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A Comparison of Some Confidence Intervals for Estimating the Kurtosis Parameter

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A COMPARISON OF SOME CONFIDENCE INTERVALS FOR ESTIMATING THE KURTOSIS PARAMETER

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To: Dean Michael R. Heithaus
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This thesis, written by Guensley Jerome, and entitled A Comparison of Some Confidence intervals for Estimating the Kurtosis Parameter, having been approved in respect to style and intellectual content, is referred to you for judgment.

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Florida International University, 2017

DEDICATION

I would like to dedicate this work to my amazing wife Nelssie and my family who unselfishly sacrificed so much for me.

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ABSTRACT OF THE THESIS

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by

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Professor B.M. Golam Kibria, Major Professor

Several methods have been proposed to estimate the kurtosis of a distribution. The three common estimators are: g_2, G_2 and b_2. This thesis addressed the performance of these estimators by comparing them under the same simulation environments and conditions. The performance of these estimators is compared through confidence intervals by determining the average width and probabilities of capturing the kurtosis parameter of a distribution. We considered and compared classical and non-parametric methods in constructing these intervals. Classical method assumes normality to construct the confidence intervals while the non-parametric methods rely on bootstrap techniques. The bootstrap techniques used are: Bias-Corrected Standard Bootstrap, Efron's Percentile Bootstrap, Hall's Percentile Bootstrap and Bias-Corrected Percentile Bootstrap. We have found significant differences in the performance of classical and bootstrap estimators. We observed that the parametric method works well in terms of coverage probability when data come from a normal distribution, while the bootstrap intervals struggled in constantly reaching a 95% confidence level. When sample data are from a distribution with negative kurtosis, both parametric and bootstrap confidence intervals performed well, although we noticed that bootstrap methods tend to have smaller intervals. When it comes to positive kurtosis, bootstrap methods perform slightly better than classical methods in coverage probability. Among the three kurtosis estimators, G_2 performed better. Among bootstrap techniques, Efron's Percentile intervals had the best coverage.

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INTRODUCTION

1.1 Kurtosis and Misconception

Kurtosis is one of the more obscure statistics parameters and has not been discussed by many. To begin, we would first want to define what kurtosis is. The historical misconception is that the kurtosis is a characterization of the peakedness of a distribution. In various books, the kurtosis is described as the "flatness or peakedness of a distribution" (Van Belle et al., 2004) when in reality, the kurtosis is directly related to the tails of a given distribution. The paper aptly titled: *Kurtosis as peakedness, 1908 - 2014, R.I.P.* (Westfall, 2014) strongly addressed said misconception. He wrote: *"Kurtosis tells you virtually nothing about the shape of the peak – its only unambiguous interpretation is in terms of tail extremity."* His claims were backed up with numerous examples of why you cannot relate the peakedness of the distribution to kurtosis. So now, we can define kurtosis: it is related to the tails of the distribution, not the peakedness or flatness. It simply measures how heavy the tail of a distribution is. With longer tails, we get more outliers while shorter tails produce a lot fewer to no outliers. Distributions with positive kurtosis, or leptokurtic, have long tails (Ex: a Student t Distribution) and distributions with negative kurtosis, or platykurtic, have short tails (Ex: Uniform Distribution). Distributions with zero kurtosis are referred to as mesokurtic (Ex: Normal Distribution) (Van Belle et al., 2004)

1.2 Population Kurtosis and Estimators

Kurtosis, κ, is known as one of the shape parameters of a probability model. The kurtosis parameter of a probability distribution was first defined by Karl Pearson in 1905 (Westfall, 2014) to measure departure from normality. He defined it:

$$\kappa(X) = \frac{\mu_4}{\sigma^4} = \frac{\mathbb{E}(X-\mu)^4}{\Big(\mathbb{E}(X-\mu)^2\Big)^2}$$

where $\mathbb{E}$ is the expectation operator, μ is the mean, μ_4 is the fourth moment about the mean, and σ is the standard deviation. The Normal distribution with a mean μ and variance σ^2 has a kurtosis of 3. Often statisticians adjust this result to zero, meaning the kurtosis minus 3 equals zero. When an adjustment is made, it is usually referred to as Excess Kurtosis. In the present thesis, excess kurtosis is defined as

$$\mathrm{Kurt}(X) = \frac{\mu_4}{\sigma^4} - 3 = \frac{\mathbb{E}(X-\mu)^4}{\Big(\mathbb{E}(X-\mu)^2\Big)^2} - 3.$$

The excess kurtosis defined above is the parameter of a given distribution. To estimate the distribution's parameter, three kurtosis estimators have been proposed. They are g_2, G_2 and b_2.

1.2.1 Estimator g_2

By replacing the population moments with sample moments, we can then define the first estimator of the excess kurtosis, usually referred to as g_2.

$$\mathrm{g}_2 = \frac{m_4}{m_2^2} - 3$$

for a given sample size n with

$$m_r = \frac{1}{n}\sum(x_i - \bar{x})^r$$

with variance

$$\mathrm{var}(\mathrm{g}_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)} \quad \text{(Cramér, 1947).}$$

Fisher showed that the excess kurtosis estimator g_2 is an biased estimator since $\mathbb{E}(g_2) = -\frac{6}{n+1}$ (Fisher, 1930). To make g_2 an unbiased estimator, we can simply apply the correction of $-\frac{n+1}{6}$, but according to Joanes and Gill(1998), it is preferred to use ratios of unbiased cumulants to construct unbiased estimators of kurtosis.

1.2.2 *Estimator* G_2

First, we describe a cumulant-generating function, $K(t)$. In statistics, cumulants are values that provide an alternative to the moments of a probability distribution. The moments can determine the cumulants and vice versa. This means that two probability distributions that have the same moments will also have the same cumulants. Before we give a more rigorous definition of the cumulant generating function, let us recall that the moment generating function for a random variable x is defined as:

$$M_X(t) = \mathbb{E}\big[e^{tX}\big] \tag{1.1}$$

$$= \mathbb{E}\left(1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right) \tag{1.2}$$

$$= \sum_{r=0}^{\infty} \frac{\mu_r \cdot t^r}{r!}, \quad \text{where} \quad \mu_r = \mathbb{E}(X^r). \tag{1.3}$$

From the moment generating function, we now define the cumulant generating function as the natural log of an MGF.

$$K_X(t) = \ln(M_X(t)).$$

From this definition, we can calculate the first cumulant k_1, as:

$$K_X'(t) = \frac{M_X'(t)}{M_X(t)}$$

at $t = 0$ we would get

$$K_X'(0) = \frac{M_X'(0)}{M_X(0)} \tag{1.4}$$

$$= M_x'(0) = \mathbb{E}(X) \tag{1.5}$$

$$= \mathbb{E}(X) = \mu_1' \tag{1.6}$$

This is easy to see:

$$M_x(0) = \mathbb{E}(1 + 0x + \frac{t^2 \cdot 0}{2!} + \dots) = 1.$$

We also can calculate the second cumulant as follows:

$$K_X''(t) = \frac{M_X(t)M_X''(t) - M_X'(t)^2}{M_X(t)^2}$$

at t=0

$$K_X''(0) = M_X''(0) - M_X'(0)^2 \tag{1.7}$$

$$= \mathbb{E}X^2 - (\mathbb{E}X)^2 \tag{1.8}$$

$$= \mu_2' - {\mu_1'}^2 \tag{1.9}$$

$$= \mathrm{Var}(X). \tag{1.10}$$

Therefore, the k-th cumulant of the k-th terms in the Taylor series expansion at 0 is

$$k_n(X) = \frac{1}{n!}\frac{d^n}{dt^n}K_X(0) \quad \text{(Watkins, 2009)}$$

Based on the general formula above, If we continue to get the cumulant generating function where we can show that

$$k_3 = 2{\mu_1'}^3 - 3\mu_1'\mu_2' + \mu_3' \tag{1.11}$$

$$k_4 = -6{\mu_1'}^4 + 12{\mu_1'}^2\mu_2' - 3{\mu_2'}^2 - 4\mu_1'\mu_3' + \mu_4'. \tag{1.12}$$

After deriving k_1 and k_2, we can see that they are equivalent to

$$K_1 = \mu$$

$$K_2 = \mu_2.$$

If we write the other cumulant generating functions in terms of the central moments, we would get:

$$K_3 = \mu_3 \quad \text{and} \tag{1.13}$$

$$K_4 = \mu_4 - 3\mu_2^2 \tag{1.14}$$

As it was previously defined, the excess kurtosis is

$$\mathrm{Kurt}(X) = \frac{\mu_4}{\sigma^4} - 3.$$

Then, in terms of the population cumulant, the excess kurtosis can also be defined as

$$\gamma = \frac{K_4}{K_2^2} - 3 \quad \text{(Joanes and Gill, 1998)}.$$

Assume an unbiased cumulant estimator, c_j, for which $\mathbb{E}(c_j) = K_j$. Then, Cramer(1947) shows that the unbiased sample cumulants c_j are

$$c_2 = \frac{n}{n-1} m_2 \tag{1.15}$$

$$c_3 = \frac{n^2}{(n-1)(n-2)} m_3 \quad \text{and} \tag{1.16}$$

$$c_4 = \frac{n^2}{(n-1)(n-2)(n-3)} \Big\{ (n+1)m_4 - 3(n-1)m_2^2 \Big\}. \tag{1.17}$$

We now construct the kurtosis estimator, G_2, solely using cumulant estimators (Joanes and Gill, 1998)

$$\mathrm{G}_2 = \frac{c_4}{c_2^2} \tag{1.18}$$

$$= \frac{n-1}{(n-2)(n-3)} \{(n+1)g_2 + 6\}. \tag{1.19}$$

G_2, estimator we derived above is the excess kurtosis estimator adopted by statistical packages such as SAS and SPSS (Bruin, 2011). It is generally biased, but unbiased for the normal distribution.

Its variance is:

$$\mathrm{var}(\mathrm{G}_2) = \mathrm{var}\left(\frac{n-1}{(n-2)(n-3)}\{(n+1)\mathrm{g}_2+6\}\right) \tag{1.20}$$

$$= \left[\frac{(n-1)(n+1)}{(n-2)(n-3)}\right]^2 \cdot \mathrm{var}(g_2). \tag{1.21}$$

The following approximation can be used to estimate the $\mathrm{var}(G_2)$: $\mathrm{var}(\mathrm{G}_2) \approx \big(1+10/n\big) \cdot \mathrm{var}(\mathrm{g}_2)$ for all $n > 3$

1.2.3 Estimator b_2

If we consider how g_2 is defined, m_2^2 derived from the sample moment, is a biased estimator of the sample standard deviation. Using the unbiased standard deviation of the sample instead would give us the third excess kurtosis estimator. We refer to it as b_2, and it is used by computer software packages such as MINITAB and BMDP (Joanes and Gill, 1998). It is defined as

$$\mathrm{b}_2 = \frac{m_4}{s^4} - 3$$

where

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}}.$$

Expanding the definition of b_2, we get

$$\mathrm{b}_2 = \frac{\frac{\sum(x_i-\bar{x})^4}{n}}{\left(\sqrt{\frac{\sum(x_i-\bar{x})^2}{n-1}}\right)^4} - 3 \tag{1.22}$$

$$= \left(\frac{n-1}{n}\right)^2 \frac{\sum(x_i-\bar{x})^4}{\left(\sum(x_i-\bar{x})^2\right)^2} - 3 \tag{1.23}$$

$$= \left(\frac{n-1}{n}\right)^2 \cdot \frac{m_4}{m_2^2} - 3 \tag{1.24}$$

which also is an alternative way of defining b_2.

In order to get the variance of b_2, let us first rewrite b_2 in terms of g_2. We would have:

$$\mathrm{b}_2 = \left(\frac{n-1}{n}\right)^2 \cdot \mathrm{g}_2 + 3\left[\left(\frac{n-1}{n}\right)^2 - 1\right]$$

Then variance of b_2 is

$$\mathrm{var}(\mathrm{b}_2) = \left(\frac{n-1}{n}\right)^4 \mathrm{var}(\mathrm{g}_2). \tag{1.25}$$

We use the following approximation $\mathrm{var}(\mathrm{b}_2) \approx \big(1 - 4/n\big) \cdot \mathrm{var}(\mathrm{g}_2)$ for all $n > 1$.

From the approximations of $\mathrm{var}(\mathrm{G}_2) \approx \big(1 + 10/n\big) \cdot \mathrm{var}(\mathrm{g}_2)$ and $\mathrm{var}(\mathrm{b}_2) \approx \big(1 - 4/n\big) \cdot \mathrm{var}(\mathrm{g}_2)$, it's easy to see that $\mathrm{var}(\mathrm{G}_2)$ will always be greater than both $\mathrm{var}(\mathrm{g}_2)$ and $\mathrm{var}(\mathrm{b}_2)$ since the term $1+10/n$ will always be a value greater than 1. In that same manner, $\mathrm{var}(b_2)$ estimation will be less than $\mathrm{var}(g_2)$ and $\mathrm{var}(G_2)$ since the term $1 - 4/n$ will always be a value between 0 and 1. Therefore, we can write

$$\mathrm{var}(b_2) \leq \mathrm{var}(g_2) \leq \mathrm{var}(G_2).$$

The objective of this paper is to compare several confidence intervals using both classical and bootstrap methods for the kurtosis and find which interval methods that would best estimate the kurtosis parameter of distributions with zero, positive or negative kurtosis. Since a theoretical comparison is not possible, a simulation study has been made. Average width and coverage probabilities are considered as criterion of good estimators. The organization of this thesis is as follows: we define both parametric and non-parametric confidence intervals in Chapter 2. Chapter 3 we discuss some distributions and compare their kurtosis. A simulation study is described in Chapter 4. Two real life data sets are analyzed in Chapter 5. Last, some concluding remarks are given in Chapter 6.

CHAPTER 2

CONFIDENCE INTERVALS

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random sample of size n from a population with mean μ and variance σ^2. Given a specific level of confidence, we can construct confidence intervals to estimate a given parameter of the distribution of concern. As we are studying kurtosis in my paper, then the excess kurtosis parameter $\mathrm{Kurt}(X)$ of the population will be the value we will want to estimate. We will rely on two main approaches, parametric and non-parametric approaches, to construct confidence intervals with $(1-\alpha)100\%$ confidence level

2.1 Parametric Approach

The general format of parametric confidence intervals is

$$\text{estimator} \pm \text{critical value} \times \text{standard error of estimator}$$

Given this general format, to construct confidence intervals for excess kurtosis parameter of a given population, we will use one of the three estimators g_2, G_2 and b_2 for a sample of size n, with their respective standard error and critical value $z_{\alpha/2}$ which is the upper $\alpha/2$ percentile of the standard normal distribution (Joanes and Gill, 1998).

- For estimator g_2 with sample size n, the $(1-\alpha)100\%$ confidence interval will be:

$$\mathrm{g}_2 \pm z_{\alpha/2} \cdot \sqrt{\mathrm{var}(\mathrm{g}_2)} \tag{2.1}$$

$$\mathrm{g}_2 \pm z_{\alpha/2} \cdot \sqrt{\frac{24n(n-2)(n-3)}{(n-1)^2(n+3)(n+5)}}. \tag{2.2}$$

- For estimator G_2 with sample size n, the $(1-\alpha)100\%$ confidence interval will be:

$$\mathrm{G}_2 \pm z_{\alpha/2} \cdot \sqrt{\mathrm{var}(\mathrm{G}_2)} \tag{2.3}$$

$$\mathrm{G}_2 \pm z_{\alpha/2} \cdot \sqrt{\frac{24n(n-1)^2}{(n-2)(n-3)(n+3)(n+5)}}. \tag{2.4}$$

- For estimator b_2 with sample size n, the $(1-\alpha)100\%$ confidence interval will be:

$$\mathrm{b}_2 \pm z_{\alpha/2}\sqrt{\frac{24n(n-1)^4(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}}.$$

2.2 Bootstrap Approach

DiCiccio and Efron (1981) argued that parametric confidence intervals can be quite inaccurate in practice since they rely on asymptotic approximation. Meaning that the sample size n used to estimate parameter of a population is assumed to grow indefinitely (Efron, 1987) while the bootstrap process does not need to worry about such assumption. The basic idea of the bootstrap is re-sampling from a sample of size n with replacement in order to derive the different bootstrap statistics. The process goes as follows: assume $x = (x_1, x_2, \cdots, x_n)$ be a sample of size n. Let there be a bootstrap sample $x* = (x_1^*, x_2^*, \ldots, x_n^*)$ obtained by randomly sampling, with replacement, from the original sample x, of size n. We then calculate the bootstrap statistics from $x*$. The bootstrap statistics in question in this paper is the kurtosis. Repeat this process B-time, where B is expected to be at least 1000 to get reliable results(Efron, 1979). The original sample where bootstrap samples are drawn through re-sampling is referred to as the empirical distribution. The bootstrap method is a non-parametric tool where we do not need to know much about the underlying distribution to make statistical inference such as constructing confidence intervals to estimate the parameter of a population. Bootstrapping process is best used through the aid of a computer since the number of bootstrap samples needed, B, are expected to be large. We will consider the following bootstrap confidence intervals.

2.2.1 *Bias-Corrected Standard Bootstrap Approach*

Let θ be one of the three point estimators for kurtosis previously defined. Then the bias-corrected standard bootstrap confidence intervals is

$$\theta - \mathrm{Bias}(\theta) \pm z_{\alpha/2}\hat{\sigma}_B$$

$$2\theta - \bar{\theta} \pm z_{\alpha/2}\hat{\sigma}_B$$

where $\hat{\sigma}_B = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}(\theta_i^* - \bar{\theta})^2}$ is the bootstrap standard deviation, $\bar{\theta} = \frac{1}{B}\sum_{i=1}^{B}\theta_i^*$ is the bootstrap mean and $\mathrm{Bias}(\theta) = \bar{\theta} - \theta$ (Sergio and Kibria, 2016).

