mathbb{C}P^n$ with respect to the double bracketed inner product $\langle\langle,\rangle\rangle$ on the space of homogeneous polynomials, via the identifications to differential operators. Indeed, we have:

$$\langle\langle h, qr\rangle\rangle = \langle\langle \partial(Q)h, r\rangle\rangle := ((\partial(\partial(Q)h)r)(0). \tag{8.2.2}$$

Since $\partial(Q)h \equiv 0$ then

$$\langle\langle h, qr\rangle\rangle = \langle\langle \partial(Q)h, r\rangle\rangle := ((\partial(\partial(Q)h)r)(0) = ((\partial(0)r)(0) = \frac{d}{dt}\bigg|_{t=0} r(0+0t) = 0. \tag{8.2.3}$$

This proves we have indeed found our orthogonal space with respect to the double bracket metric $\langle\langle,\rangle\rangle$, defined at the beginning of the this chapter.

$$H(\mathbb{C}^{n+1}) = \bigoplus_{p,q\geq 1} H^{p,q}(\mathbb{C}^{n+1})$$

can be written as a direct sum with subsets $H^{p,q}(\mathbb{C}^{n+1}) = \operatorname{span}\left\{h^{p,q}_{a,b} \;\middle|\; (a,b) = 0\right\}$ are the orthogonal complements to the ideal $I\left(\langle q\rangle\right)$ generated by $q(z,w) = (z,w)$, which is the polynomial whose vanishing set defines the geodesic flow of M biholomorphically to the quadric $X \subset \mathbb{C}P^n$.

Endowing the set $\mathcal{S}(V)$ of polynomial functions on $\mathbb{C}^{n+1}$ with the inner product $\langle\langle,\rangle\rangle$, and using the results from this section, we can *explicitly describe* the space of holomorphic line bundles along the quadric $X = \{([a],[b]) \in V \times V \mid (a,b) = 0\}$.

8.3 Quaternionic projective space $\mathbb{H}P^n$

Now we consider the functions of $z = (z_0, ..., z_n)$ and $w = (w_0, ..., w_n)$ such that $q_i = z_i + w_i j$ are the quaternionic coordinates of $\mathbb{H}^{n+1}$. The harmonic polynomials on $\mathbb{C}^{n+1}$ which induce the eigenfunctions on $\mathbb{C}P^n$ are precisely the ones on $\mathbb{R}^{2n+2}$, which are invariant under S^1. And the invariant functions which generate these polynomials are exactly $z_i\overline{z_j}$, which also fits the Cartan embedding interpretation. Similarly, for $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4} = \mathbb{C}^{2n+2}$, the harmonic polynomials which determine the eigenfunctions on $\mathbb{H}P^n$ are the ones right-invariant under $Sp(1) = SU(2)$, and they are functions of the variables $q_l\overline{q_k} = z_l\overline{z_k} + w_l\overline{w_k} + (z_k w_l - z_l w_k)j$. In particular, we see that the following functions, together with their conjugates, span the $Sp(1)$ invariant quadratic harmonic polynomials:

$$(a,z)(b,\overline{z}) + (a,w)(b,\overline{w})$$

$$(a,z)(b,w) - (a,w)(b,z)$$

with $(a,b) = 0$.

Now we relate the functions to the quantization space which is the symplectic isotropic Grassmannian $\mathbb{F}_{is} = \mathbb{F}_{is}(2, 2n+2)$. Using the construction in Section 3.2, $\mathbb{F}_{is}$ is embedded in the regular Grassmannian by a hyperplane section given by the holomorphic symplectic form denoted by $I(z,w)$. Then the Grassmannian is embedded in $P(\Lambda^2(\mathbb{C}^{2n+2}))$ by Plucker relations. Since the Picard group of $\mathbb{F}_{is}(2, 2n+2)$ is $\mathbb{Z}$, every line bundle is of type $\mathcal{O}(k)$ for some power of the hyperplane section bundle $\mathcal{O}_{\mathbb{F}_{is}}(1)$, which is the restriction of $\mathcal{O}_{P(\Lambda^2\mathbb{C}^{2n+2})}(1)$. In particular we can consider the section as a homogeneous polynomials of degree k on the variables $U_iV_j - U_jV_i$ for $U, V \in \mathbb{C}^{2n+2}$, which are orthogonal to $\sum U_iV_{n+i+1} - U_{n+i+1}V_i = I(U,V)$ relative to $\langle\langle \cdot, \cdot \rangle\rangle$. Alternatively, these polynomials are the ones in the kernel of $\Box = \sum(\frac{\partial^2}{\partial U_i \partial V_{n+i+1}} - \frac{\partial^2}{\partial V_i \partial U_{n+i+1}})$. Using similar reasoning as the one in the case of the generalized flag manifold $\mathbb{F}$ related to $\mathbb{C}P^n$, we observe that all such polynomials are linear combinations of $p(U,V) = l_{AB}(U,V)^k$, where

$$l_{AB}(U,V) = (A,U)(B,V) - (B,U)(A,V)$$

for A, B satisfying certain conditions that will be determined later.

To describe the conditions, we first introduce some notation and conventions. With a slight abuse of notation, consider the elements in $\mathbb{C}^{2n+2}$ as pairs $\{a,b\}$ of elements $a, b \in \mathbb{C}^{n+1}$. Then if $A = \{a,b\}$ and $B = \{c,d\}$ are in $\mathbb{C}^{2n+1}$, $I(A,B) = I(\{a,b\},\{c,d\}) = (a,d) - (b,c)$ and $(A,B) = (\{a,b\},\{c,d\})) = (a,c) + (b,d)$. If $U = \{z,w\}, V = \{u,v\}$, then

$$\Box(l_{AB}(U,V)) = (a,d) - (b,c) = I(A,B).$$

To calculate $\Box(l_{AB}(U,V)^k)$, we use U, V, A, B as above and

$$\frac{\partial^2}{\partial U_i \partial V_{n+i+1}}(l(U,V)^k) = \frac{\partial^2}{\partial z_i \partial v_i}(l_{AB}(U,V)^k) =$$

$$\frac{\partial^2}{\partial z_i \partial v_i}([(a,z)+(b,w)][(c,u)+(d,v)] - [(a,u)+(b,v)][(c,z)+(d,w)])^k =$$

$$\frac{\partial}{\partial z_i}k(l_{AB}(U,V)^{k-1})(d_i[(a,z)+(b,w)] - b_i[(c,z)+(d,w)] =$$

$$k(k-1)(l_{AB}(U,V)^{k-2})(a_i[(c,u)+(d,v)]-c_i[(a,u)+(b,v)])(d_i[(a,z)+(b,w)]-b_i[(c,z)+(d.w)])+$$

$$k(l_{AB}(U,V)^{k-1})(a_id_i - b_ic_i).$$

Similarly,

$$\frac{\partial^2}{\partial V_i \partial U_{n+i+1}} l_{AB}(U,V)^k = \frac{\partial^2}{\partial w_i \partial u_i} l_{AB}(U,V)^k =$$

$$k(k-1)(l_{AB}(U,V)^{k-2})(b_i[(c,u) - d_[(a,u)+(b,v)])(c_i[(a,z)+(b,w)] - a_i[(c,z)+(d,w)])+$$

$$k(l_{AB}(U,V)^{k-1})(c_i b_i - a_i d_i).$$

Now subtracting the two identities leads to:

$$\Box(l_{AB}(U,V)^k) = (k^2+k)(l_{AB}(U,V)^{k-1})\sum_i (a_i d_i - b_i c_i) = (k^2+k)(l_{AB}(U,V)^{k-1})((a,d)-(b,c)).$$

Hence, we have the following result.

Lemma 8.3.1. *The holomorphic sections of the quantization bundle $\mathcal{O}_{\mathbb{F}_{is}}(k)$ are the linear combinations of the polynomials $p(U,V) = l_{AB}(U,V)^k$ satisfying*

$$(a,d) - (b,c) = 0.$$

Now for the functions which restrict to harmonic polynomials on $\mathbb{H}^n$ we take

$$V = jU = \{\overline{w}, -\overline{z}\}.$$

This transforms $\Box$ into the Laplacian of $\mathbb{H}^n$, so that $p(U, jU)$ are pull-backs of the eigenfunctions for the k-th eigenvalue λ_k. More precisely we have:

Theorem 8.3.2. *The eigenspace of the Laplace-Beltrami operator on $\mathbb{H}P^n$ corresponding to the kth eigenvalue is the span of the functions whose pull-back to $\mathbb{H}^{n+1}$ equals*

$$p(U,jU) = ([(a,z)+(b,w)][(c,\overline{w})-(d,\overline{z})] - [(c,z)+(d,w)][(a,\overline{w})-(b,\overline{z})])^k,$$

where $(a,d) - (b,c) = 0$.