2.2.2 *Efron's Percentile Bootstrap Approach*

Introduced by Efron (1981) , Efron's Percentile Bootstrap approach is to construct a $100(1-\alpha)\%$ percentile confidence interval. Let $\theta^*_{L,\alpha/2}$ represents the value for which $(\alpha/2)\%$ bootstrap estimates are less than and $\theta^*_{H,\alpha/2}$ the value for which $(\alpha/2)\%$ bootstrap estimates exceed. Then the confidence interval would have the following lower and upper bounds:

$$L = \theta^*_{(\alpha/2)\times B} \quad \text{and} \quad U = \theta^*_{1-(\alpha/2)\times B},$$

in order to get the following interval

$$[\theta^*_{L,\alpha/2}, \theta^*_{H,\alpha/2}].$$

2.2.3 *Hall's Percentile Bootstrap Approach*

Introduced by Hall (1992), the method uses the bootstrap on distribution of $\theta^* - \theta$. For any of the estimators previously defined, we sample from the empirical distribution, calculate estimates from the B bootstrap samples

$$\theta_1^*, \theta_2^*, \theta_3^*, \ldots, \theta_B^*.$$

The difference between each bootstrap estimate and the population parameter is taken to get

$$\theta_1^* - \theta, \theta_2^* - \theta, \theta_3^* - \theta, \ldots, \theta_B^* - \theta$$

We can label each $\theta_i^* - \theta$ as δ_i^* to have

$$\delta_1^*, \delta_2^*, \delta_3^*, \cdots, \delta_B^*$$

Like Efron's method, for a value $\delta_{L,\alpha/2}^*$ which $(\alpha/2)\%$ of the δs are less than and for a value $\delta_{H,\alpha/2}^*$ for which $(\alpha/2)\%$ of the δs exceed. The lower and upper bound of the confidence interval will be given by:

$$L = 2\theta - \theta_{(1-\alpha/2)\times B}^* \quad \text{and} \quad U = 2\theta - \theta_{\alpha/2\times B}^*.$$

2.2.4 Bias Corrected Percentile Bootstrap

Efron (1981) proposed the method when sample estimators consistently under or over estimate its parameter. Efron suggested that instead of using the usual 0.025 and 0.975 percentiles of the bootstraps, we should use $b_{0.025}$ and $b_{0.975}$ instead. They are calculated as:

$$b_{0.975} = \Phi\Big(p^* + \frac{p^* + 1.96}{1 - a(p^* + 1.96)}\Big) \quad \text{and} \quad b_{0.025} = \Phi\Big(p^* + \frac{p^* - 1.96}{1 - a(p^* - 1.96)}\Big),$$

where:

- $\Phi(\cdot)$ is the standard normal cumulative distribution function (CDF)
- p^* is the bias-correction that is calculated as $\Phi^{-1}\left(\frac{|\theta_i^* < \theta|}{B}\right)$ which is the inverse normal cdf of the proportion of bootstrap statistics values that are less than the empirical sample statistics.
- a is the "acceleration factor". For normal bootstrap processes, $a = 0.000$

We calculate the confidence intervals as:

$$L = \theta_{\Phi(2p^* - 1.96)}^* \quad \text{and} \quad U = \theta_{\Phi(2p^* + 1.96)}^*$$

CHAPTER 3

DISTRIBUTIONS AND THEIR KURTOSIS

To compare the performance of the kurtosis estimators previously defined, we want to construct confidence intervals using either parametric or bootstrap methods. The data that are to be used will be coming from different distributions with kurtosis of zero, positive and negative to properly gauge the performance of g_2, G_2 and b_2 kurtosis estimators. Recall that we are concentrating with finding the Excess Kurtosis, $\mathrm{Kurt}(X) = \kappa - 3$. We know that sample size is an important factor in constructing confidence intervals, so we consider performing our simulation using a range of possible sample sizes. We use $n = 10, 20, 30, 50, 100$ and 300, which represent small to large sample sizes. Since we want to capture positive, zero and negative kurtosis, the distributions used are the following:

- Zero Kurtosis:
  - **Normal Distribution:** $X \sim \mathcal{N}(\mu, \sigma)$
    - Mean: μ
    - Variance: σ^2
    - Excess Kurtosis: 0

    A Normal Distribution With Zero Excess Kurtosis Is Shown In Figure 3.1

- Negative Kurtosis:
  - **Uniform Distribution** $X \sim \mathcal{U}(a, b)$
    - Mean: $\frac{1}{2}(a+b)$
    - Excess Kurtosis: $-\frac{6}{5}$

    A uniform distribution with excess kurtosis is shown in Figure 3.2
  - **Beta Distribution: X** $\sim \mathrm{Beta}(2, 2)$
    - Shape Parameter: 2
    - Shape Parameter: 2
    - Excess Kurtosis: -0.8571429

    A beta distribution with excess kurtosis of -0.8571429 is shown in Figure 3.3a
  - **Beta Distribution: X** $\sim \mathrm{Beta}(2, 5)$
    - Shape Parameter: 2
    - Shape Parameter: 5
    - Excess Kurtosis: -0.12

    A beta distribution with excess kurtosis of -0.12 is shown in Figure 3.3
- Positive Kurtosis:
  - **Logistic Distribution: X** $\sim \mathrm{logis}(\mu, \sigma)$
    - Location Parameter: μ
    - Scale Parameter: σ
    - Excess Kurtosis: $\frac{6}{5}$

    A logistic distribution with excess kurtosis of $6/5$ is shown in Figure 3.5

- **Student t-distribution** $X \sim T_{\nu=10}$
  - Mean: 0, for $\nu > 0$
  - Degree of Freedom: 10
  - Excess Kurtosis: 1 for $\nu > 4$

  A t-distribution with excess kurtosis of 10 is shown in Figure 3.6a

- **Student t-Distribution** $X \sim T_{\nu=64}$
  - Mean: 0, for $\nu > 0$
  - Degree of Freedom: 64
  - Excess Kurtosis: 0.1 for $\nu > 4$

  A t-distribution with excess kurtosis of 64 is shown in Figure 3.6b

- **Double Exponential** $X \sim \mathrm{DExp}(\mu, \beta)$
  - Location Parameter: μ
  - Scale Parameter: β
  - Excess Kurtosis: 3

  A double exponential distribution with excess kurtosis of 3 is shown in Figure 3.4

3.1 Zero Kurtosis

The excess kurtosis was defined so that the Kurtosis of the normal distribution is 0. Therefore the only distribution that will be presented under this section is that of the normal distribution.

3.1.1 Normal Distribution

The normal distribution is probably the most commonly used and well studied probability distribution in statistics. Given a mean μ and variance σ^2, the normal distribution is defined as follows:

A random variable X has a normal distribution if and only if its probability density is given by

$$\phi(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\qquad -\infty<x<\infty,\quad -\infty<\mu<\infty,\quad \sigma>0.$$

We refer to standard normal distribution, a normal distribution with mean $\mu=0$ and variance $\sigma^2=1$ written as:

$$\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

Then, from basic derivative of exponential functions we have:

$$\phi'(z)=-z\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}=-z\cdot\phi(z).$$

From above, we can show some properties of the standard normal.

Property 3.1.1.1: $\phi(z)\to 0$ as $z\to\pm\infty$

Proof. It is clear to see that as $z\to\pm\infty$, then

$$\lim_{z\to\pm\infty}\phi(z)=\frac{1}{\sqrt{2\pi}}\cdot\lim_{z\to\pm\infty}e^{-\frac{1}{2}z^2}=0$$

□

Property 3.1.1.2: For a standard normal distribution Z and for n $\in\mathbb{N}_+$ then $\mathbb{E}(Z^{n+1})=n\mathbb{E}(Z^{n-1})$

Proof. Recall that $\phi'(z)=-z\cdot\phi(z)$ and $\phi(z)\to 0$ as $z\to\pm\infty$
Calculating the expected value of Z^{n+1} of the standard normal distribution gives us:

$$\begin{aligned}\mathbb{E}(Z^{n+1})&=\int_{-\infty}^{\infty}z^{n+1}\phi(z)dz &(3.1)\\ &=\int_{-\infty}^{\infty}z^{n}z\phi(z)dz &(3.2)\\ &=-\int_{-\infty}^{\infty}z^{n}\phi'(z)dz &(3.3)\\ & &(3.4)\end{aligned}$$

let $u=z^n$ and $dv=\phi'(z)dz$, integrating by part gives us

$$\begin{aligned}\mathbb{E}(Z^{n+1})&=-z^n\phi(z)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}nz^{z-1}\phi(z)dz &(3.5)\\ &=n\mathbb{E}(Z^{n-1}) &(3.6)\end{aligned}$$

□

From proving the first two properties, we now can show that the excess kurtosis of a normal distribution is $\mathrm{Kurt}(X) = 0$

Proof. Recall that any normal distribution can be written as a standard normal distribution. Then for

$$X \sim \mathcal{N}(\mu, \sigma) \to Z = \frac{X - \mu}{\sigma}$$

with

$$Z \sim \mathcal{N}(0, 1)$$

then Kurtosis definition is $\mathrm{Kurt}(Z) = \mathbb{E}(Z^4)$ then

$$\mathbb{E}(Z^4) = 3\mathbb{E}(Z^2) \qquad \text{from } \textbf{Property 3.1.1.2}$$

and because

$$\mathrm{var}(Z) = 1 \tag{3.7}$$

$$\mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 = 1 \tag{3.8}$$

because $\mathbb{E}(Z) = 0$, then $\mathbb{E}(Z^2) = 1$,therefore,

$$\mathbb{E}(Z^4) = 3 \times 1 = 3$$

With excess kurtosis, then

$$\mathbb{E}(Z^4) = 3 - 3 = 0$$

□

3.2 Negative Kurtosis

We now consider the different distributions with negative excess kurtosis. Recall that, a distribution that has a $\mathrm{Kurt}(X) < 0$ are distributions with little to no outliers. Here are some of the distributions we studied in this paper with negative kurtosis.

3.2.1 Uniform Distribution

In statistics, the continuous uniform distribution such that for each member of the family, all intervals of the same length on the distribution's support are equally probable. The support is defined by the two parameters, a and b, which are its minimum and maximum values. The distribution is

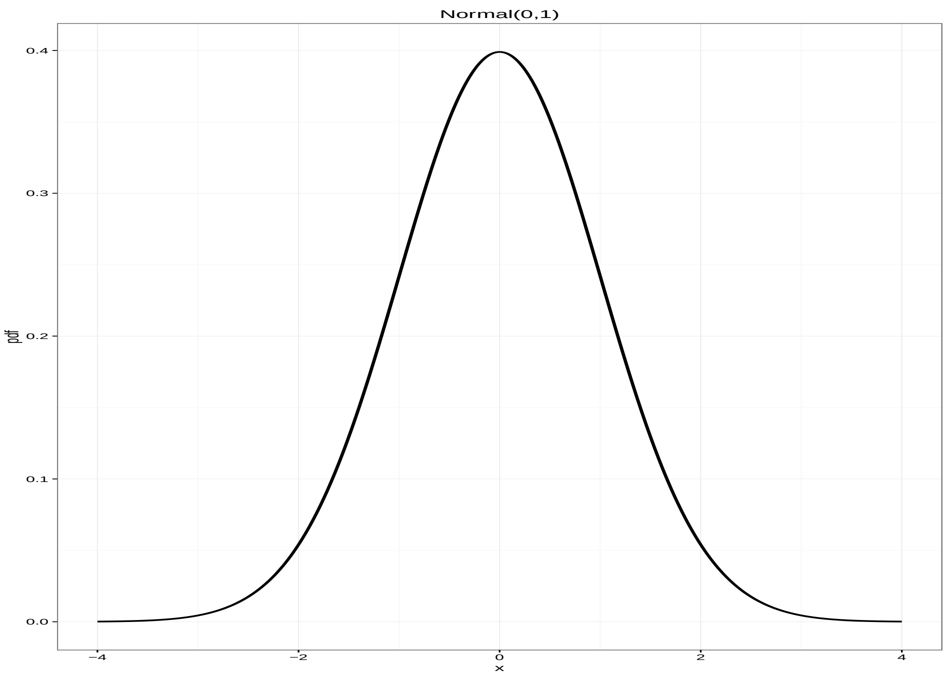


FIGURE 3.1: A Normal Distribution Illustrating Zero Excess Kurtosis

defined as:

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } x \in \text{[a,b]} \qquad -\infty < a < b < \infty. \\ \\ 0, & \text{otherwise} \end{cases}$$

As for this paper, we concentrate on the standard uniform distribution defined as

$$f(x) = 1 \qquad 0 \leq x \leq 1.$$

We prove the following

Property 3.2.1.1 If $X \sim \mathcal{U}(0,1)$, then the *n*th moment is $\mathbb{E}(X^n) = \frac{1}{n+1}$

Proof. Since $X \in [0,1]$ for a standard uniform, then

$$\mathbb{E}(X^n) = \int_0^1 x^n dx \tag{3.9}$$

$$= \frac{x^{n+1}}{n+1}\Big|_0^1 \tag{3.10}$$

$$= \frac{1}{n+1}. \tag{3.11}$$

□

It is also easy to see that the $\mathbb{E}(X)$ of the standard uniform distribution is $\frac{1}{2}$ since

$$\mathbb{E}(X) = \int_0^1 x dx = \frac{1}{2}$$

and its variance is:

$$\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 \tag{3.12}$$

$$= \int_0^1 x^2 dx - 1/4 \tag{3.13}$$

$$= 1/3 - 1/4 \tag{3.14}$$

$$= 1/12. \tag{3.15}$$

Next, we can show that the excess kurtosis of the standard uniform distribution is: -6/5. First, we begin with the definition of the excess kurtosis:

$$\text{Kurt}(X) = \frac{\mathbb{E}(X-\mu)^4}{\sigma^4} - 3$$

We evaluate the numerator of the excess kurtosis definition to have:

$$\mathrm{E}(X-\mu)^4 = 12^2 \int_0^1 (x-1/2)^4 dx$$

with a U-substitution, having $u = x - 1/2$, then

$$= 12^2 \int_{-1/2}^{1/2} u^4 du \tag{3.16}$$

$$= 12^2 \cdot \left.\frac{u^5}{5}\right|_{-1/2}^{1/2} \tag{3.17}$$

$$= \frac{2}{160} = \frac{1}{80}. \tag{3.18}$$

$$\tag{3.19}$$

Therefore

$$\frac{\frac{1}{80}}{\left(\frac{1}{12}\right)^2} = \frac{9}{5}$$

which leads to an excess kurtosis of $-\frac{6}{5}$

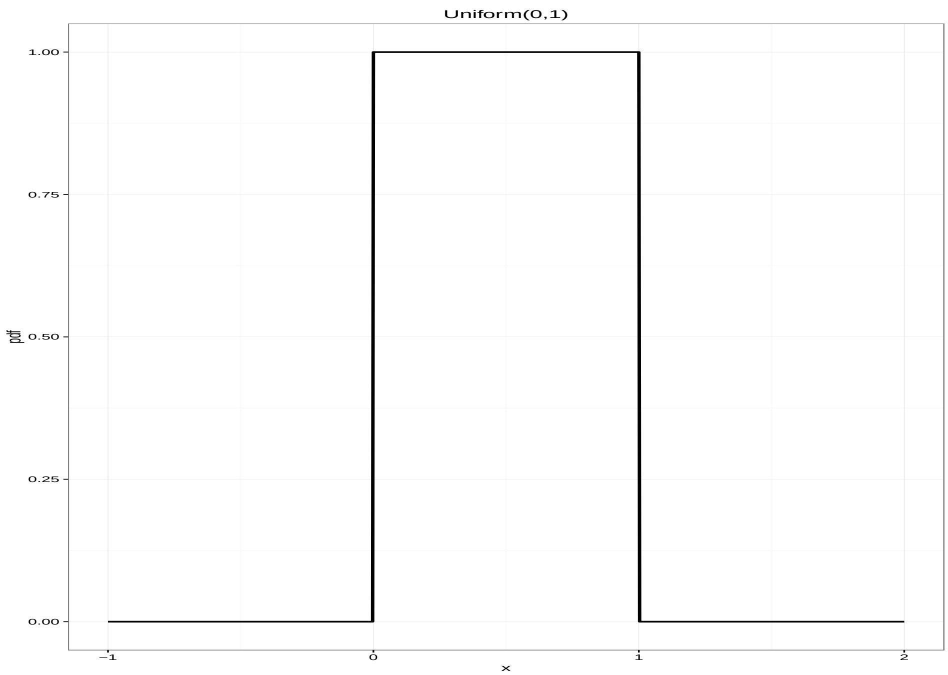


FIGURE 3.2: A Uniform Distribution With Kurtosis = -6/5

3.2.2 *Beta Distribution*

In probability theory and statistics, the beta distribution is a family of continuous probability distributions defined on the interval $[0, 1]$ parametrized by two positive shape parameters, denoted by α and β, which control the shape of the distribution. It is defined as

$$f(x) = \frac{x^{a-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and for any given variable z

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx.$$

We will omit showing the proof of the kurtosis of the beta distribution, but a sketch of calculating its kurtosis is to generate $\mathbb{E}(x^n)$ moments for $n \in \{1, 2, 3, 4\}$.

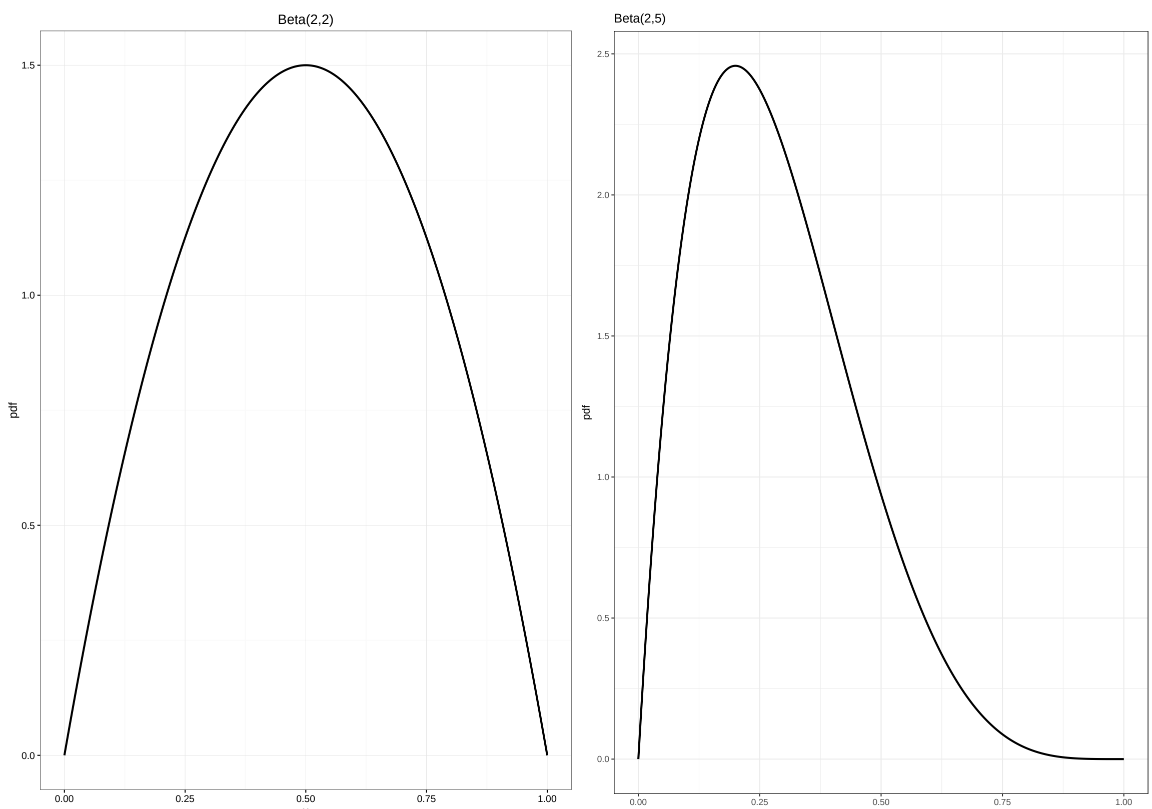
The excess kurtosis of a beta distribution with parameters α and β is the following (Weisstein, 2003):

$$\mathrm{Kurt}(X) = \frac{6[(\alpha-\beta)^2(\alpha+\beta+1) - \alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}.$$

To obtain a negative kurtosis from this, we chose our parameters $\alpha = \beta = 2$, which yielded an excess kurtosis

$$\mathrm{Kurt}(X) = -0.8571429.$$

And to obtain an excess Kurtosis value of -0.12, we choose $\alpha = 2$ and $\beta = 5$



(A) A Beta Distribution Illustrating Short Tails or Negative Excess Kurtosis (Beta(2,2))

(B) A Beta Distribution Illustrating Short Tails or Negative Excess Kurtosis (Beta(2,5))

FIGURE 3.3: Two Beta Distributions With Kurtosis of -0.8571429 and -0.12 Respectively

3.3 Positive Kurtosis

Distributions with positive kurtosis are those that have long tails, which subsequently yield many outliers. They are the distribution with excess kurtosis greater than zero. Here are some of the distributions analyzed in this paper with positive kurtosis.

3.3.1 Double Exponential Distribution

Double exponential distribution also known as Laplace distribution. This distribution is often referred to as Laplace's first law of errors. Given a location parameter μ and scale parameter $\beta > 0$, a double exponential Distribution is defined as

$$f(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{\beta}} \quad x \in (-\infty, \infty).$$

We refer to a standard double exponential distribution that with location parameter $\mu = 0$ and scale parameter $\beta = 1$. We want to derive the kurtosis of the Laplace distribution but first, we define the following property.

Property 3.3.1.1: Assume $X \sim \mathrm{DExp}(\mu, \beta)$ for parameters μ and β. For any n even $\in \mathbb{N}$, then its moment $\mathbb{E}(X^n) = n!$

Proof. Given a function $f(X)$, the moment of X about μ of order n is defined as

$$\mathbb{E}[(X - \mu)]^n.$$

Because the location parameter of a standard double exponential function is zero and its scale parameter $\beta = 1$, then the double exponential distribution can be rewritten as:

$$f(x) = \frac{1}{2} e^{-|x|}.$$

Its moment about the mean $\mu = 0$ gives us:

$$\mathbb{E}(X^n) = \frac{1}{2} \int_{-\infty}^{\infty} x^n e^{-|x|} dx.$$

Because of the symmetric nature of the standard double exponential, n and the existence of the absolute value, we must split the function into two parts since the function is increasing on the left side of 0 and decreasing in the right side of zero. We have:

$$\mathbb{E}(X^n) = \frac{1}{2}\int_{-\infty}^{0} x^n e^x dx + \frac{1}{2}\int_{0}^{\infty} x^n e^{-x} dx$$

$$= 2 * \frac{1}{2}\int_{0}^{\infty} x^n e^{-x} \qquad \text{Due to symmetry.}$$

And we recognize the above function as the gamma function which is also equal to

$$\int_{0}^{\infty} x^n e^{-x} = n! \qquad \text{(Miller, 2004a)}$$

□

From there, we may now derive its excess kurtosis value.

Property 3.3.1.2: $X \sim \mathrm{DExp}(0,1)$**, then its excess kurtosis is** $\mathrm{Kurt}(X) = 3$.