Proof. Using the same reasoning as in [43] for the $\mathbb{C}P^n$, it remains to prove that the span $\mathcal{H}$ of all $p(U,jU)$ equals all of the space of harmonic polynomials on $\mathbb{H}^n$. For this, we adopt the proofs of Theorem 14.2 and 14.4 in [43], to the case of $\mathbb{H}^n$ and the quantization bundle $\mathcal{O}_{\mathbb{F}_{is}}(k)$. Lemma 8.3.1 provides a description of the space of holomorphic sections of $\mathcal{O}_{\mathbb{F}_{is}}(k)$ as a span of a set of functions. This space is a simple $Sp(n+1)$-module of highest weight $k\Lambda_2$, where Λ_2 is the second fundamental weight of $Sp(n+1)$. On the other hand, the λ_k-eignespace is also a simple $Sp(n+1)$-module of highest weight $k\Lambda_2$. Therefore the two spaces are isomorphic and set defined in the statement in the theorem corresponds to the spanning set defined in Lemma 8.3.1. $\square$

We note that a similar set of harmonic polynomials for $\mathbb{H}^{n+1}$ defining eigenfunctions on $\mathbb{H}P^n$ was found in [17], Proposition 3.4, but its span was not explicitly discussed there.

8.4 The space $SU(3)/SO(3)$

In this example we extend the results from [29] and find a generating set for all eigenfunctions of the Laplace-Beltrami operator on $SU(3)/SO(3)$. Following the notations there we denote by z, w etc. matrices in the Lie groups $SU(n)$ or $SO(n)$ (so $z \in SU(n)$). If $z\bar{z}^T = I$ with I being the identity matrix) and by Z, W matrices in the corresponding Lie algebras. Denote by z_{ij} the entries of z. The standard metric on $SU(n)$ is given by $g(Z, W) = \mathrm{Re}\left((Z\overline{W}^T)\right)$. Then the Laplace-Beltrami operator Δ on $SU(n)$ (denoted by τ in [29]) satisfies $\Delta(fg) = \Delta(f)g + k(f, g) + \Delta(g)f$ where $k(f, g) = g(\nabla f, \nabla g)$.

For $a \in \mathbb{C}^n$ we set

$$\phi_a = \sum_{j,\alpha} a_j a_\alpha \Phi_{j\alpha} = (z^T a a^T z) = (z^T a, z^T a).$$

By Proposition 4.1 in [29], ϕ_a is an $SO(n)$-invariant Δ-eigenfunction on $SU(n)$. Using the reasoning of [29], we may show that

$$\tilde{\phi}_a(z) = \sum_{j,\alpha} a_j a_\alpha \overline{\Phi_{j\alpha}} = (\bar{z}^T a a^T \bar{z}) = (\bar{z}^T a, \bar{z}^T a)$$

is also an $SO(n)$-invariant Δ-eigenfunction on $SU(n)$ with the same eigenvalue as ϕ_a.

More generally, for $a, b \in \mathbb{C}^n$ with $(a, b) = 0$ and $p, q \geq 0$, we have that $\phi_a^p \tilde{\phi}_b^q$ is also an $SO(n)$-invariant Δ-eigenfunction on $SU(n)$ of the same eigenvalue. Indeed, we can show that

$$k(\Phi_{j\alpha}, \overline{\Phi_{k\beta}}) = -2\delta_{r\alpha}\delta_{j\beta} - 2\delta_{\alpha\beta}\delta_{kj} + \frac{4}{n}\Phi_{j\alpha}\overline{\Phi_{k\beta}}$$

which implies that $k(\phi_a, \tilde{\phi}_b) = -4(a, b)^2 + \frac{4}{n}\phi_a\tilde{\phi}_b$. Then using that $(a, b) = 0$ and the general formula $k(f^p, h^q) = pqf^{p-1}h^{q-1}k(f, h)$ and the fact that $k(\phi_a, \tilde{\phi}_b) = 0$, we prove that $\phi_a^p \tilde{\phi}_b^q$

are eigenfunctions. Denote by $\lambda_{p,q}$ the eigenvalue of $\phi_a^p\tilde{\phi}_b^q$, $(a,b)=0$. Formulas for $\lambda_{p,q}$ in the case $n=3$ are listed in Example 5.0.3.