Proof. By definition,

$$\mathrm{Kurt}(X) = \frac{\mathbb{E}(X-\mu)^4}{\sigma^4} - 3$$

For a standard double exponential distribution, it suffices to show that

$$\frac{\mathbb{E}(X^4)}{[\mathbb{E}(X^2)]^2} - 3 = 3.$$

From **Property 3.3.1.1**, we showed $\mathbb{E}(X^n) = n!$. Then

$$\frac{\mathbb{E}(X^4)}{[\mathbb{E}(X^2)]^2} - 3.$$

$$= \frac{4!}{(2!)^2} - 3 = 3$$

□

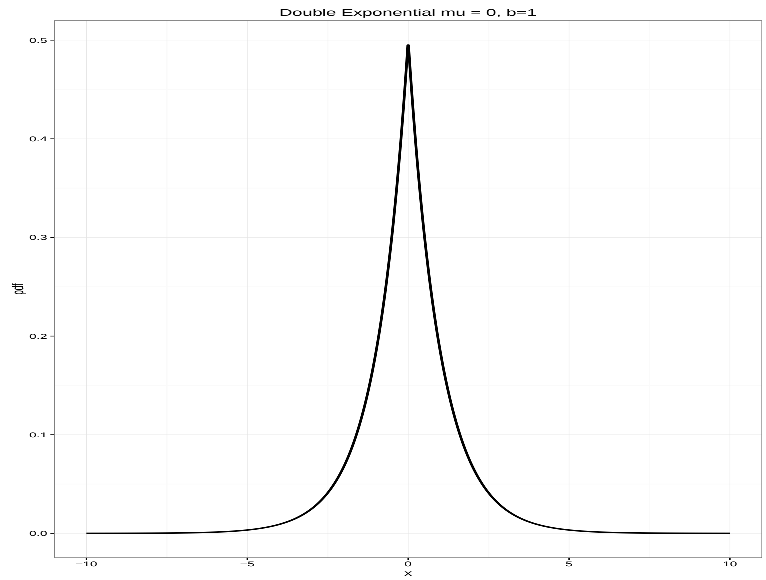


FIGURE 3.4: A Double Exponential Distribution of Kurtosis = 3

3.3.2 *Logistic Distribution*

In probability theory and statistics, the logistic distribution is a continuous probability distribution which resembles the normal distribution in shape but has heavier tails (higher kurtosis). For a location parameter μ and scale parameter $\sigma > 0$, then the logistic distribution is defined as

$$f(x) = \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma\left(1 + e^{-\frac{x-\mu}{\sigma}}\right)^2}.$$

In this paper, we consider the standard logistic distribution for a location parameter $\mu = 0$ and scale parameter $\sigma = 1$, which we write as

$$f(x) = \frac{e^x}{(1 + e^x)^2}.$$

The excess kurtosis of the standard logistic distribution is

$$\text{Kurt}(X) = \frac{6}{5} \qquad \text{(Gupta and Kundu, 2010)}$$

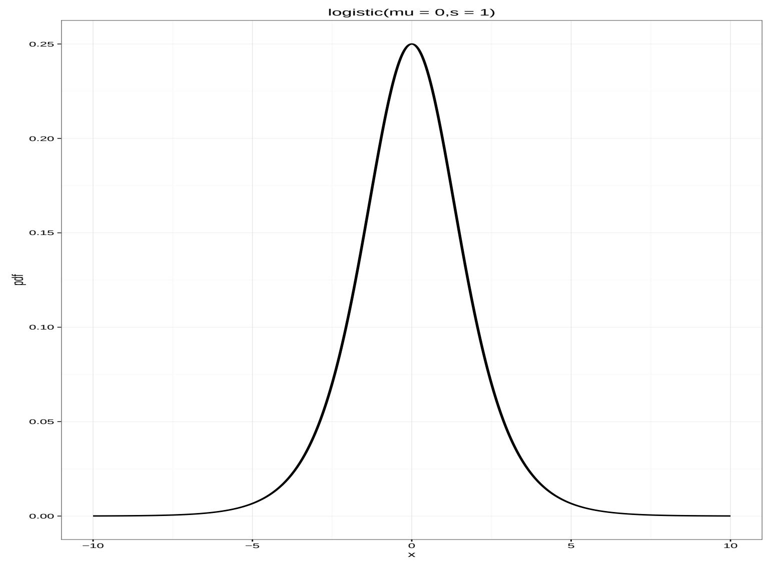


FIGURE 3.5: A Standard Logistic Distribution of Kurtosis = 6/5

3.3.3 *Student's t Distribution*

In probability and statistics, Student's t-distribution (or simply the t-distribution) is any member of a family of continuous probability distributions that arises when estimating the mean of a normally distributed population in situations where the sample size is small and population standard deviation is unknown. Let Z has a standard normal distribution and V a chi-squared distribution with n degrees of freedom with $n \in (-\infty, \infty)$ and Z and V are independent. Then for a random variable X for which

$$X = \frac{Z}{\sqrt{V/n}}$$

with n degrees of freedom is defined as:

$$f(x) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\Gamma(n/2)}\left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} \quad x \in (-\infty, \infty) \qquad \text{(Miller, 2004b).}$$

To find the excess kurtosis of t-distribution, we first define the gamma distribution as follows: a random variable X is referred to as a gamma distribution if and only if its probability density function is

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad \text{for} \quad x > 0, \quad \alpha > 0 \quad \text{and} \quad \beta > 0$$

For $\beta = 2$ and $\alpha = n/2$ the gamma distribution we get from substituting these values is called a chi-square distribution since its probability distribution is

$$\frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} \quad \text{for} \quad x > 0, \quad \alpha > 0 \quad \text{and} \quad \beta > 0$$

where n is referred to as the degree of freedom. The *k*th moment about the origin of the gamma distribution is given by

$$\mu_k' = \frac{\beta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \qquad \text{(Miller, 2004b)}$$

which directly implies that the *k*th moment for a chi-square distribution with n degree of freedom is

$$\mathbb{E}(V^k) = 2^k \frac{\Gamma(n/2 + k)}{\Gamma(n/2)}.$$

It is easy to show that the t-distribution has a mean of 0 since, by independence of Z and V,

$$E(T) = \mathbb{E}(Z) \cdot \sqrt{n} \cdot \mathbb{E}\Big(V^{-1/2}\Big)$$

and because the mean of the standard normal is 0, then

$$\mathbb{E}(t) = 0 \cdot \sqrt{n}\mathbb{E}\Big(V^{-1/2}\Big) = 0$$

as long as $n > 1$ to satisfy the restriction on $\mathbb{E}(V^{-1/2})$

We now derive the *k*th moment of the t-distribution

$$\mathbb{E}(t^k) = \mathbb{E}\left(\frac{Z}{\sqrt{V/n}}\right)^k$$

then by independence, we get

$$E(t^k) = n^{k/2}\mathbb{E}(Z^k)\mathbb{E}(V^{-k/2}).$$

First, we can quickly show that

$$\mathbb{E}(V^{-k/2}) = 2^{-k/2}\frac{\Gamma(n/2 - k/2)}{\Gamma(n/2)}$$

from the *k*th moment of the chi-square previously shown. Next we work with $\mathbb{E}(Z^k)$

Recall that we showed that

$$\forall n \in \mathbb{N}, \mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^{n-1}) \qquad \textbf{Property 3.1.1.2}$$

then:

$$\mathbb{E}(Z^k) = (k-1)\cdot\mathbb{E}\Big(Z^{k-2}\Big) \tag{3.20}$$

$$= (k-1)\cdot(k-3)\mathbb{E}\Big(Z^{k-4}\Big) \tag{3.21}$$

$$= (k-1)\cdot(k-3)\cdot(k-5)\cdot\mathbb{E}\Big(Z^{k-6}\Big) \tag{3.22}$$

and so on to have

$$= (k-1)\cdot(k-3)\cdot(k-5)\cdot\ldots\cdot\mathbb{E}\Big(Z^{k-2l}\Big) \quad \text{for} \quad l \in \mathbb{N}$$

From there, we can see that for any k odd, $\mathbb{E}(Z^k)$ will always be 0. For k even, we get

$$1\cdot 3\cdot 5\cdot\ldots\cdot(k-1)$$

We finally get to the *k*th moment of the t-distribution with n degree of freedom as:

$$\mathbb{E}(T^k) = \frac{n^{k/2}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(k-1)\cdot\Gamma((n-k)/2)}{2^{(k/2)}\Gamma(n/2)}.$$

Let there be a random variable X that has a t-distribution with n degree of freedom, then its excess kurtosis is given as

$$\mathrm{Kurt}(X) = \frac{6}{n-4}$$

Proof.

$$\mathrm{Kurt}(X) = \frac{\mathbb{E}(X-\mu)^4}{(\sigma^2)^2} - 3.$$

We did show that the mean of a t-distribution must be 0 for $n > 1$. We can then rewrite the kurtosis as

$$\begin{aligned}
\mathrm{Kurt}(X) &= \frac{\mathbb{E}(X)^4}{(\sigma^2)^2} - 3 && (3.23)\\
&= \frac{\frac{n^2 \cdot 1 \cdot 3 \cdot \Gamma((n-4)/2)}{4 \cdot \Gamma(n/2)}}{\left(\frac{n \cdot 1 \Gamma((n-2)/2)}{2 \cdot \Gamma(n/2)}\right)} - 3 && (3.24)\\
&= \frac{3(n-2)^2 \Gamma[(n-4)/2]}{4\Gamma(n/2)} - 3. && (3.25)
\end{aligned}$$

One of the well known property of the gamma function is

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \qquad \alpha > 0$$

Then

$$\begin{aligned}
\Gamma(n/2) &= (n/2-1)\Gamma(n/2-1) && (3.26)\\
&= (n/2-1)(n/2-2)\Gamma(n/2-2). && (3.27)
\end{aligned}$$

Substitute equation (3.27) in equation (3.25), we get:

$$\begin{aligned}
\mathrm{Kurt(X)} &= \frac{3(n-2)^2}{4(n/2-1)(n/2-2)} - 3 && (3.28)\\
&= \frac{3(n-2)}{(n-4)} - 3 && (3.29)\\
&= \frac{6}{n-4}. && (3.30)
\end{aligned}$$

□

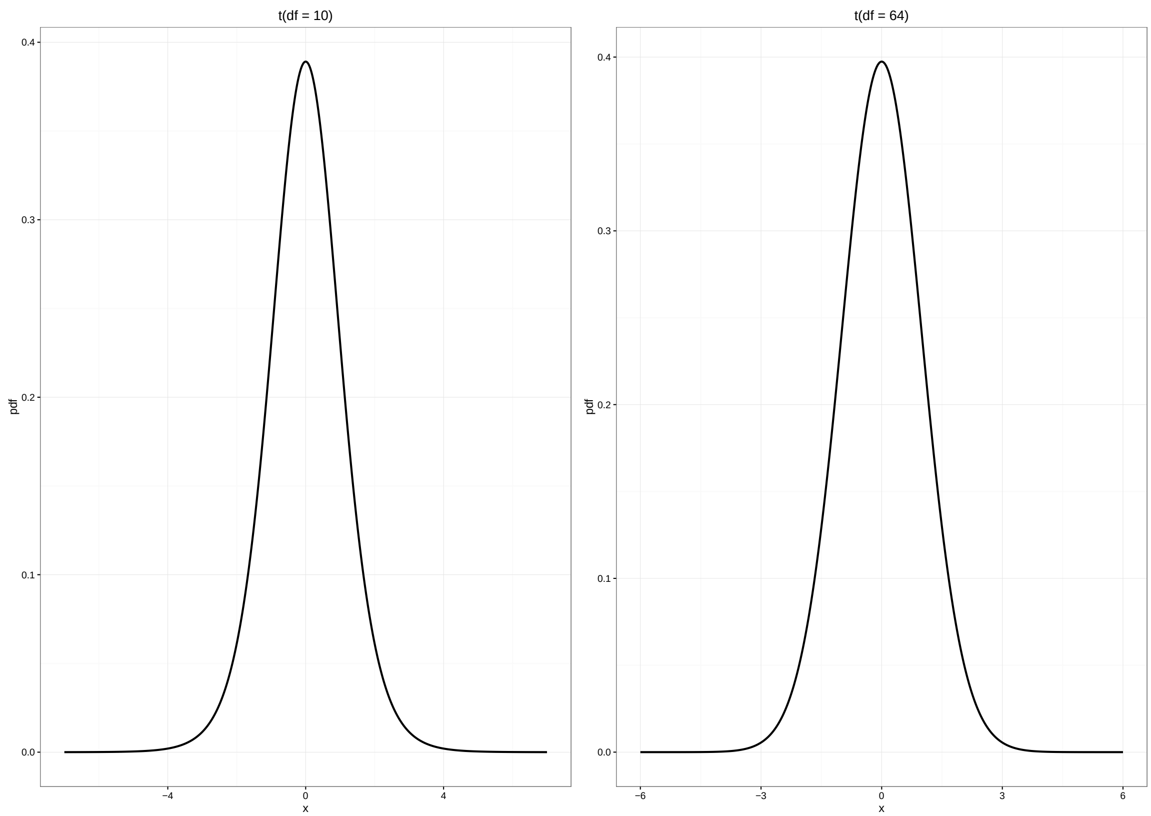
We simulated the following t-distributions

$$X \sim t_{\mathrm{df}=64}$$

to get an excess kurtosis of $\mathrm{Kurt(X)} = 0.1$ (figure 3.6b) and

$$X \sim t_{\mathrm{df}=10}$$

To get an excess kurtosis of $\mathrm{Kurt}(X) = 1$ (See figure 3.6a).



(A) A T-distribution Illustrating Fat Tails or Positive Excess Kurtosis $T_{df=10}$

(B) A T-distribution Illustrating Moderately Fat Tails or Positive Excess Kurtosis $T_{df=64}$

FIGURE 3.6: T-Distribution Kurtosis

CHAPTER 4

SIMULATION STUDIES

Since a theoretical comparison among estimators is outside the scope of my thesis, a simulation study to compare the performance of each estimators in capturing the true kurtosis parameter is conducted in this chapter. We aim to compute confidence estimates using each estimator and then compare coverage probability and mean width of these intervals for each one of the distributions we introduced in Chapter 3 in capturing either zero, negative or positive excess kurtosis.

4.1 Simulation Techniques

The main objective of this paper is to compare the performance of the estimators. The criteria in judging performance is derived from the coverage probability and average width of constructed confidence intervals. In order to get these intervals, we had to simulate our dataset. Simulation was done the following way:

- For sample size $n = 10, 20, 30, 50, 100, 300$ we generate the distributions discussed in Chapter 3 using the Statistical Software R.
- Standard normal distribution to capture zero kurtosis
- Beta(2,2), Beta(2,5) and standard uniform to capture negative kurtosis
- Standard logistic, standard double exponential and two Student t distributions with respective degrees of freedom 10 and 64 to capture positive kurtosis.

In constructing confidence intervals with 95% confidence level using the parametric method, for any of the given distribution we discussed in Chapter 3, we generate n sample size for each of the sample sizes mentioned above. Confidence intervals are calculated for each of the estimators g_2, G_2

and b_2. The data was simulated $3,000$ times to generate $3,000$ lower and upper bound values for each of the three estimators. We then take the average width of each estimators and then calculate the percentage of times when the true kurtosis parameter of a given distribution is within the 3000 constructed intervals.

For the construction of confidence intervals with 95% confidence level using the bootstrap method, from any of the distributions discussed in Chapter 3, given a sample size n and an estimator θ, we generate the bootstrap confidence intervals using $1,000$ bootstrap statistics. Based on one of on the three estimators. We then simulate the process 3000 times to construct the bootstrap intervals using the various bootstrap confidence interval techniques we discussed in Chapter 2. We then take the average width of each intervals as well as the percent coverage every time the true kurtosis parameter is within the 3000 constructed bootstrap intervals. Refer to Kibria and Banik (2001) for more on simulation techniques.

4.2 Results and Discussion

As mentioned, for a given estimator, we are to construct confidence intervals using both parametric and bootstrap methods. We would then calculate the coverage probability as well as the average width of these intervals as our criteria to compare the performance of these interval estimators. We constructed confidence intervals for all 7 distributions we discussed in Chapter 3 as they were chosen to capture zero excess kurtosis, positive and negative excess kurtosis. R-Software was used to complete the simulation procedures.

4.2.1 Standard Normal Distribution: Zero Kurtosis

The average width and coverage probability for all confidence intervals when data are generated from $\mathcal{N}(0,1)$ were reported in table 4.1 and figure 4.1. As expected, the larger the sample sizes, the smaller the average width of the intervals regardless of confidence interval methods. On the other hand, we observed that the only time the average widths of the intervals using classical method is smaller is for when sample size is $n = 10$. In all other sample sizes, the classical method does have higher average width comparing to all other non-parametric method. Another observation is that when we compare all three estimators by confidence interval construction methods, in every case, the average width of b_2 is always less than or equal to that of g_2. Furthermore, the average with of g_2 is also always less than or equal to G_2, regardless of sample sizes. Such inequality was first mentioned in equation (2.5), where I showed that $\text{var}(b_2) \leq \text{var}(g_2) \leq \text{var}(G_2)$. As for the coverage intervals, the classical method started achieving 95% coverage for sample sizes $n = 30$ or higher, for all three estimators although we should mention that the estimator G_2 has achieved 94% coverage or higher on every method, regardless of sample sizes. We also noticed that the classical method does show higher coverage probability comparing to all non parametric intervals for sample sizes 50 or higher. And as sample sizes increase, coverage probability of parametric methods slightly decreases. Last, we see that G_2 tends to also have the highest coverage or ties for highest coverage comparing to the other two estimators regardless of sample sizes as well as confidence interval construction method. g_2 performs the worst every time. From these observations, we can say that for the normal distribution, the best method in estimating the true kurtosis parameter is to use the classical method with G_2 estimator.

4.2.2 Negative Kurtosis

To assess performance of estimators with negative kurtosis, we simulated data from $X \sim \text{Beta}(2, 2)$ with excess kurtosis $\text{Kurt}(X) = -0.8571429$. We also simulated data from $X \sim \text{Beta}(2, 5)$ with excess kurtosis $\text{Kurt}(X) = -0.12$. Last from a standard uniform $X \sim \mathcal{U}[0, 1]$ distribution with excess kurtosis $\text{Kurt}(X) = -6/5$. All results are reported on Tables 4.3, 4.4 and Figures 4.2 and 4.3 for the Beta(2,2) and Beta(2,5) respectively. In Table 4.2 and Figure 4.4 for the standard uniform distribution. As for interval average width, whether it is from either the Beta(2,2), Beta(2,5) or Uniform[0,1], their behavior is similar. The higher the sample size, the shorter the intervals, as expected. Also, regardless of methods, the average width of $b_2 \leq g_2 \leq G_2$. For large sample sizes (n > 30), the parametric method has higher average width than non parametric methods

In terms of coverage probability, if we look at the Standard Uniform distribution, the classical method reached at least 95% threshold for all three estimators regardless of sample sizes or the interval construction method. If we now look at the Beta(2,2) Distribution, Efron's Percentile Bootstrap performs well and sometimes better than the classical method. The advantage of Efron's Percentile Bootstrap is that its average interval is always less than that of the classical method regardless of sample size or estimators. Therefore, from these observations, we can say that for the uniform distribution, the classical method is better for constantly reaching that 95% threshold, with Efron's percentile bootstrap being a close second. For Beta(2,2), Efron's Percentile Bootstrap is better because of to the fact that, in comparison to the classical method, its average interval widths is smaller while constantly reaching that 95% coverage threshold. As for Beta(2,5), the classical method appears to be the best approach in constructing confidence intervals as bootstrap methods struggle to constantly get to that 95% threshold.

4.2.3 Positive Kurtosis

To assess performance of estimators with positive kurtosis, we simulated data from t distribution with degree of freedom $n = 10, 64$ respectively (See Tables 4.5, 4.6 and Figures 4.6 and 4.5). We also simulated data from the standard logistic distribution (see Figure 4.7 and Table 4.7). Last from double exponential (see Figure 4.8 and Table 4.8)

To address performance of estimators with positive kurtosis. Data from standard double exponential, logistic, and t-distributions were simulated to achieve that goal. We first look at $X \sim \mathrm{T}_{\mathrm{df}=64}$ which yielded an excess kurtosis value of $\mathrm{Kurt}(X) = 0.1$. The t-distribution was specifically chosen to see how well our confidence interval methods would properly capture the true kurtosis parameter as t-distribution with 64 degrees of freedom. The excess kurtosis of $t_{df=64}$ is close to a normal distribution with a kurtosis of 0. Our observation does suggest that the interval constructions with coverage parameter reflects the results we get from the normal distribution. Like that of the kurtosis of a normal distribution, average width of the classical method is longer comparing to all other bootstrap confidence interval methods for large sample size $(n \leq 50)$ and its coverage probability is also slightly higher than all other bootstrap method. We need to mention that the coverage of such method is significantly lower than the marginal 95% level, even for large sample sizes. In constructing bootstrap intervals, G_2 is always greater than or equal to the next highest estimator in terms of coverage probability. In terms of average width, for all other estimators, we noticed that, for large sample $n \geq 50$, the classical method performs a lot worst comparing to all other bootstrap methods, if we compare similar estimators. And in every case, we see that the coverage probability rarely meets its 95% threshold only occasional for small sample size. But, with small sample sizes, interval lengths are expected to be quite wide, thus the possible chance of capture the true kurtosis parameter many times. Comparing the classical method and bootstrap methods, we did notice that the bootstrap methods do have higher coverage probability, but none of these confidence interval methods consistently meet their 95% threshold. So, when it comes to positive kurtosis parameter, if sample size is small, it is best that estimator G_2 with Efron's Percentile Method is used. For large sample size, there are not a clear winner since most of them failed to meet the 95% threshold due to poor performance. But Efron's method as well as Bias Corrected Percentile bootstrap does get closer than most. Last, in choosing an estimator, it is recommended to always use G_2 since it is consistently higher than all other estimators even when they all perform poorly. g_2 performs the worst is almost all cases.