Consider the space $\mathcal{H}^{p,q} = \left\{\phi_a^p\tilde{\phi}_b^q \mid a,b\in\mathbb{C}^n, (a,b)=0\right\}$. Then $\mathcal{H}^{p,q}$ has an $SU(n)$-module structure via the formula

$$A\cdot(\phi_a^p\tilde{\phi}_b^q) = \phi_{\bar{A}a}^p\tilde{\phi}_{Ab}^q \tag{8.4.1}$$

(recall that $\bar{A}=\left(A^T\right)^{-1}$). We retain the notation from the $\mathbb{C}P^n$ case above, and example 5.0.3. In particular, we consider the flag $\mathbb{F} = SU(n)/S(U(1)\times U(1)\times U(n-2))$ to be embedded as a quadratic hypersurface in $n-1\times n-1$. Also, the holomorphic line bundles over $\mathbb{F}$ are denoted by L_{k_1,k_2} (see the proof of Theorem 6.3.5). Denote by Λ_i the ith fundamental weight of $SU(n)$.

Theorem 8.4.1. *The spaces $\mathcal{H}^{p,q}$ and $H^0(\mathbb{F},\mathcal{O}(L_{2p,2q}))$ are both isomorphic to the simple highest weight $SU(n)$-module of highest weight $2q\Lambda_1+2p\Lambda_{n-1}$. In the case $n=3$, we have that the Δ-eigenspace of $SU(3)/SO(3)$ of eigenvalue λ_k is spanned by the functions $\phi_a^p\tilde{\phi}_b^q$ for which $(a,b)=0$ and $\lambda_{p,q}=\lambda_k$.*

Proof. Recall from Section 8.2 that

$$H^0(\mathbb{F},\mathcal{O}(L_{2p,2q})) \simeq \mathcal{S}^{2p,2q}/((z,w)\mathcal{S}^{2p-1,2q-1}).$$

where $\mathcal{S}^{2p,2q}$ is space of polynomials in $z,w\in\mathbb{C}^n$ of homogeneous degree $(2p,2q)$.The space on the right hand side is a simple $SU(n)$-module of highest weight $2q\Lambda_1+2p\Lambda_{n-1}$, which is isomorphic to the space spanned by the functions $h_{a,b}^{p,q}$, $a,b\in\mathbb{C}^n$ with $(a,b)=0$, where

$$h_{a,b}^{p,q}(z,w) = (a,z)^p(b,w)^q.$$

This follows from Theorem 14.4 in [43]. Note that the discussion in [43] concerns polynomials in z and $\bar{z}$, but since they are treated as independent variables, the same results apply for polynomials in z,w. The action of $A\in SU(n)$ on $h_{a,b}^{p,q}$ is similar to the one in (8.4.1):

$$A\cdot h_{a,b}^{p,q} = h_{\bar{A}a,Ab}^{p,q}$$

. After verifying that the weights of $\phi_a^p \tilde{\phi}_b^q$ and $h_{a,b}^{2p,2q}$ coincide, we see that the map $\phi_a^p \tilde{\phi}_b^q \mapsto h_{a,b}^{2p,2q}$ leads to the desired isomorphism $\mathcal{H}^{p,q} \simeq H^0(\mathbb{F}, \mathcal{O}(L_{2p,2q}))$.

Since by the Borel-Weil Theorem $h_{a,b}^{m,n}$ generate all irreducible $SU(3)$-modules, from which we get that the eigenspaces of the Laplace-Beltrami operators on Riemannian symmetric spaces are finite sums of such modules, we obtain the result concerning $n = 3$. $\square$

CHAPTER 9

CONCLUSIONS, OPEN QUESTIONS, AND FUTURE DIRECTIONS

The results about the explicit form of harmonic polynomials leading to eigenfunctions on symmetric spaces $\mathbb{HP}^n$ and $SU(3)/SO(3)$ are new, but this is only the beginning of the program. The natural question is how to extend them to the other irreducible symmetric spaces. The first step is to extend the description to the space $SU(n)/SO(n)$ and the other symmetric spaces of maximal rank. In the near future, the first basic examples to consider are the Grasmannians - real complex and quaternionic, which are in fundamental in many questions. They are not all of maximal rank or rank one. Some indications on how to construct explicitly the eigenfunctions are coming from the papers by S. Gudmundsson and his collaborators. After comparing with the holomorphic side, we expect to also provide many more of the eigenfunctions which they have constructed and have the property that product of two of them is again an eigenfunction. In general, it is likely that all of the compact Riemannian symmetric spaces will have this type of descriptions.