Table 4.1
Average Width and Coverage Probability of The Intervals When The Data Are Generated from $\mathcal{N}(0, 1)$

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.79	3.60	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.93	6.36	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.56	2.91	b_2	10.00
4	Bias Corrected Percentile Bootstrap	0.91	3.79	g_2	10.00
5	Bias Corrected Percentile Bootstrap	0.97	6.71	G_2	10.00
6	Bias Corrected Percentile Bootstrap	0.80	3.08	b_2	10.00
7	Classical	0.94	2.96	g_2	10.00
8	Classical	0.95	5.23	G_2	10.00
9	Classical	0.53	2.40	b_2	10.00
10	Efron's Percentile Bootstrap	1.00	3.43	g_2	10.00
11	Efron's Percentile Bootstrap	1.00	6.05	G_2	10.00
12	Efron's Percentile Bootstrap	0.86	2.77	b_2	10.00
13	Hall's Percentile Bootstrap	0.57	3.44	g_2	10.00
14	Hall's Percentile Bootstrap	0.77	6.09	G_2	10.00
15	Hall's Percentile Bootstrap	0.37	2.79	b_2	10.00
16	Bias Corrected Standard Bootstrap	0.80	2.80	g_2	20.00
17	Bias Corrected Standard Bootstrap	0.88	3.65	G_2	20.00
18	Bias Corrected Standard Bootstrap	0.67	2.53	b_2	20.00
19	Bias Corrected Percentile Bootstrap	0.91	3.44	g_2	20.00
20	Bias Corrected Percentile Bootstrap	0.93	4.49	G_2	20.00
21	Bias Corrected Percentile Bootstrap	0.85	3.11	b_2	20.00

Table 4.1
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
22	Classical	0.96	2.98	g_2	20.00
23	Classical	0.95	3.89	G_2	20.00
24	Classical	0.93	2.69	b_2	20.00
25	Efron's Percentile Bootstrap	0.94	2.80	g_2	20.00
26	Efron's Percentile Bootstrap	0.99	3.66	G_2	20.00
27	Efron's Percentile Bootstrap	0.84	2.53	b_2	20.00
28	Hall's Percentile Bootstrap	0.67	2.79	g_2	20.00
29	Hall's Percentile Bootstrap	0.79	3.64	G_2	20.00
30	Hall's Percentile Bootstrap	0.55	2.52	b_2	20.00
31	Bias Corrected Standard Bootstrap	0.80	2.38	g_2	30.00
32	Bias Corrected Standard Bootstrap	0.87	2.83	G_2	30.00
33	Bias Corrected Standard Bootstrap	0.72	2.22	b_2	30.00
34	Bias Corrected Percentile Bootstrap	0.88	2.76	g_2	30.00
35	Bias Corrected Percentile Bootstrap	0.92	3.29	G_2	30.00
36	Bias Corrected Percentile Bootstrap	0.83	2.59	b_2	30.00
37	Classical	0.96	2.75	g_2	30.00
38	Classical	0.95	3.26	G_2	30.00
39	Classical	0.96	2.57	b_2	30.00
40	Efron's Percentile Bootstrap	0.90	2.37	g_2	30.00
41	Efron's Percentile Bootstrap	0.96	2.81	G_2	30.00
42	Efron's Percentile Bootstrap	0.82	2.21	b_2	30.00
43	Hall's Percentile Bootstrap	0.74	2.35	g_2	30.00

Table 4.1
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
44	Hall's Percentile Bootstrap	0.82	2.79	G_2	30.00
45	Hall's Percentile Bootstrap	0.63	2.20	b_2	30.00
46	Bias Corrected Standard Bootstrap	0.83	1.94	g_2	50.00
47	Bias Corrected Standard Bootstrap	0.87	2.15	G_2	50.00
48	Bias Corrected Standard Bootstrap	0.78	1.86	b_2	50.00
49	Bias Corrected Percentile Bootstrap	0.88	2.12	g_2	50.00
50	Bias Corrected Percentile Bootstrap	0.91	2.34	G_2	50.00
51	Bias Corrected Percentile Bootstrap	0.85	2.03	b_2	50.00
52	Classical	0.96	2.34	g_2	50.00
53	Classical	0.95	2.59	G_2	50.00
54	Classical	0.96	2.25	b_2	50.00
55	Efron's Percentile Bootstrap	0.88	1.90	g_2	50.00
56	Efron's Percentile Bootstrap	0.92	2.10	G_2	50.00
57	Efron's Percentile Bootstrap	0.82	1.82	b_2	50.00
58	Hall's Percentile Bootstrap	0.76	1.85	g_2	50.00
59	Hall's Percentile Bootstrap	0.81	2.04	G_2	50.00
60	Hall's Percentile Bootstrap	0.70	1.77	b_2	50.00
61	Bias Corrected Standard Bootstrap	0.84	1.46	g_2	100.00
62	Bias Corrected Standard Bootstrap	0.87	1.54	G_2	100.00
63	Bias Corrected Standard Bootstrap	0.81	1.43	b_2	100.00
64	Bias Corrected Percentile Bootstrap	0.89	1.52	g_2	100.00
65	Bias Corrected Percentile Bootstrap	0.91	1.60	G_2	100.00

Table 4.1
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
66	Bias Corrected Percentile Bootstrap	0.87	1.49	b_2	100.00
67	Classical	0.96	1.78	g_2	100.00
68	Classical	0.96	1.88	G_2	100.00
69	Classical	0.96	1.75	b_2	100.00
70	Efron's Percentile Bootstrap	0.87	1.43	g_2	100.00
71	Efron's Percentile Bootstrap	0.91	1.51	G_2	100.00
72	Efron's Percentile Bootstrap	0.85	1.41	b_2	100.00
73	Hall's Percentile Bootstrap	0.82	1.43	g_2	100.00
74	Hall's Percentile Bootstrap	0.85	1.51	G_2	100.00
75	Hall's Percentile Bootstrap	0.78	1.41	b_2	100.00
76	Bias Corrected Standard Bootstrap	0.88	0.96	g_2	300.00
77	Bias Corrected Standard Bootstrap	0.89	0.98	G_2	300.00
78	Bias Corrected Standard Bootstrap	0.87	0.96	b_2	300.00
79	Bias Corrected Percentile Bootstrap	0.91	0.97	g_2	300.00
80	Bias Corrected Percentile Bootstrap	0.91	0.99	G_2	300.00
81	Bias Corrected Percentile Bootstrap	0.90	0.97	b_2	300.00
82	Classical	0.95	1.08	g_2	300.00
83	Classical	0.95	1.10	G_2	300.00
84	Classical	0.95	1.07	b_2	300.00
85	Efron's Percentile Bootstrap	0.89	0.93	g_2	300.00
86	Efron's Percentile Bootstrap	0.90	0.95	G_2	300.00
87	Efron's Percentile Bootstrap	0.87	0.92	b_2	300.00

Table 4.1
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
88	Hall's Percentile Bootstrap	0.86	0.95	g_2	300.00
89	Hall's Percentile Bootstrap	0.88	0.96	G_2	300.00
90	Hall's Percentile Bootstrap	0.85	0.94	b_2	300.00

Table 4.2
Average Width and Coverage Probability of The Intervals When The Data Are Generated from $\mathcal{U}[0, 1]$

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.98	3.33	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.98	5.89	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.99	2.70	b_2	10.00
4	Bias Corrected Standard Bootstrap	0.97	0.54	g_2	100.00
5	Bias Corrected Standard Bootstrap	0.97	0.57	G_2	100.00
6	Bias Corrected Standard Bootstrap	0.94	0.53	b_2	100.00
7	Bias Corrected Standard Bootstrap	1.00	1.96	g_2	20.00
8	Bias Corrected Standard Bootstrap	1.00	2.55	G_2	20.00
9	Bias Corrected Standard Bootstrap	0.95	1.77	b_2	20.00
10	Bias Corrected Standard Bootstrap	0.99	1.37	g_2	30.00
11	Bias Corrected Standard Bootstrap	0.99	1.63	G_2	30.00
12	Bias Corrected Standard Bootstrap	0.94	1.28	b_2	30.00
13	Bias Corrected Standard Bootstrap	0.96	0.28	g_2	300.00
14	Bias Corrected Standard Bootstrap	0.96	0.28	G_2	300.00

Table 4.2
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
15	Bias Corrected Standard Bootstrap	0.95	0.28	b_2	300.00
16	Bias Corrected Standard Bootstrap	0.97	0.89	g_2	50.00
17	Bias Corrected Standard Bootstrap	0.97	0.98	G_2	50.00
18	Bias Corrected Standard Bootstrap	0.93	0.85	b_2	50.00
19	Bias Corrected Percentile Bootstrap	0.95	3.05	g_2	10.00
20	Bias Corrected Percentile Bootstrap	0.94	5.39	G_2	10.00
21	Bias Corrected Percentile Bootstrap	0.93	2.47	b_2	10.00
22	Bias Corrected Percentile Bootstrap	0.96	0.52	g_2	100.00
23	Bias Corrected Percentile Bootstrap	0.96	0.55	G_2	100.00
24	Bias Corrected Percentile Bootstrap	0.95	0.51	b_2	100.00
25	Bias Corrected Percentile Bootstrap	0.96	1.79	g_2	20.00
26	Bias Corrected Percentile Bootstrap	0.96	2.34	G_2	20.00
27	Bias Corrected Percentile Bootstrap	0.94	1.62	b_2	20.00
28	Bias Corrected Percentile Bootstrap	0.96	1.24	g_2	30.00
29	Bias Corrected Percentile Bootstrap	0.96	1.47	G_2	30.00
30	Bias Corrected Percentile Bootstrap	0.95	1.16	b_2	30.00
31	Bias Corrected Percentile Bootstrap	0.95	0.27	g_2	300.00
32	Bias Corrected Percentile Bootstrap	0.95	0.28	G_2	300.00
33	Bias Corrected Percentile Bootstrap	0.95	0.27	b_2	300.00
34	Bias Corrected Percentile Bootstrap	0.96	0.82	g_2	50.00
35	Bias Corrected Percentile Bootstrap	0.96	0.91	G_2	50.00
36	Bias Corrected Percentile Bootstrap	0.96	0.79	b_2	50.00

Table 4.2
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
37	Classical	0.96	2.96	g_2	10.00
38	Classical	0.96	5.23	G_2	10.00
39	Classical	0.98	2.40	b_2	10.00
40	Classical	1.00	1.78	g_2	100.00
41	Classical	1.00	1.88	G_2	100.00
42	Classical	1.00	1.75	b_2	100.00
43	Classical	0.99	2.98	g_2	20.00
44	Classical	0.99	3.89	G_2	20.00
45	Classical	1.00	2.69	b_2	20.00
46	Classical	1.00	2.75	g_2	30.00
47	Classical	1.00	3.26	G_2	30.00
48	Classical	1.00	2.57	b_2	30.00
49	Classical	1.00	1.08	g_2	300.00
50	Classical	1.00	1.10	G_2	300.00
51	Classical	1.00	1.07	b_2	300.00
52	Classical	1.00	2.34	g_2	50.00
53	Classical	1.00	2.59	G_2	50.00
54	Classical	1.00	2.25	b_2	50.00
55	Efron's Percentile Bootstrap	1.00	3.26	g_2	10.00
56	Efron's Percentile Bootstrap	1.00	5.77	G_2	10.00
57	Efron's Percentile Bootstrap	1.00	2.64	b_2	10.00
58	Efron's Percentile Bootstrap	0.96	0.54	g_2	100.00

Table 4.2
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
59	Efron's Percentile Bootstrap	0.96	0.56	G_2	100.00
60	Efron's Percentile Bootstrap	0.97	0.53	b_2	100.00
61	Efron's Percentile Bootstrap	0.99	1.93	g_2	20.00
62	Efron's Percentile Bootstrap	0.99	2.52	G_2	20.00
63	Efron's Percentile Bootstrap	1.00	1.74	b_2	20.00
64	Efron's Percentile Bootstrap	0.98	1.32	g_2	30.00
65	Efron's Percentile Bootstrap	0.98	1.57	G_2	30.00
66	Efron's Percentile Bootstrap	1.00	1.24	b_2	30.00
67	Efron's Percentile Bootstrap	0.96	0.28	g_2	300.00
68	Efron's Percentile Bootstrap	0.96	0.28	G_2	300.00
69	Efron's Percentile Bootstrap	0.96	0.28	b_2	300.00
70	Efron's Percentile Bootstrap	0.97	0.88	g_2	50.00
71	Efron's Percentile Bootstrap	0.97	0.97	G_2	50.00
72	Efron's Percentile Bootstrap	0.99	0.84	b_2	50.00
73	Hall's Percentile Bootstrap	0.87	3.23	g_2	10.00
74	Hall's Percentile Bootstrap	0.88	5.71	G_2	10.00
75	Hall's Percentile Bootstrap	0.64	2.61	b_2	10.00
76	Hall's Percentile Bootstrap	0.95	0.54	g_2	100.00
77	Hall's Percentile Bootstrap	0.95	0.56	G_2	100.00
78	Hall's Percentile Bootstrap	0.91	0.53	b_2	100.00
79	Hall's Percentile Bootstrap	0.91	1.89	g_2	20.00
80	Hall's Percentile Bootstrap	0.91	2.47	G_2	20.00

Table 4.2
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
81	Hall's Percentile Bootstrap	0.77	1.70	b_2	20.00
82	Hall's Percentile Bootstrap	0.93	1.33	g_2	30.00
83	Hall's Percentile Bootstrap	0.93	1.58	G_2	30.00
84	Hall's Percentile Bootstrap	0.83	1.24	b_2	30.00
85	Hall's Percentile Bootstrap	0.95	0.28	g_2	300.00
86	Hall's Percentile Bootstrap	0.95	0.28	G_2	300.00
87	Hall's Percentile Bootstrap	0.94	0.28	b_2	300.00
88	Hall's Percentile Bootstrap	0.94	0.88	g_2	50.00
89	Hall's Percentile Bootstrap	0.94	0.97	G_2	50.00
90	Hall's Percentile Bootstrap	0.87	0.84	b_2	50.00

Table 4.3
Average Width and Coverage Probability of The Intervals When The Data Are Generated from Beta(2,2)

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.98	3.33	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.97	5.89	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.86	2.70	b_2	10.00
4	Bias Corrected Percentile Bootstrap	0.97	3.32	g_2	10.00
5	Bias Corrected Percentile Bootstrap	0.96	5.87	G_2	10.00
6	Bias Corrected Percentile Bootstrap	0.91	2.69	b_2	10.00
7	Classical	0.97	2.96	g_2	10.00

Table 4.3
Average Width and Coverage Probability(Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
8	Classical	0.96	5.23	G_2	10.00
9	Classical	0.99	2.40	b_2	10.00
10	Efron's Percentile Bootstrap	1.00	3.24	g_2	10.00
11	Efron's Percentile Bootstrap	1.00	5.71	G_2	10.00
12	Efron's Percentile Bootstrap	1.00	2.62	b_2	10.00
13	Hall's Percentile Bootstrap	0.80	3.22	g_2	10.00
14	Hall's Percentile Bootstrap	0.87	5.70	G_2	10.00
15	Hall's Percentile Bootstrap	0.56	2.61	b_2	10.00
16	Bias Corrected Standard Bootstrap	0.97	2.17	g_2	20.00
17	Bias Corrected Standard Bootstrap	0.98	2.83	G_2	20.00
18	Bias Corrected Standard Bootstrap	0.89	1.96	b_2	20.00
19	Bias Corrected Percentile Bootstrap	0.95	2.29	g_2	20.00
20	Bias Corrected Percentile Bootstrap	0.96	2.98	G_2	20.00
21	Bias Corrected Percentile Bootstrap	0.93	2.07	b_2	20.00
22	Classical	0.99	2.98	g_2	20.00
23	Classical	0.99	3.89	G_2	20.00
24	Classical	0.99	2.69	b_2	20.00
25	Efron's Percentile Bootstrap	1.00	2.11	g_2	20.00
26	Efron's Percentile Bootstrap	1.00	2.75	G_2	20.00

Table 4.3
Average Width and Coverage Probability(Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
27	Efron's Percentile Bootstrap	1.00	1.90	b_2	20.00
28	Hall's Percentile Bootstrap	0.87	2.10	g_2	20.00
29	Hall's Percentile Bootstrap	0.91	2.75	G_2	20.00
30	Hall's Percentile Bootstrap	0.73	1.90	b_2	20.00
31	Bias Corrected Standard Bootstrap	0.96	1.58	g_2	30.00
32	Bias Corrected Standard Bootstrap	0.97	1.88	G_2	30.00
33	Bias Corrected Standard Bootstrap	0.90	1.48	b_2	30.00
34	Bias Corrected Percentile Bootstrap	0.96	1.62	g_2	30.00
35	Bias Corrected Percentile Bootstrap	0.95	1.92	G_2	30.00
36	Bias Corrected Percentile Bootstrap	0.94	1.51	b_2	30.00
37	Classical	1.00	2.75	g_2	30.00
38	Classical	1.00	3.26	G_2	30.00
39	Classical	1.00	2.57	b_2	30.00
40	Efron's Percentile Bootstrap	1.00	1.56	g_2	30.00
41	Efron's Percentile Bootstrap	0.99	1.86	G_2	30.00
42	Efron's Percentile Bootstrap	0.99	1.46	b_2	30.00
43	Hall's Percentile Bootstrap	0.90	1.57	g_2	30.00
44	Hall's Percentile Bootstrap	0.93	1.86	G_2	30.00
45	Hall's Percentile Bootstrap	0.79	1.46	b_2	30.00

Table 4.3
Average Width and Coverage Probability(Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
46	Bias Corrected Standard Bootstrap	0.96	1.11	g_2	50.00
47	Bias Corrected Standard Bootstrap	0.97	1.23	G_2	50.00
48	Bias Corrected Standard Bootstrap	0.92	1.07	b_2	50.00
49	Bias Corrected Percentile Bootstrap	0.96	1.11	g_2	50.00
50	Bias Corrected Percentile Bootstrap	0.95	1.23	G_2	50.00
51	Bias Corrected Percentile Bootstrap	0.95	1.06	b_2	50.00
52	Classical	1.00	2.34	g_2	50.00
53	Classical	1.00	2.59	G_2	50.00
54	Classical	1.00	2.25	b_2	50.00
55	Efron's Percentile Bootstrap	0.99	1.10	g_2	50.00
56	Efron's Percentile Bootstrap	0.99	1.22	G_2	50.00
57	Efron's Percentile Bootstrap	0.98	1.05	b_2	50.00
58	Hall's Percentile Bootstrap	0.92	1.09	g_2	50.00
59	Hall's Percentile Bootstrap	0.94	1.21	G_2	50.00
60	Hall's Percentile Bootstrap	0.85	1.05	b_2	50.00
61	Bias Corrected Standard Bootstrap	0.95	0.71	g_2	100.00
62	Bias Corrected Standard Bootstrap	0.96	0.75	G_2	100.00
63	Bias Corrected Standard Bootstrap	0.93	0.70	b_2	100.00
64	Bias Corrected Percentile Bootstrap	0.95	0.71	g_2	100.00

Table 4.3
Average Width and Coverage Probability(Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
65	Bias Corrected Percentile Bootstrap	0.95	0.75	G_2	100.00
66	Bias Corrected Percentile Bootstrap	0.95	0.70	b_2	100.00
67	Classical	1.00	1.78	g_2	100.00
68	Classical	1.00	1.88	G_2	100.00
69	Classical	1.00	1.75	b_2	100.00
70	Efron's Percentile Bootstrap	0.97	0.71	g_2	100.00
71	Efron's Percentile Bootstrap	0.97	0.74	G_2	100.00
72	Efron's Percentile Bootstrap	0.96	0.69	b_2	100.00
73	Hall's Percentile Bootstrap	0.94	0.71	g_2	100.00
74	Hall's Percentile Bootstrap	0.95	0.75	G_2	100.00
75	Hall's Percentile Bootstrap	0.90	0.70	b_2	100.00
76	Bias Corrected Standard Bootstrap	0.95	0.38	g_2	300.00
77	Bias Corrected Standard Bootstrap	0.95	0.39	G_2	300.00
78	Bias Corrected Standard Bootstrap	0.94	0.38	b_2	300.00
79	Bias Corrected Percentile Bootstrap	0.95	0.38	g_2	300.00
80	Bias Corrected Percentile Bootstrap	0.95	0.39	G_2	300.00
81	Bias Corrected Percentile Bootstrap	0.95	0.38	b_2	300.00
82	Classical	1.00	1.08	g_2	300.00
83	Classical	1.00	1.10	G_2	300.00

Table 4.3
Average Width and Coverage Probability(Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
84	Classical	1.00	1.07	b_2	300.00
85	Efron's Percentile Bootstrap	0.95	0.38	g_2	300.00
86	Efron's Percentile Bootstrap	0.95	0.39	G_2	300.00
87	Efron's Percentile Bootstrap	0.95	0.38	b_2	300.00
88	Hall's Percentile Bootstrap	0.94	0.38	g_2	300.00
89	Hall's Percentile Bootstrap	0.94	0.39	G_2	300.00
90	Hall's Percentile Bootstrap	0.93	0.38	b_2	300.00

Table 4.4
Average Width and Coverage Probability of The Intervals When The Data Are Generated from Beta(2,5)

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Classical	0.94	2.96	g_2	10.00
2	Classical	0.94	5.23	G_2	10.00
3	Classical	0.54	2.40	b_2	10.00
4	Classical	0.95	2.98	g_2	20.00
5	Classical	0.93	3.89	G_2	20.00
6	Classical	0.92	2.69	b_2	20.00
7	Classical	0.95	2.75	g_2	30.00
8	Classical	0.94	3.26	G_2	30.00
9	Classical	0.95	2.57	b_2	30.00
10	Classical	0.95	2.34	g_2	50.00
11	Classical	0.94	2.59	G_2	50.00
12	Classical	0.96	2.25	b_2	50.00
13	Classical	0.95	1.78	g_2	100.00
14	Classical	0.95	1.88	G_2	100.00
15	Classical	0.95	1.75	b_2	100.00
16	Classical	0.95	1.08	g_2	300.00
17	Classical	0.95	1.10	G_2	300.00
18	Classical	0.95	1.07	b_2	300.00
19	Bias Corrected Standard Bootstrap	0.79	3.69	g_2	10.00
20	Bias Corrected Standard Bootstrap	0.94	6.52	G_2	10.00
21	Bias Corrected Standard Bootstrap	0.57	2.99	b_2	10.00

Table 4.4
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
22	Bias Corrected Standard Bootstrap	0.78	3.16	g_2	20.00
23	Bias Corrected Standard Bootstrap	0.86	4.12	G_2	20.00
24	Bias Corrected Standard Bootstrap	0.67	2.85	b_2	20.00
25	Bias Corrected Standard Bootstrap	0.80	2.65	g_2	30.00
26	Bias Corrected Standard Bootstrap	0.86	3.15	G_2	30.00
27	Bias Corrected Standard Bootstrap	0.73	2.48	b_2	30.00
28	Bias Corrected Standard Bootstrap	0.80	2.13	g_2	50.00
29	Bias Corrected Standard Bootstrap	0.84	2.36	G_2	50.00
30	Bias Corrected Standard Bootstrap	0.75	2.04	b_2	50.00
31	Bias Corrected Standard Bootstrap	0.86	1.64	g_2	100.00
32	Bias Corrected Standard Bootstrap	0.88	1.72	G_2	100.00
33	Bias Corrected Standard Bootstrap	0.83	1.61	b_2	100.00
34	Bias Corrected Standard Bootstrap	0.90	1.03	g_2	300.00
35	Bias Corrected Standard Bootstrap	0.91	1.05	G_2	300.00
36	Bias Corrected Standard Bootstrap	0.89	1.02	b_2	300.00
37	Hall's Percentile Bootstrap	0.54	3.52	g_2	10.00
38	Hall's Percentile Bootstrap	0.71	6.21	G_2	10.00
39	Hall's Percentile Bootstrap	0.34	2.85	b_2	10.00
40	Hall's Percentile Bootstrap	0.66	2.98	g_2	20.00
41	Hall's Percentile Bootstrap	0.77	3.88	G_2	20.00
42	Hall's Percentile Bootstrap	0.54	2.68	b_2	20.00
43	Hall's Percentile Bootstrap	0.69	2.58	g_2	30.00

Table 4.4
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
44	Hall's Percentile Bootstrap	0.77	3.07	G_2	30.00
45	Hall's Percentile Bootstrap	0.60	2.41	b_2	30.00
46	Hall's Percentile Bootstrap	0.75	2.09	g_2	50.00
47	Hall's Percentile Bootstrap	0.79	2.31	G_2	50.00
48	Hall's Percentile Bootstrap	0.69	2.01	b_2	50.00
49	Hall's Percentile Bootstrap	0.81	1.59	g_2	100.00
50	Hall's Percentile Bootstrap	0.84	1.67	G_2	100.00
51	Hall's Percentile Bootstrap	0.78	1.55	b_2	100.00
52	Hall's Percentile Bootstrap	0.87	1.03	g_2	300.00
53	Hall's Percentile Bootstrap	0.88	1.04	G_2	300.00
54	Hall's Percentile Bootstrap	0.86	1.02	b_2	300.00
55	Efron's Percentile Bootstrap	1.00	3.50	g_2	10.00
56	Efron's Percentile Bootstrap	1.00	6.17	G_2	10.00
57	Efron's Percentile Bootstrap	0.93	2.83	b_2	10.00
58	Efron's Percentile Bootstrap	0.93	2.97	g_2	20.00
59	Efron's Percentile Bootstrap	0.99	3.88	G_2	20.00
60	Efron's Percentile Bootstrap	0.83	2.68	b_2	20.00
61	Efron's Percentile Bootstrap	0.90	2.60	g_2	30.00
62	Efron's Percentile Bootstrap	0.95	3.09	G_2	30.00
63	Efron's Percentile Bootstrap	0.83	2.43	b_2	30.00
64	Efron's Percentile Bootstrap	0.89	2.09	g_2	50.00
65	Efron's Percentile Bootstrap	0.93	2.31	G_2	50.00

Table 4.4
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
66	Efron's Percentile Bootstrap	0.85	2.01	b_2	50.00
67	Efron's Percentile Bootstrap	0.89	1.60	g_2	100.00
68	Efron's Percentile Bootstrap	0.91	1.69	G_2	100.00
69	Efron's Percentile Bootstrap	0.87	1.57	b_2	100.00
70	Efron's Percentile Bootstrap	0.92	1.02	g_2	300.00
71	Efron's Percentile Bootstrap	0.93	1.04	G_2	300.00
72	Efron's Percentile Bootstrap	0.92	1.02	b_2	300.00
73	Bias Corrected Percentile Bootstrap	0.90	3.74	g_2	10.00
74	Bias Corrected Percentile Bootstrap	0.95	6.61	G_2	10.00
75	Bias Corrected Percentile Bootstrap	0.79	3.03	b_2	10.00
76	Bias Corrected Percentile Bootstrap	0.88	3.65	g_2	20.00
77	Bias Corrected Percentile Bootstrap	0.91	4.76	G_2	20.00
78	Bias Corrected Percentile Bootstrap	0.82	3.29	b_2	20.00
79	Bias Corrected Percentile Bootstrap	0.89	3.07	g_2	30.00
80	Bias Corrected Percentile Bootstrap	0.91	3.66	G_2	30.00
81	Bias Corrected Percentile Bootstrap	0.84	2.88	b_2	30.00
82	Bias Corrected Percentile Bootstrap	0.90	2.38	g_2	50.00
83	Bias Corrected Percentile Bootstrap	0.91	2.63	G_2	50.00
84	Bias Corrected Percentile Bootstrap	0.87	2.28	b_2	50.00
85	Bias Corrected Percentile Bootstrap	0.89	1.70	g_2	100.00
86	Bias Corrected Percentile Bootstrap	0.91	1.78	G_2	100.00
87	Bias Corrected Percentile Bootstrap	0.88	1.66	b_2	100.00

Table 4.4
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
88	Bias Corrected Percentile Bootstrap	0.91	1.04	g_2	300.00
89	Bias Corrected Percentile Bootstrap	0.92	1.06	G_2	300.00
90	Bias Corrected Percentile Bootstrap	0.91	1.03	b_2	300.00

Table 4.5
Average Width and Coverage Probability of the intervals when the data are generated from $t_{(df=10)}$

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.53	3.75	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.81	6.63	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.34	3.04	b_2	10.00
4	Bias Corrected Standard Bootstrap	0.65	2.47	g_2	100.00
5	Bias Corrected Standard Bootstrap	0.68	2.60	G_2	100.00
6	Bias Corrected Standard Bootstrap	0.62	2.42	b_2	100.00
7	Bias Corrected Standard Bootstrap	0.57	3.41	g_2	20.00
8	Bias Corrected Standard Bootstrap	0.72	4.44	G_2	20.00
9	Bias Corrected Standard Bootstrap	0.46	3.07	b_2	20.00
10	Bias Corrected Standard Bootstrap	0.58	3.15	g_2	30.00
11	Bias Corrected Standard Bootstrap	0.67	3.75	G_2	30.00
12	Bias Corrected Standard Bootstrap	0.51	2.95	b_2	30.00
13	Bias Corrected Standard Bootstrap	0.71	2.16	g_2	300.00
14	Bias Corrected Standard Bootstrap	0.73	2.20	G_2	300.00

Table 4.5
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
15	Bias Corrected Standard Bootstrap	0.70	2.15	b_2	300.00
16	Bias Corrected Standard Bootstrap	0.60	2.77	g_2	50.00
17	Bias Corrected Standard Bootstrap	0.67	3.07	G_2	50.00
18	Bias Corrected Standard Bootstrap	0.56	2.66	b_2	50.00
19	Bias Corrected Percentile Bootstrap	0.79	4.00	g_2	10.00
20	Bias Corrected Percentile Bootstrap	0.92	7.06	G_2	10.00
21	Bias Corrected Percentile Bootstrap	0.64	3.24	b_2	10.00
22	Bias Corrected Percentile Bootstrap	0.72	2.45	g_2	100.00
23	Bias Corrected Percentile Bootstrap	0.75	2.58	G_2	100.00
24	Bias Corrected Percentile Bootstrap	0.69	2.40	b_2	100.00
25	Bias Corrected Percentile Bootstrap	0.78	4.25	g_2	20.00
26	Bias Corrected Percentile Bootstrap	0.87	5.53	G_2	20.00
27	Bias Corrected Percentile Bootstrap	0.70	3.83	b_2	20.00
28	Bias Corrected Percentile Bootstrap	0.73	3.78	g_2	30.00
29	Bias Corrected Percentile Bootstrap	0.82	4.50	G_2	30.00
30	Bias Corrected Percentile Bootstrap	0.67	3.53	b_2	30.00
31	Bias Corrected Percentile Bootstrap	0.77	2.03	g_2	300.00
32	Bias Corrected Percentile Bootstrap	0.78	2.06	G_2	300.00
33	Bias Corrected Percentile Bootstrap	0.75	2.01	b_2	300.00
34	Bias Corrected Percentile Bootstrap	0.72	3.14	g_2	50.00
35	Bias Corrected Percentile Bootstrap	0.77	3.48	G_2	50.00
36	Bias Corrected Percentile Bootstrap	0.68	3.01	b_2	50.00

Table 4.5
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
37	Classical	0.44	2.96	g_2	10.00
38	Classical	0.92	5.23	G_2	10.00
39	Classical	0.17	2.40	b_2	10.00
40	Classical	0.57	1.78	g_2	100.00
41	Classical	0.63	1.88	G_2	100.00
42	Classical	0.54	1.75	b_2	100.00
43	Classical	0.61	2.98	g_2	20.00
44	Classical	0.86	3.89	G_2	20.00
45	Classical	0.39	2.69	b_2	20.00
46	Classical	0.64	2.75	g_2	30.00
47	Classical	0.82	3.26	G_2	30.00
48	Classical	0.50	2.57	b_2	30.00
49	Classical	0.52	1.08	g_2	300.00
50	Classical	0.54	1.10	G_2	300.00
51	Classical	0.51	1.07	b_2	300.00
52	Classical	0.63	2.34	g_2	50.00
53	Classical	0.73	2.59	G_2	50.00
54	Classical	0.54	2.25	b_2	50.00
55	Efron’s Percentile Bootstrap	0.78	3.58	g_2	10.00
56	Efron’s Percentile Bootstrap	1.00	6.32	G_2	10.00
57	Efron’s Percentile Bootstrap	0.43	2.90	b_2	10.00
58	Efron’s Percentile Bootstrap	0.65	2.35	g_2	100.00

Table 4.5
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
59	Efron's Percentile Bootstrap	0.70	2.47	G_2	100.00
60	Efron's Percentile Bootstrap	0.62	2.30	b_2	100.00
61	Efron's Percentile Bootstrap	0.65	3.22	g_2	20.00
62	Efron's Percentile Bootstrap	0.83	4.21	G_2	20.00
63	Efron's Percentile Bootstrap	0.53	2.91	b_2	20.00
64	Efron's Percentile Bootstrap	0.65	3.07	g_2	30.00
65	Efron's Percentile Bootstrap	0.77	3.66	G_2	30.00
66	Efron's Percentile Bootstrap	0.56	2.87	b_2	30.00
67	Efron's Percentile Bootstrap	0.71	1.99	g_2	300.00
68	Efron's Percentile Bootstrap	0.73	2.02	G_2	300.00
69	Efron's Percentile Bootstrap	0.70	1.97	b_2	300.00
70	Efron's Percentile Bootstrap	0.64	2.67	g_2	50.00
71	Efron's Percentile Bootstrap	0.72	2.96	G_2	50.00
72	Efron's Percentile Bootstrap	0.58	2.57	b_2	50.00
73	Hall's Percentile Bootstrap	0.36	3.55	g_2	10.00
74	Hall's Percentile Bootstrap	0.61	6.28	G_2	10.00
75	Hall's Percentile Bootstrap	0.21	2.87	b_2	10.00
76	Hall's Percentile Bootstrap	0.59	2.34	g_2	100.00
77	Hall's Percentile Bootstrap	0.63	2.46	G_2	100.00
78	Hall's Percentile Bootstrap	0.56	2.29	b_2	100.00
79	Hall's Percentile Bootstrap	0.45	3.26	g_2	20.00
80	Hall's Percentile Bootstrap	0.60	4.25	G_2	20.00

Table 4.5
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
81	Hall's Percentile Bootstrap	0.35	2.94	b_2	20.00
82	Hall's Percentile Bootstrap	0.50	2.98	g_2	30.00
83	Hall's Percentile Bootstrap	0.59	3.54	G_2	30.00
84	Hall's Percentile Bootstrap	0.42	2.79	b_2	30.00
85	Hall's Percentile Bootstrap	0.69	1.97	g_2	300.00
86	Hall's Percentile Bootstrap	0.71	2.00	G_2	300.00
87	Hall's Percentile Bootstrap	0.68	1.96	b_2	300.00
88	Hall's Percentile Bootstrap	0.53	2.69	g_2	50.00
89	Hall's Percentile Bootstrap	0.61	2.98	G_2	50.00
90	Hall's Percentile Bootstrap	0.48	2.58	b_2	50.00

Table 4.6
Average Width and Coverage Probability of The Intervals When The Data Are Generated from $t_{(df=64)}$

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.78	3.58	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.94	6.34	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.54	2.90	b_2	10.00
4	Bias Corrected Standard Bootstrap	0.83	1.55	g_2	100.00
5	Bias Corrected Standard Bootstrap	0.86	1.63	G_2	100.00
6	Bias Corrected Standard Bootstrap	0.80	1.52	b_2	100.00
7	Bias Corrected Standard Bootstrap	0.79	2.93	g_2	20.00

Table 4.6
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
8	Bias Corrected Standard Bootstrap	0.88	3.81	G_2	20.00
9	Bias Corrected Standard Bootstrap	0.66	2.64	b_2	20.00
10	Bias Corrected Standard Bootstrap	0.78	2.47	g_2	30.00
11	Bias Corrected Standard Bootstrap	0.85	2.94	G_2	30.00
12	Bias Corrected Standard Bootstrap	0.69	2.31	b_2	30.00
13	Bias Corrected Standard Bootstrap	0.88	1.06	g_2	300.00
14	Bias Corrected Standard Bootstrap	0.88	1.08	G_2	300.00
15	Bias Corrected Standard Bootstrap	0.86	1.05	b_2	300.00
16	Bias Corrected Standard Bootstrap	0.79	2.02	g_2	50.00
17	Bias Corrected Standard Bootstrap	0.84	2.23	G_2	50.00
18	Bias Corrected Standard Bootstrap	0.73	1.94	b_2	50.00
19	Bias Corrected Percentile Bootstrap	0.89	3.86	g_2	10.00
20	Bias Corrected Percentile Bootstrap	0.96	6.83	G_2	10.00
21	Bias Corrected Percentile Bootstrap	0.79	3.13	b_2	10.00
22	Bias Corrected Percentile Bootstrap	0.88	1.62	g_2	100.00
23	Bias Corrected Percentile Bootstrap	0.89	1.71	G_2	100.00
24	Bias Corrected Percentile Bootstrap	0.86	1.59	b_2	100.00
25	Bias Corrected Percentile Bootstrap	0.89	3.54	g_2	20.00
26	Bias Corrected Percentile Bootstrap	0.93	4.62	G_2	20.00
27	Bias Corrected Percentile Bootstrap	0.83	3.20	b_2	20.00
28	Bias Corrected Percentile Bootstrap	0.88	2.87	g_2	30.00
29	Bias Corrected Percentile Bootstrap	0.91	3.41	G_2	30.00

Table 4.6
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
30	Bias Corrected Percentile Bootstrap	0.82	2.68	b_2	30.00
31	Bias Corrected Percentile Bootstrap	0.90	1.07	g_2	300.00
32	Bias Corrected Percentile Bootstrap	0.91	1.09	G_2	300.00
33	Bias Corrected Percentile Bootstrap	0.88	1.06	b_2	300.00
34	Bias Corrected Percentile Bootstrap	0.86	2.19	g_2	50.00
35	Bias Corrected Percentile Bootstrap	0.89	2.42	G_2	50.00
36	Bias Corrected Percentile Bootstrap	0.83	2.10	b_2	50.00
37	Classical	0.91	2.96	g_2	10.00
38	Classical	0.95	5.23	G_2	10.00
39	Classical	0.45	2.40	b_2	10.00
40	Classical	0.95	1.78	g_2	100.00
41	Classical	0.95	1.88	G_2	100.00
42	Classical	0.95	1.75	b_2	100.00
43	Classical	0.96	2.98	g_2	20.00
44	Classical	0.95	3.89	G_2	20.00
45	Classical	0.89	2.69	b_2	20.00
46	Classical	0.96	2.75	g_2	30.00
47	Classical	0.95	3.26	G_2	30.00
48	Classical	0.93	2.57	b_2	30.00
49	Classical	0.93	1.08	g_2	300.00
50	Classical	0.94	1.10	G_2	300.00
51	Classical	0.93	1.07	b_2	300.00

Table 4.6
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
52	Classical	0.96	2.34	g_2	50.00
53	Classical	0.95	2.59	G_2	50.00
54	Classical	0.95	2.25	b_2	50.00
55	Efron's Percentile Bootstrap	1.00	3.44	g_2	10.00
56	Efron's Percentile Bootstrap	1.00	6.08	G_2	10.00
57	Efron's Percentile Bootstrap	0.82	2.79	b_2	10.00
58	Efron's Percentile Bootstrap	0.86	1.53	g_2	100.00
59	Efron's Percentile Bootstrap	0.89	1.61	G_2	100.00
60	Efron's Percentile Bootstrap	0.82	1.50	b_2	100.00
61	Efron's Percentile Bootstrap	0.91	2.83	g_2	20.00
62	Efron's Percentile Bootstrap	0.98	3.69	G_2	20.00
63	Efron's Percentile Bootstrap	0.80	2.55	b_2	20.00
64	Efron's Percentile Bootstrap	0.88	2.39	g_2	30.00
65	Efron's Percentile Bootstrap	0.94	2.84	G_2	30.00
66	Efron's Percentile Bootstrap	0.80	2.23	b_2	30.00
67	Efron's Percentile Bootstrap	0.88	1.03	g_2	300.00
68	Efron's Percentile Bootstrap	0.90	1.05	G_2	300.00
69	Efron's Percentile Bootstrap	0.87	1.02	b_2	300.00
70	Efron's Percentile Bootstrap	0.87	1.98	g_2	50.00
71	Efron's Percentile Bootstrap	0.92	2.19	G_2	50.00
72	Efron's Percentile Bootstrap	0.80	1.90	b_2	50.00
73	Hall's Percentile Bootstrap	0.55	3.45	g_2	10.00