9.1 Harmonic analysis on compact symmetric spaces

Following S. Helgason [33] and more recently S. Gindikin [20], using exactly the same spaces of our considerations and taking advantage of the natural connections of the dual fibrations and a Cauchy-type integral operator defined on these spaces called the Radon transform, and their applications to harmonic analysis on compact symmetric spaces. First we have double fibration of the (complex) Lie group G and its (closed Lie-) subgroups K and H, the the dual fibration in the form of the natural double fibration for the same homogeneous flag space G/L which appears as our quantization space, over the symmetric space, $mathbbF = G/L \xrightarrow{\pi_1} Z = G/K$, with $M = Z^{\mathbb{R}} = G^{\mathbb{R}}/K^{\mathbb{R}}$, and simultaneously a natural fibration over the corresponding horospherical manifold $\mathbb{F} \xrightarrow{\pi_2} \Theta = G/H$. In this set-up, L can be interpreted as $L = K \cap H$, and can be summarized by the dual double fibration diagrams below:

$$
\begin{tikzcd}
 & G \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
Z = G/K & & \Theta = G/H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{F} = G/L \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Z = G/K & & \Theta = G/H
\end{tikzcd}
$$

This natural connection along with Gindikin's horospherical Cauchy transform, an integral transform of Radon type, $R : \mathcal{O}(Z) \to \mathcal{O}(\Theta)$, with a corresponding inversion formula to prove the following theorem and suggest the development of a theory of harmonic analysis on the complex symmetric Stein manifolds $Z = G/K$ (which admits holomorphic functions ans contains horospheres, unlike its dual the *compact* Riemannian symmetric space $M := Z^{\mathbb{R}} = G^{\mathbb{R}}/K^{\mathbb{R}}$) which clearly does not. However, being canonical dual objects, the holomorphic functions on Z can be associated to M, the real form of Z, and allow for the analysis of harmonic functions on Z and therefore M, or in plain English, the study of harmonic analysis on compact Riemannian symmetric spaces, which with all the tools developed in the last few decades, is yelling to be studied rigorously, and everything I learned in the process of completing this dissertation may allow me to contribute whatever I can to this area of geometry and analysis gaining momentum and at its peak of current interest.

Theorem 9.1.1. *(S. Gindikin 2006, Theorem 1 [20])*

The spaces of holomorphic functions $\mathcal{O}(Z)$ and $\mathcal{O}(\Theta)$ are isomorphic as G-modules.

This approach uses the holomorphic language of analytic cohomology, with a very simple way to transfer to Dolbeault cohomology, as well as natural parallels with the Penrose transform which associates analytical cohomology by integration along complex cycles with some holomorphic functions. In fact, $\mathcal{O}(Z)$ can be considered as the intersection of $H^{n-l}(Z \setminus X(v), \mathcal{O})$ with $v \in V$, where the set V of cycles $X(v)$ is taken to be the set of all compact forms of the Stein symmetric manifold Z. The study of harmonic analysis on compact Riemannian symmetric spaces is a very promising area for further research.

9.2 Tabulation of Laplace-Beltrami eigenspectra of higher rank Riemannian symmetric spaces

The classification of the Laplacian eigenspectra of compact Riemannian symmetric spaces is summed up below, and can be found in many places within the relevant literature. The Laplace eigenspectra of CROSSes is well known (rank one), and can be found in many places within the relevant literature. Their calculation is substantially easier because the spectrums are one dimensional (generated by one element). The (same) table of the eigenspectra of the Laplacian operator Δ_M on the compact rank one symmetric spaces (CROSSes) M also mentioned in the introduction, and can be found in A. Besse [4] is again compiled below, for comparison of very clear research that remains to be done, and is within the scope of the author, for further future research.

What Tsanov, D. Grantcharov, and G. Grantcharov noticed was a relation between the Laplace eigenspectra $\frac{1}{2}\Delta_M$ on $M = G/K$ and the geometric quantization of $\mathrm{Geod}(M) = G/L$. The authors completed the CROSS case (i.e. rank 1), but the higher rank cases present a clear path forward for further research. The first higher rank example found is that of $M = SU(3)/SO(3)$ of rank two, and in this dissertation the table of rank two compact Riemannian symmetric spaces been completed, but still leaves plenty of work for the future on the tabulation of higher rank symmetric spaces $M = G/K$ with $rk(M) \geq 3$.

Table 4: The Laplace eigenspectrum of compact rank one Riemannian symmetric spaces

CROSS $M = G/K$	**Eigenspectrum Spec**(Δ_M)
$S^n = {}^{SO(n+1)}\diagup_{SO(n)}$	$\mathrm{Spec}(\Delta_{S^n}) = \{\lambda_k = k(k+n-1) \mid k \geq 0\}$
$\mathbb{R}P^n = {}^{SO(n+1)}\diagup_{SO(n) \times SO(1)}$	$\mathrm{Spec}(\Delta_{\mathbb{R}P^n}) = \{\lambda_k = 2k(2k+n-1) \mid k \geq 0\}$
$\mathbb{C}P^n = {}^{SU(n+1)}\diagup_{S(U(n) \times U(1))}$	$\mathrm{Spec}(\Delta_{\mathbb{C}P^n}) = \{\lambda_k = 4k(k+n) \mid k \geq 0\}$
$\mathbb{H}P^n = {}^{Sp(n+1)}\diagup_{Sp(n) \times Sp(1)}$	$\mathrm{Spec}(\Delta_{\mathbb{H}P^n}) = \{\lambda_k = 4k(k+2n+1) \mid k \geq 0\}$
$CaP^2 = {}^{F_4}\diagup_{Spin(9)}$	$\mathrm{Spec}(\Delta_{CaP^2}) = \{\lambda_k = 4k(k+11) \mid k \geq 0\}$

A similar table can be created for the quantization space G/L of higher rank spaces from the results of [26] and this dissertation. Table 5 below summarizes the rank one case. Compiling a similar table of the flag manifolds representing the quantization spaces of higher rank Riemannian symmetric spaces would also be fruitful avenue of further research.

This dissertation contains the results to construct the table of Laplace eigenspectra of compact rank two Riemannian symmetric spaces. Also the method for computing the eigenspectra for rank two spaces is presented, allowing for the tabulation of rank two spaces and their eigenspectra, a direct analogy of Table 3 for spaces one rank higher. The second column would no longer signify the geodesic flow but the higher dimensional generalization of the quantization space, which mentioned much earlier, is the space (which has the structure of a genealized manifold) parametrizing all maximal flat totally geodesic tori, roughly sitting inside the Lie algebra. This is another clear avenue of further future research would be the tabulation of the quantization space for the higher rank compact Riemannian symmetric space.

Table 5: The geodesic flow of compact rank one Riemannian symmetric space

CROSS $M = G/K$	**Quantization space of geodesic flow** $G/L = \mathrm{Geod}(M)$
$S^n = \frac{SO(n+1)}{SO(n)}$	$\mathrm{Geod}(S^n) = \frac{SO(n+1)}{SO(n)\times U(1)}$
$\mathbb{R}P^n = \frac{SO(n+1)}{S(O(n)\times O(1))}$	$\mathrm{Geod}(\mathbb{R}P^n) = \frac{SO(n+1)}{S(O(1)\times O(n-1)\times U(1))}$
$\mathbb{C}P^n = \frac{SU(n+1)}{S(U(n)\times U(1))}$	$\mathrm{Geod}(\mathbb{C}P^n) = \frac{SU(n+1)}{S(U(1)\times U(n-1)\times U(1))}$
$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$	$\mathrm{Geod}(\mathbb{H}P^n) = \frac{Sp(n+1)}{Sp(n-1)\times Sp(1)\times U(1)}$
$\mathbb{C}aP^2 = \frac{F_4}{Spin(9)}$	$\mathrm{Geod}(\mathbb{C}aP^2) = \frac{F_4}{SO(7)\times U(1)}$

9.3 Conclusion

We end this statement by noting that the amount of work remaining is quite large, but well within the scope of our investigations. Ideally, one would like to continue the current project until completion, including higher rank versions of Tables 2 and 3, as well as explicit expressions for the Laplace-Beltrami eigenfunctions for higher rank spaces, in terms of harmonic polynomials and Borel-Weil-Bott theory as we did for the first few examples. Then there is also the other related work like developing a theory of harmonic analysis on compact Riemannian symmetric spaces in the way mentioned briefly in the previous section. It will definitely be challenging work, but ultimately I am confident the program will work and is within the purview of myself and my advisor and collaborators, and beside the current paper in progress, there is at the very least a few more papers that can spring out of the investigations outlined in this chapter.

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