Table 4.6
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
74	Hall's Percentile Bootstrap	0.75	6.11	G_2	10.00
75	Hall's Percentile Bootstrap	0.35	2.80	b_2	10.00
76	Hall's Percentile Bootstrap	0.81	1.54	g_2	100.00
77	Hall's Percentile Bootstrap	0.84	1.62	G_2	100.00
78	Hall's Percentile Bootstrap	0.77	1.51	b_2	100.00
79	Hall's Percentile Bootstrap	0.66	2.87	g_2	20.00
80	Hall's Percentile Bootstrap	0.79	3.75	G_2	20.00
81	Hall's Percentile Bootstrap	0.52	2.59	b_2	20.00
82	Hall's Percentile Bootstrap	0.71	2.42	g_2	30.00
83	Hall's Percentile Bootstrap	0.80	2.88	G_2	30.00
84	Hall's Percentile Bootstrap	0.62	2.26	b_2	30.00
85	Hall's Percentile Bootstrap	0.86	1.04	g_2	300.00
86	Hall's Percentile Bootstrap	0.87	1.06	G_2	300.00
87	Hall's Percentile Bootstrap	0.84	1.03	b_2	300.00
88	Hall's Percentile Bootstrap	0.75	1.96	g_2	50.00
89	Hall's Percentile Bootstrap	0.80	2.17	G_2	50.00
90	Hall's Percentile Bootstrap	0.68	1.88	b_2	50.00

Table 4.7
Average Width and Coverage Probability of The Intervals When The Data Are Generated from Logistic(0,1)

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.50	3.80	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.80	6.71	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.30	3.07	b_2	10.00
4	Bias Corrected Standard Bootstrap	0.66	2.65	g_2	100.00
5	Bias Corrected Standard Bootstrap	0.69	2.79	G_2	100.00
6	Bias Corrected Standard Bootstrap	0.63	2.60	b_2	100.00
7	Bias Corrected Standard Bootstrap	0.57	3.62	g_2	20.00
8	Bias Corrected Standard Bootstrap	0.70	4.73	G_2	20.00
9	Bias Corrected Standard Bootstrap	0.47	3.27	b_2	20.00
10	Bias Corrected Standard Bootstrap	0.58	3.33	g_2	30.00
11	Bias Corrected Standard Bootstrap	0.68	3.96	G_2	30.00
12	Bias Corrected Standard Bootstrap	0.51	3.11	b_2	30.00
13	Bias Corrected Standard Bootstrap	0.76	2.15	g_2	300.00
14	Bias Corrected Standard Bootstrap	0.78	2.19	G_2	300.00
15	Bias Corrected Standard Bootstrap	0.75	2.14	b_2	300.00
16	Bias Corrected Standard Bootstrap	0.62	3.09	g_2	50.00
17	Bias Corrected Standard Bootstrap	0.69	3.42	G_2	50.00
18	Bias Corrected Standard Bootstrap	0.57	2.97	b_2	50.00
19	Bias Corrected Percentile Bootstrap	0.79	4.10	g_2	10.00
20	Bias Corrected Percentile Bootstrap	0.91	7.24	G_2	10.00
21	Bias Corrected Percentile Bootstrap	0.63	3.32	b_2	10.00

Table 4.7
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
22	Bias Corrected Percentile Bootstrap	0.74	2.71	g_2	100.00
23	Bias Corrected Percentile Bootstrap	0.77	2.85	G_2	100.00
24	Bias Corrected Percentile Bootstrap	0.71	2.65	b_2	100.00
25	Bias Corrected Percentile Bootstrap	0.76	4.39	g_2	20.00
26	Bias Corrected Percentile Bootstrap	0.85	5.73	G_2	20.00
27	Bias Corrected Percentile Bootstrap	0.67	3.97	b_2	20.00
28	Bias Corrected Percentile Bootstrap	0.75	4.04	g_2	30.00
29	Bias Corrected Percentile Bootstrap	0.82	4.82	G_2	30.00
30	Bias Corrected Percentile Bootstrap	0.69	3.78	b_2	30.00
31	Bias Corrected Percentile Bootstrap	0.79	2.16	g_2	300.00
32	Bias Corrected Percentile Bootstrap	0.81	2.19	G_2	300.00
33	Bias Corrected Percentile Bootstrap	0.78	2.14	b_2	300.00
34	Bias Corrected Percentile Bootstrap	0.73	3.38	g_2	50.00
35	Bias Corrected Percentile Bootstrap	0.78	3.75	G_2	50.00
36	Bias Corrected Percentile Bootstrap	0.68	3.25	b_2	50.00
37	Classical	0.37	2.96	g_2	10.00
38	Classical	0.89	5.23	G_2	10.00
39	Classical	0.13	2.40	b_2	10.00
40	Classical	0.57	1.78	g_2	100.00
41	Classical	0.61	1.88	G_2	100.00
42	Classical	0.53	1.75	b_2	100.00
43	Classical	0.54	2.98	g_2	20.00

Table 4.7
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
44	Classical	0.80	3.89	G_2	20.00
45	Classical	0.36	2.69	b_2	20.00
46	Classical	0.59	2.75	g_2	30.00
47	Classical	0.76	3.26	G_2	30.00
48	Classical	0.46	2.57	b_2	30.00
49	Classical	0.51	1.08	g_2	300.00
50	Classical	0.53	1.10	G_2	300.00
51	Classical	0.49	1.07	b_2	300.00
52	Classical	0.58	2.34	g_2	50.00
53	Classical	0.69	2.59	G_2	50.00
54	Classical	0.50	2.25	b_2	50.00
55	Efron's Percentile Bootstrap	0.71	3.61	g_2	10.00
56	Efron's Percentile Bootstrap	1.00	6.37	G_2	10.00
57	Efron's Percentile Bootstrap	0.36	2.92	b_2	10.00
58	Efron's Percentile Bootstrap	0.68	2.55	g_2	100.00
59	Efron's Percentile Bootstrap	0.72	2.69	G_2	100.00
60	Efron's Percentile Bootstrap	0.65	2.50	b_2	100.00
61	Efron's Percentile Bootstrap	0.65	3.43	g_2	20.00
62	Efron's Percentile Bootstrap	0.82	4.48	G_2	20.00
63	Efron's Percentile Bootstrap	0.51	3.09	b_2	20.00
64	Efron's Percentile Bootstrap	0.62	3.23	g_2	30.00
65	Efron's Percentile Bootstrap	0.74	3.84	G_2	30.00

Table 4.7
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
66	Efron's Percentile Bootstrap	0.54	3.02	b_2	30.00
67	Efron's Percentile Bootstrap	0.75	2.04	g_2	300.00
68	Efron's Percentile Bootstrap	0.76	2.08	G_2	300.00
69	Efron's Percentile Bootstrap	0.74	2.03	b_2	300.00
70	Efron's Percentile Bootstrap	0.64	2.91	g_2	50.00
71	Efron's Percentile Bootstrap	0.72	3.23	G_2	50.00
72	Efron's Percentile Bootstrap	0.58	2.80	b_2	50.00
73	Hall's Percentile Bootstrap	0.34	3.62	g_2	10.00
74	Hall's Percentile Bootstrap	0.61	6.41	G_2	10.00
75	Hall's Percentile Bootstrap	0.20	2.93	b_2	10.00
76	Hall's Percentile Bootstrap	0.62	2.53	g_2	100.00
77	Hall's Percentile Bootstrap	0.66	2.66	G_2	100.00
78	Hall's Percentile Bootstrap	0.59	2.48	b_2	100.00
79	Hall's Percentile Bootstrap	0.46	3.44	g_2	20.00
80	Hall's Percentile Bootstrap	0.61	4.49	G_2	20.00
81	Hall's Percentile Bootstrap	0.36	3.10	b_2	20.00
82	Hall's Percentile Bootstrap	0.51	3.24	g_2	30.00
83	Hall's Percentile Bootstrap	0.61	3.85	G_2	30.00
84	Hall's Percentile Bootstrap	0.44	3.03	b_2	30.00
85	Hall's Percentile Bootstrap	0.72	2.07	g_2	300.00
86	Hall's Percentile Bootstrap	0.74	2.11	G_2	300.00
87	Hall's Percentile Bootstrap	0.72	2.06	b_2	300.00

Table 4.7
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
88	Hall's Percentile Bootstrap	0.56	2.95	g_2	50.00
89	Hall's Percentile Bootstrap	0.63	3.27	G_2	50.00
90	Hall's Percentile Bootstrap	0.52	2.84	b_2	50.00

Table 4.8
Average Width and Coverage Probability of The Intervals When The Data Are Generated from Exponential(0,1)

	Method	Coverage Probability	Width	Estimator	Sample Size
1	Bias Corrected Standard Bootstrap	0.33	4.31	g_2	10.00
2	Bias Corrected Standard Bootstrap	0.65	7.61	G_2	10.00
3	Bias Corrected Standard Bootstrap	0.18	3.49	b_2	10.00
4	Bias Corrected Standard Bootstrap	0.61	4.36	g_2	100.00
5	Bias Corrected Standard Bootstrap	0.66	4.59	G_2	100.00
6	Bias Corrected Standard Bootstrap	0.59	4.28	b_2	100.00
7	Bias Corrected Standard Bootstrap	0.45	4.83	g_2	20.00
8	Bias Corrected Standard Bootstrap	0.60	6.30	G_2	20.00
9	Bias Corrected Standard Bootstrap	0.38	4.36	b_2	20.00
10	Bias Corrected Standard Bootstrap	0.51	4.91	g_2	30.00
11	Bias Corrected Standard Bootstrap	0.60	5.84	G_2	30.00
12	Bias Corrected Standard Bootstrap	0.45	4.58	b_2	30.00
13	Bias Corrected Standard Bootstrap	0.70	3.71	g_2	300.00
14	Bias Corrected Standard Bootstrap	0.72	3.77	G_2	300.00

Table 4.8
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
15	Bias Corrected Standard Bootstrap	0.70	3.69	b_2	300.00
16	Bias Corrected Standard Bootstrap	0.55	4.66	g_2	50.00
17	Bias Corrected Standard Bootstrap	0.62	5.16	G_2	50.00
18	Bias Corrected Standard Bootstrap	0.51	4.47	b_2	50.00
19	Bias Corrected Percentile Bootstrap	0.61	4.48	g_2	10.00
20	Bias Corrected Percentile Bootstrap	0.87	7.92	G_2	10.00
21	Bias Corrected Percentile Bootstrap	0.26	3.63	b_2	10.00
22	Bias Corrected Percentile Bootstrap	0.68	4.56	g_2	100.00
23	Bias Corrected Percentile Bootstrap	0.72	4.81	G_2	100.00
24	Bias Corrected Percentile Bootstrap	0.66	4.48	b_2	100.00
25	Bias Corrected Percentile Bootstrap	0.69	5.88	g_2	20.00
26	Bias Corrected Percentile Bootstrap	0.82	7.67	G_2	20.00
27	Bias Corrected Percentile Bootstrap	0.62	5.32	b_2	20.00
28	Bias Corrected Percentile Bootstrap	0.67	5.95	g_2	30.00
29	Bias Corrected Percentile Bootstrap	0.74	7.08	G_2	30.00
30	Bias Corrected Percentile Bootstrap	0.62	5.57	b_2	30.00
31	Bias Corrected Percentile Bootstrap	0.75	3.55	g_2	300.00
32	Bias Corrected Percentile Bootstrap	0.76	3.61	G_2	300.00
33	Bias Corrected Percentile Bootstrap	0.74	3.53	b_2	300.00
34	Bias Corrected Percentile Bootstrap	0.68	5.52	g_2	50.00
35	Bias Corrected Percentile Bootstrap	0.73	6.13	G_2	50.00
36	Bias Corrected Percentile Bootstrap	0.64	5.32	b_2	50.00

Table 4.8
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
37	Classical	0.12	2.96	g_2	10.00
38	Classical	0.50	5.23	G_2	10.00
39	Classical	0.03	2.40	b_2	10.00
40	Classical	0.28	1.78	g_2	100.00
41	Classical	0.32	1.88	G_2	100.00
42	Classical	0.24	1.75	b_2	100.00
43	Classical	0.21	2.98	g_2	20.00
44	Classical	0.43	3.89	G_2	20.00
45	Classical	0.15	2.69	b_2	20.00
46	Classical	0.25	2.75	g_2	30.00
47	Classical	0.38	3.26	G_2	30.00
48	Classical	0.19	2.57	b_2	30.00
49	Classical	0.27	1.08	g_2	300.00
50	Classical	0.28	1.10	G_2	300.00
51	Classical	0.26	1.07	b_2	300.00
52	Classical	0.29	2.34	g_2	50.00
53	Classical	0.35	2.59	G_2	50.00
54	Classical	0.24	2.25	b_2	50.00
55	Efron’s Percentile Bootstrap	0.31	4.01	g_2	10.00
56	Efron’s Percentile Bootstrap	0.83	7.09	G_2	10.00
57	Efron’s Percentile Bootstrap	0.03	3.25	b_2	10.00
58	Efron’s Percentile Bootstrap	0.59	4.07	g_2	100.00

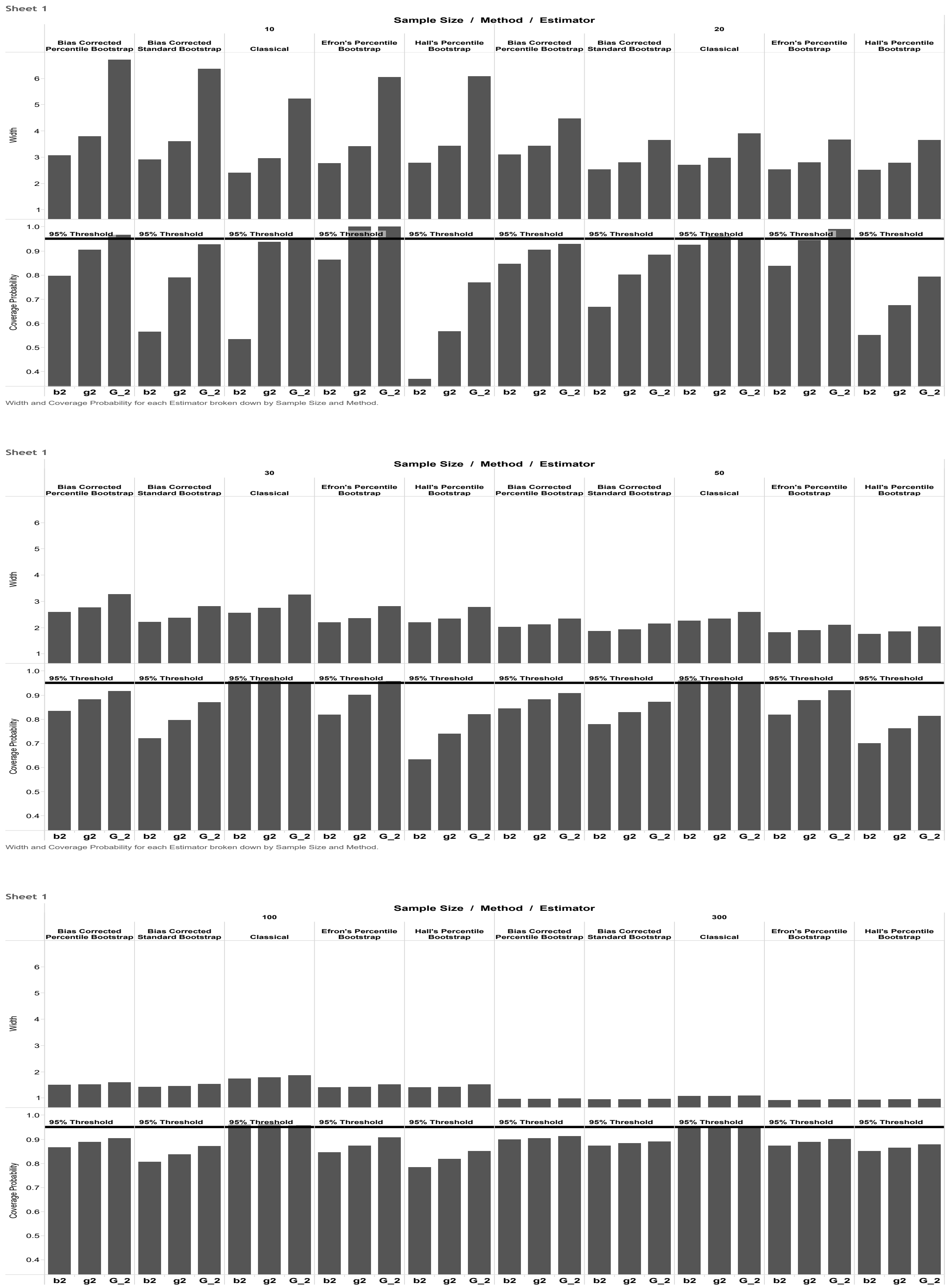
Table 4.8
Average Width and Coverage Probability (Continued)

	Method	Coverage Probability	Width	Estimator	Sample Size
59	Efron's Percentile Bootstrap	0.64	4.29	G_2	100.00
60	Efron's Percentile Bootstrap	0.57	3.99	b_2	100.00
61	Efron's Percentile Bootstrap	0.49	4.67	g_2	20.00
62	Efron's Percentile Bootstrap	0.69	6.08	G_2	20.00
63	Efron's Percentile Bootstrap	0.38	4.21	b_2	20.00
64	Efron's Percentile Bootstrap	0.51	4.67	g_2	30.00
65	Efron's Percentile Bootstrap	0.64	5.56	G_2	30.00
66	Efron's Percentile Bootstrap	0.44	4.37	b_2	30.00
67	Efron's Percentile Bootstrap	0.71	3.54	g_2	300.00
68	Efron's Percentile Bootstrap	0.73	3.60	G_2	300.00
69	Efron's Percentile Bootstrap	0.70	3.52	b_2	300.00
70	Efron's Percentile Bootstrap	0.56	4.51	g_2	50.00
71	Efron's Percentile Bootstrap	0.63	4.99	G_2	50.00
72	Efron's Percentile Bootstrap	0.51	4.33	b_2	50.00
73	Hall's Percentile Bootstrap	0.23	4.01	g_2	10.00
74	Hall's Percentile Bootstrap	0.48	7.08	G_2	10.00
75	Hall's Percentile Bootstrap	0.13	3.24	b_2	10.00
76	Hall's Percentile Bootstrap	0.56	4.09	g_2	100.00
77	Hall's Percentile Bootstrap	0.60	4.30	G_2	100.00
78	Hall's Percentile Bootstrap	0.54	4.00	b_2	100.00
79	Hall's Percentile Bootstrap	0.37	4.69	g_2	20.00
80	Hall's Percentile Bootstrap	0.51	6.11	G_2	20.00

Table 4.8
Average Width and Coverage Probability (Continued)

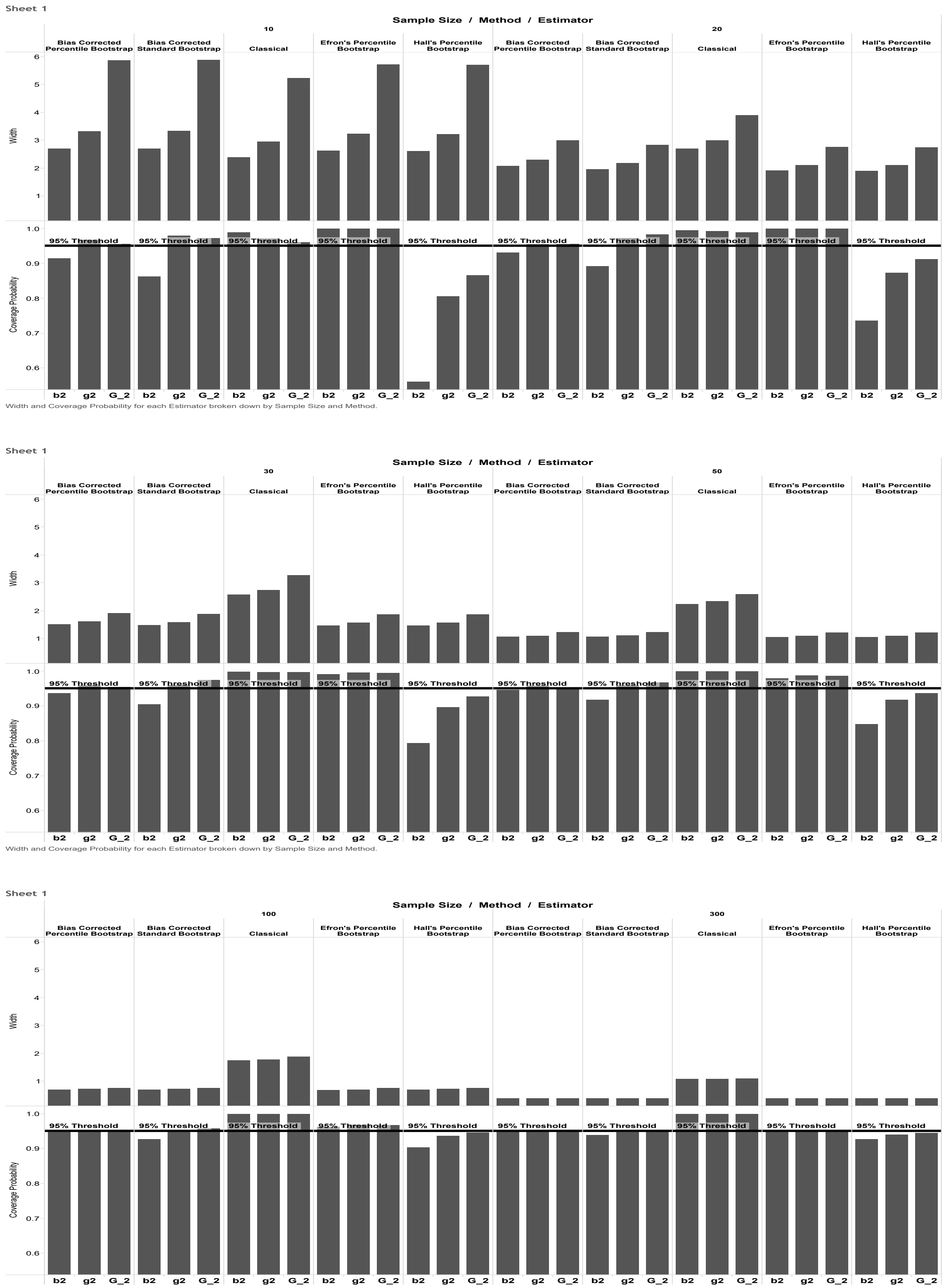
	Method	Coverage Probability	Width	Estimator	Sample Size
81	Hall's Percentile Bootstrap	0.30	4.23	b_2	20.00
82	Hall's Percentile Bootstrap	0.42	4.65	g_2	30.00
83	Hall's Percentile Bootstrap	0.53	5.53	G_2	30.00
84	Hall's Percentile Bootstrap	0.37	4.35	b_2	30.00
85	Hall's Percentile Bootstrap	0.68	3.49	g_2	300.00
86	Hall's Percentile Bootstrap	0.69	3.55	G_2	300.00
87	Hall's Percentile Bootstrap	0.67	3.47	b_2	300.00
88	Hall's Percentile Bootstrap	0.49	4.41	g_2	50.00
89	Hall's Percentile Bootstrap	0.56	4.88	G_2	50.00
90	Hall's Percentile Bootstrap	0.46	4.23	b_2	50.00

FIGURE 4.1: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from Standard Normal Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300



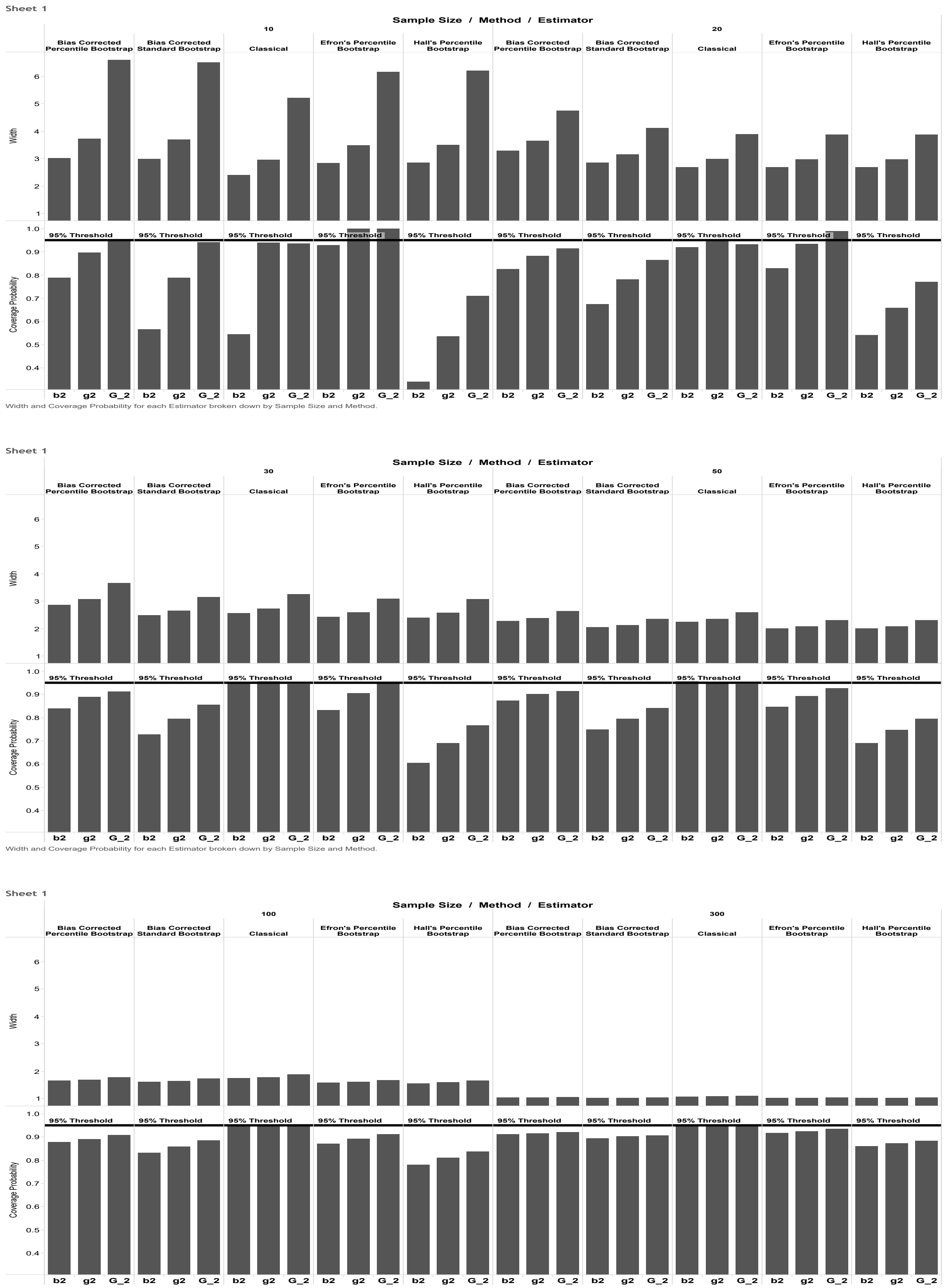
Normal(0,1)

FIGURE 4.2: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from Beta(2,2) Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300



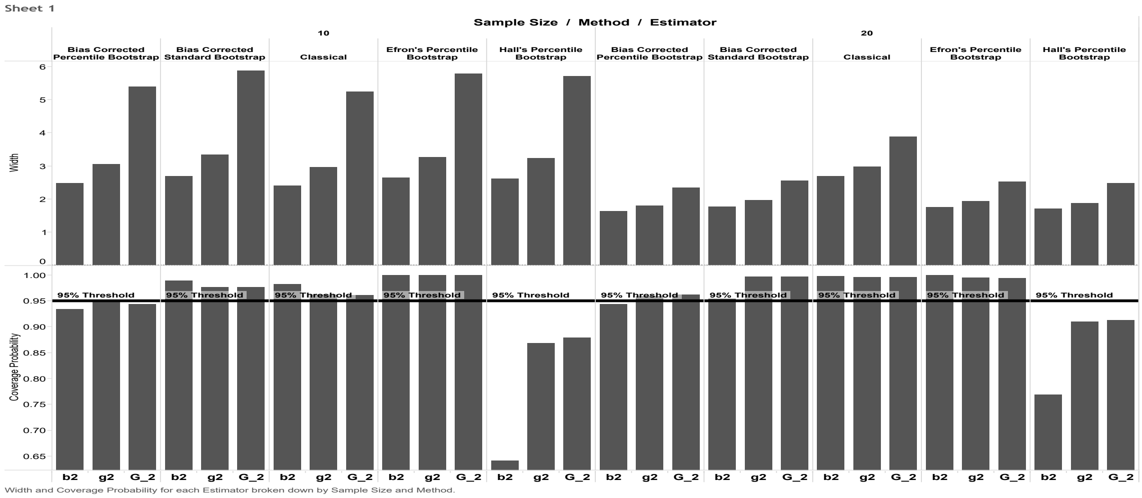
Beta(2,2)

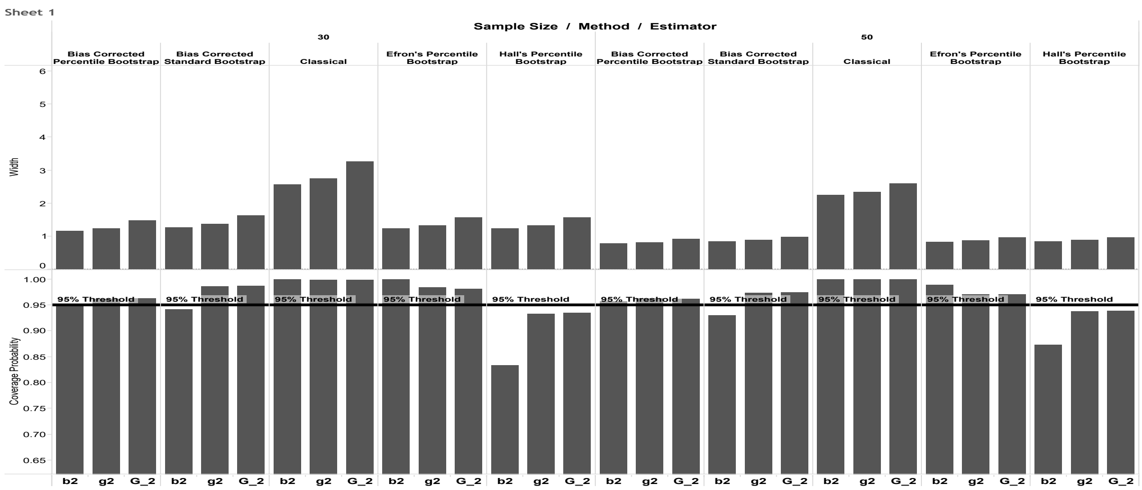
FIGURE 4.3: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from Beta(2,5) Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300

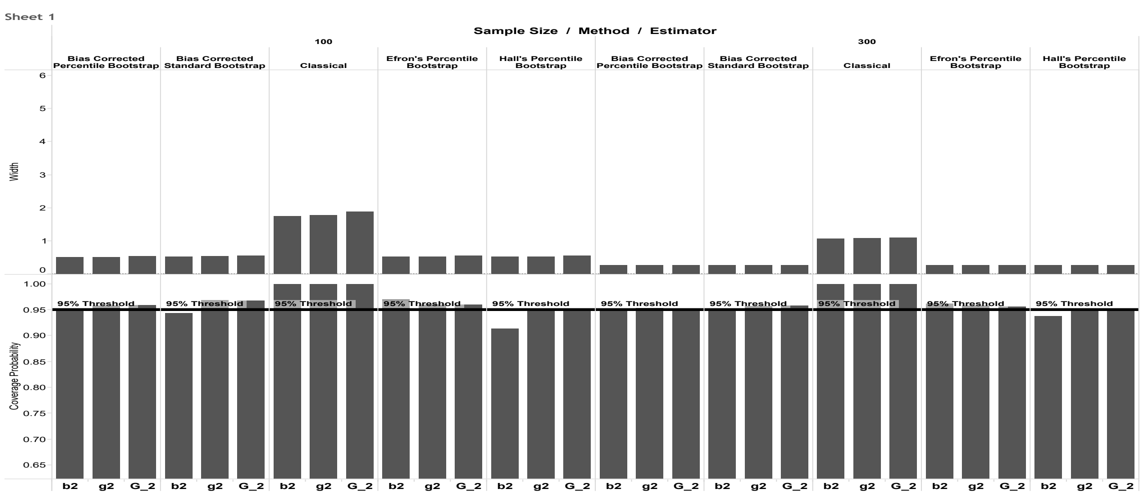


Beta(2,5)

FIGURE 4.4: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from Standard Uniform Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300

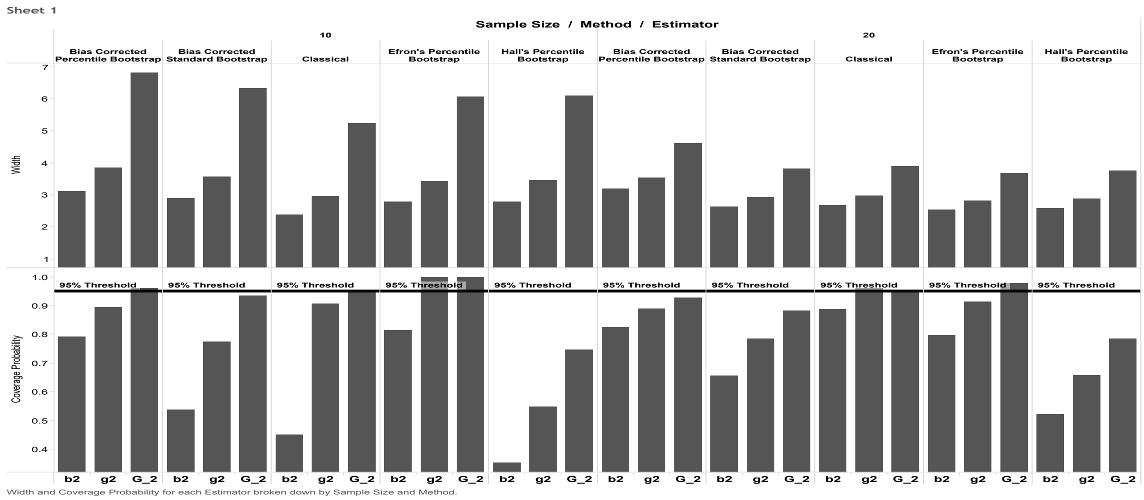


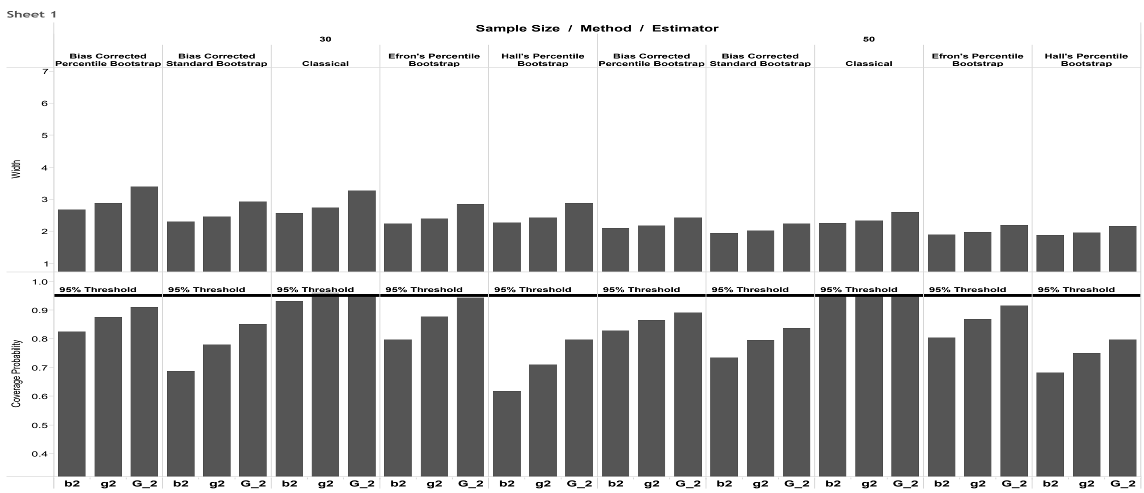


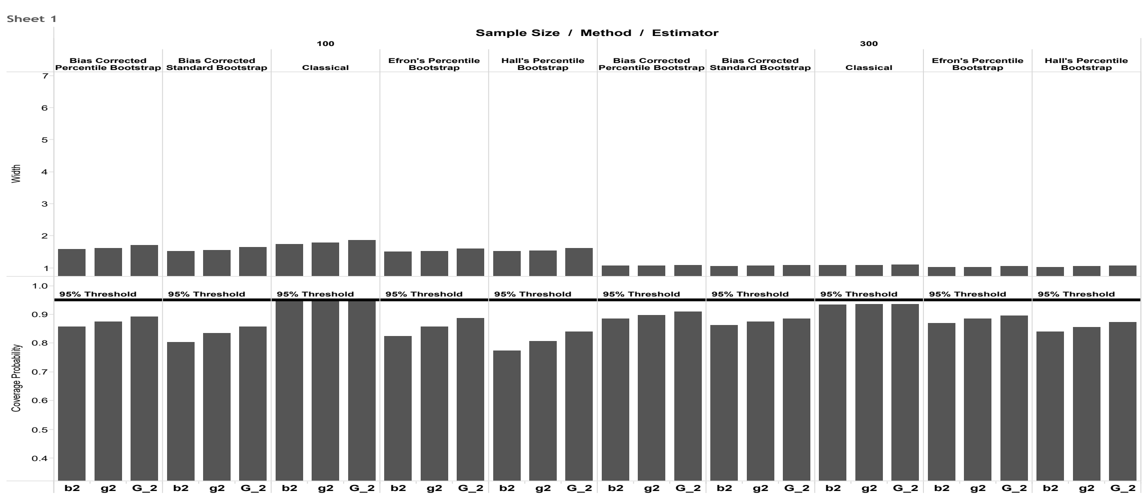


Uniform[0,1]

FIGURE 4.5: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from T(df=64) Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300

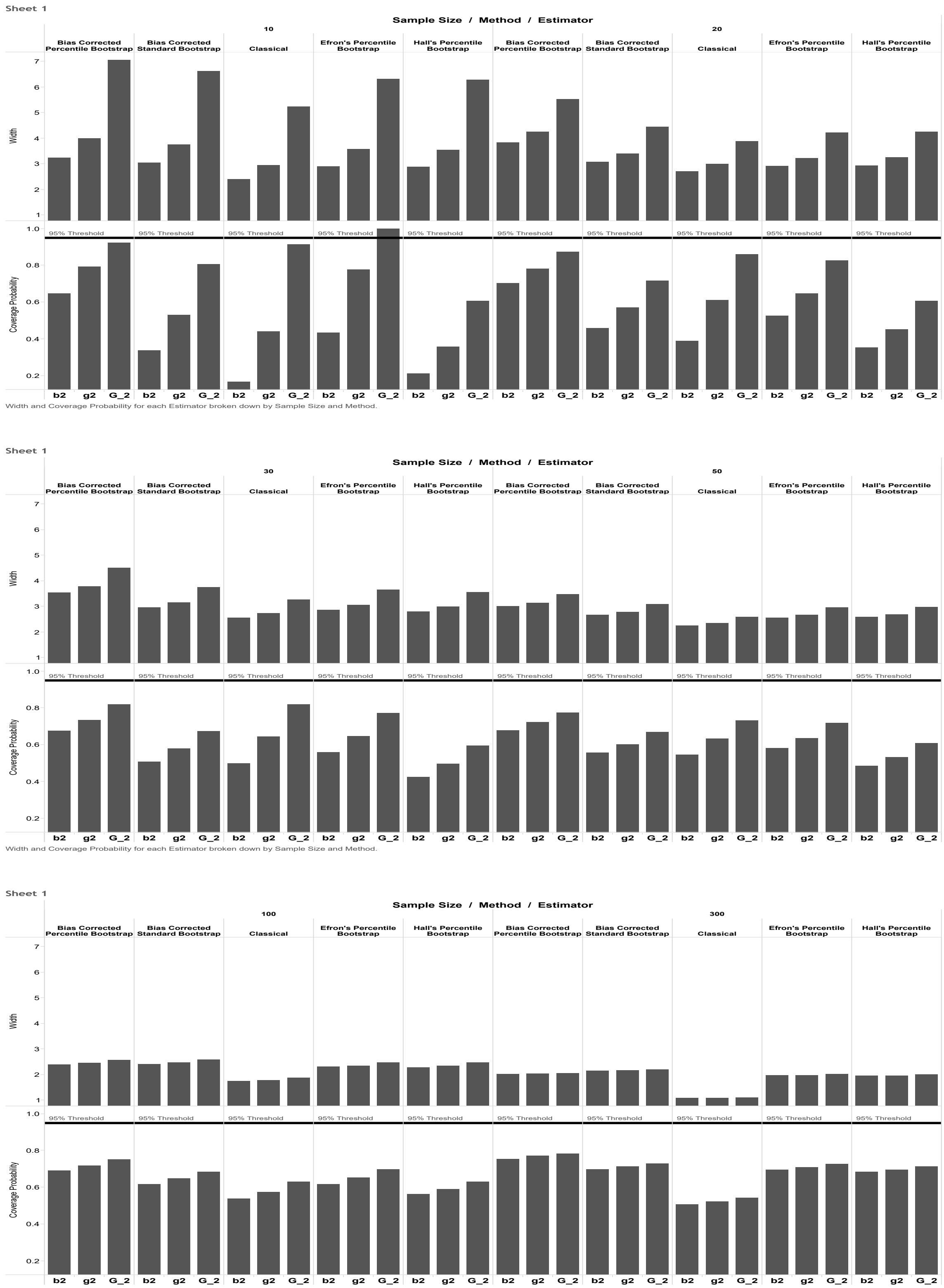






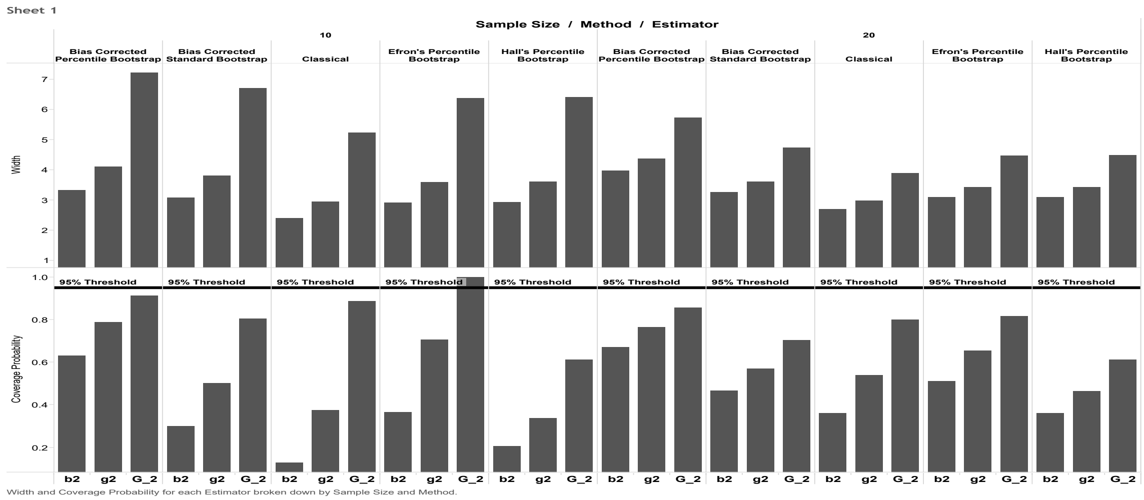
$T_{df=64}$

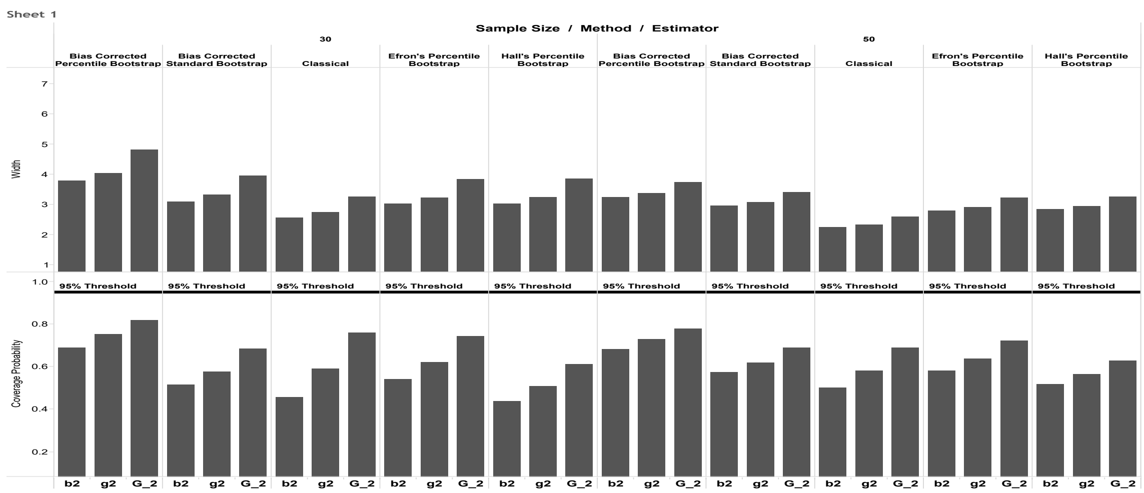
FIGURE 4.6: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from T(df=10) Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300

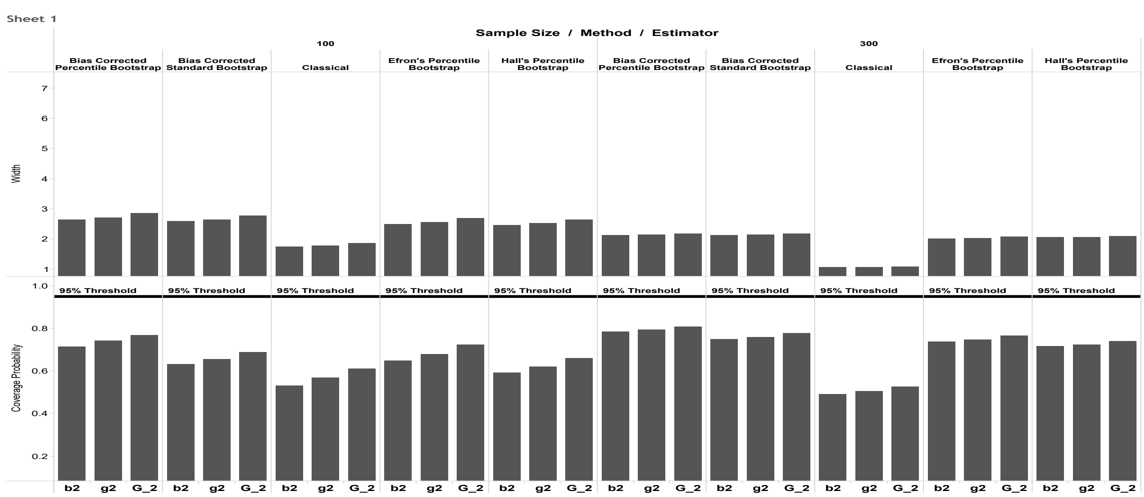


$T_{df=10}$

FIGURE 4.7: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from Standard Logistic Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300

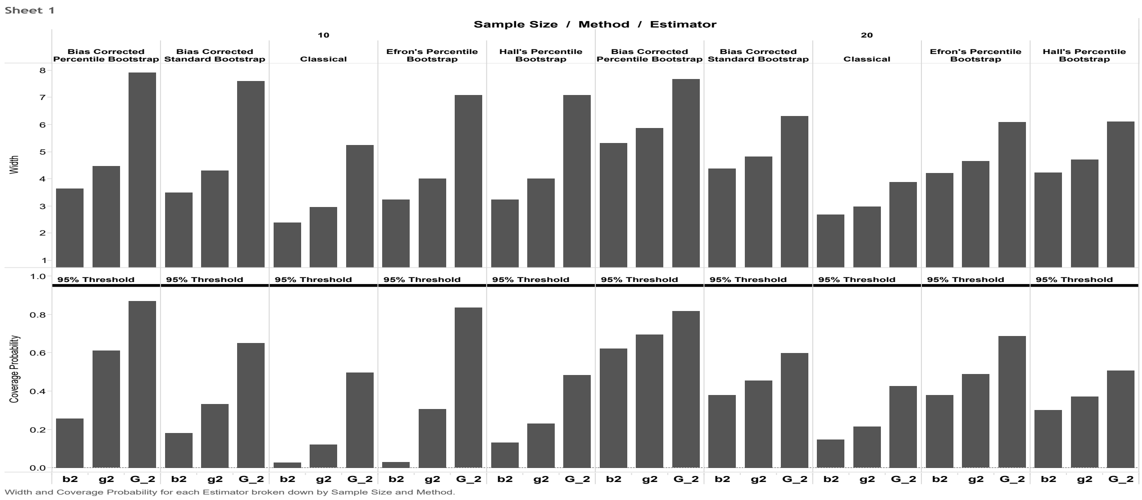


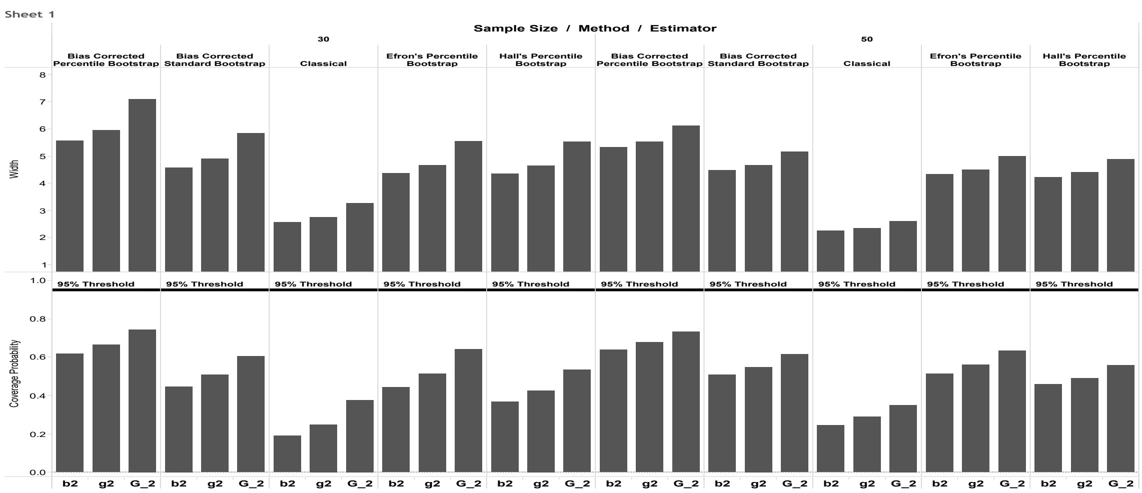


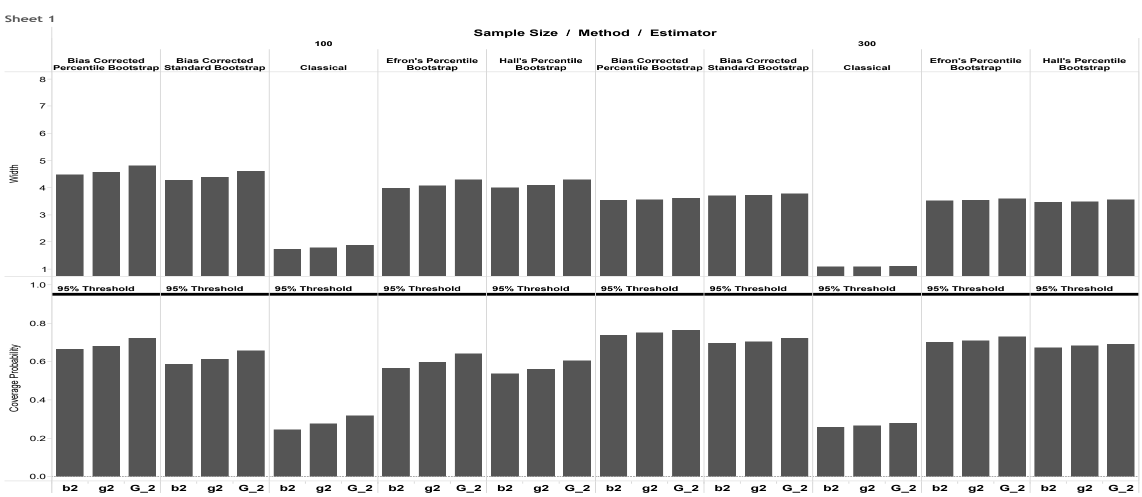


Logistic(0,1)

FIGURE 4.8: Average Width and Coverage Probability of The Confidence Intervals When Data Were Generated from Standard Exponential Distribution of Sample Size n = 10, 20, 30, 50, 100 and 300







Exponential(0,1)

CHAPTER 5

APPLICATIONS

5.1 Box Office Documentary Films

To illustrate the findings of this research, data set of lifetime gross revenue of documentaries are analyzed in this chapter. The following data are of the top 40 highest grossing movies of all time according to Box Office Mojo (Data Source:*Documentary*). The goal is to find the kurtosis of the sample.

\$119194771.00	\$77437223.00	73013910.00	\$72091016.00	\$33449086.00	32011576.00
\$28972764.00	\$28873374.00	\$25326071.00	\$24540079.00	\$24146161.00	\$21576018.00
\$19422319.00	\$17780194.00	\$16432322.00	\$15428747.00	\$15012935.00	\$14444502.00
\$14363397.00	\$13099931.00	\$13011160.00	\$11689053.00	\$11536423.00	10101037.00
\$8413144.00	\$8117961.00	\$8020721.00	\$7830611.00	\$7720487.00	\$7718961.00
\$7320323.00	\$7128031.00	\$7033803.00	\$6706368.00	\$6417135.00	\$6206566.00
\$6047363.00	\$5728581.00	\$5705874.00			

Some preliminary data analysis showed that the data is heavily skewed with major outliers in the tails, creating the case of positive kurtosis. Even with simply glancing at the available values, we see that the accumulated income of the first five entries is higher than that of the next 20 movies. We were able to fit a 3-parameter gamma distribution with shape and rate values of $\alpha = 0.5563$, $\beta = 1/27787166$ and a threshold of 5677456 (See Figure 5.1). We tested our hypothesis with a Komolgorov Smirnov test yielding a ks statistics of $ks = 0.0094$ with a p-value of $p = 0.7724$

The mean is \$$21136186$ and standard deviation \$$23927505$ with sample kurtosis 6.40537. Since we fit a gamma distribution to the sample, then its kurtosis parameter is that of gamma which is $\frac{6}{\sqrt{\alpha}} = 8.044$

We noticed that in Table (5.1) only the classical method had cases where the population kurtosis was not captured and this was to be expected. When we analyzed the results of our simulations, we noticed that classical cases had a harder time performing when it comes to capturing kurtosis

	Method	Lower Bound	Upper Bound	Width
1	Bias Corrected g2	-1.3299	15.0839	16.4138
2	Bias Corrected G2	-1.3744	17.3646	18.7389
3	Bias Corrected b2	-1.1783	14.0568	15.2351
4	Hall's Percentile g2	-2.086	12.3123	14.3983
5	Hall's Percentile G2	-2.4384	14.0096	16.4479
6	Hall's Percentile b2	-2.7883	11.5547	14.343
7	Efron's Perntile g2	0.2838	15.5396	15.2559
8	Efron's Percentile B2	0.6955	17.0933	16.3978
9	Efron's Percentile b2	0.1357	14.2165	14.0808
10	Bias Corrected Percentile g2	1.7751	23.3973	21.6222
11	Bias Corrected Percentile G2	2.3075	26.3801	24.0727
12	Bias Corrected Percentile b2	1.5728	22.4045	20.8318
13	Classical g2	5.1428	7.6679	2.5251
14	Classical G2	6.0152	8.8869	2.8717
15	Classical b2	4.7408	7.1412	2.4004

TABLE 5.1:
capitalisewordsUpper and Lower Confidence Interval Bounds trying to estimate the population parameter 8.044

parameters of distributions with positive kurtosis. Classical G_2 had the smallest interval that contained the kurtosis parameter followed by Efron's Percentile b_2. Bias Corrected Percentile G_2 had the largest interval.

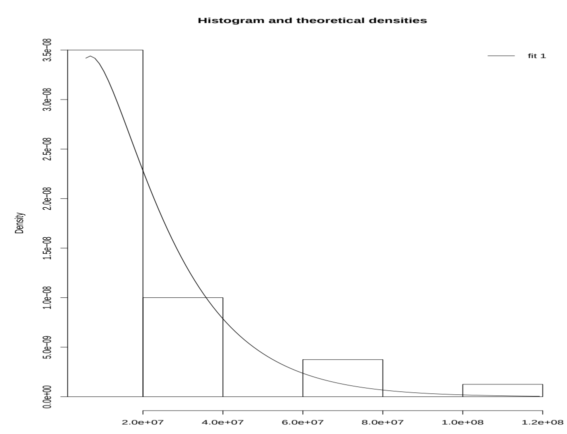


FIGURE 5.1: Histogram of Top 40 Highest Gross Documentaries of All Time

5.2 Bonds Return Over Time

Another real life example is to model the data from stock market and to calculate the average gain per day during a certain period. The funds we considered was a bond named GMO Opportunistic Income Class IV (Ticker name: GMODX). Table 5.2 represents the daily growth of the funds from August 8, 2016 to the September 26, 2016—A total of 35 business days within that period.

24.71	24.71	24.72	24.73	24.73	24.73	24.74	24.75
24.75	24.76	24.77	24.77	24.79	24.80	24.81	24.81
24.82	24.83	24.84	24.85	24.85	24.86	24.87	24.90
24.89	24.92	24.93	24.93	24.96	24.96	24.96	24.96
24.94	24.95	25.00					

TABLE 5.2: GMODX Data from 08/08/2016 to 09/26/2016

Preliminary data analysis suggests that the data can be modeled by a uniform distribution since there is little to no growth between the time period we looked at. A ks test yielded a p-value of 0.7952, meaning that we do not have enough evidence against the assumption that our sample data is uniform. See Figure (5.2) for a histogram representation of the data. Since we were able to fit a uniform distribution, then its kurtosis parameter is -6/5. As we first mentioned, when it comes to distributions that are fairly symmetric, the confidence interval methods used in this paper do a great job in estimating the population parameter. Based on the results obtained in Table 5.3, we see that all methods, parametric or non-parametric, have the kurtosis -6/5 within the constructed intervals. Bias Corrected Percentile b_2 had the lowest interval width while classical G_2 had the highest.

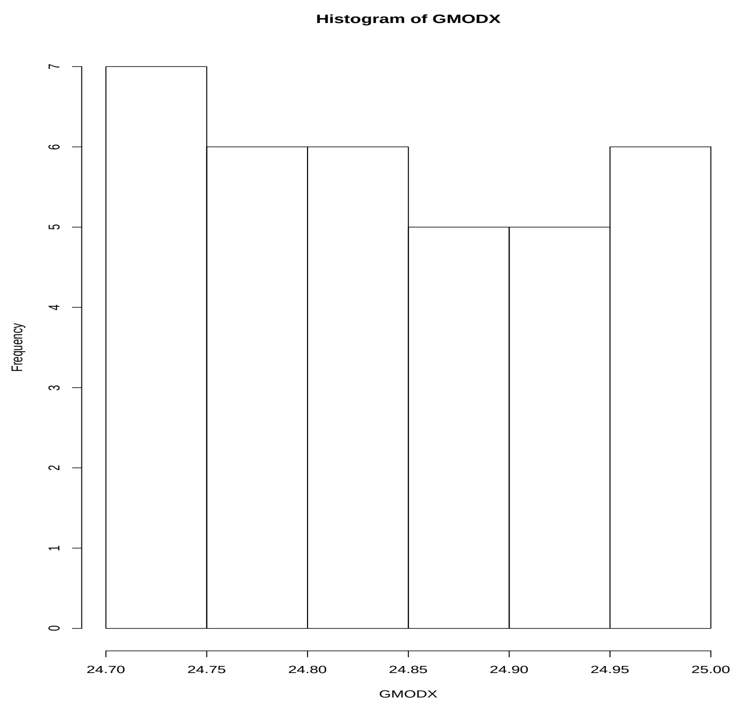


FIGURE 5.2: Histogram of Daily Return Between 08/08/2016 to 09/26/2016

	Method	Lower.Bound	Upper.Bound	Width
1	Bias Corrected g2	-1.8431	-0.8587	0.9844
2	Bias Corrected G2	-1.9629	-0.7821	1.1808
3	Bias Corrected b2	-1.9141	-0.9857	0.9284
4	Hall's Percentile g2	-1.968	-0.9801	0.9878
5	Hall's Percentile G2	-2.0721	-0.9514	1.1207
6	Hall's Percentile b2	-2.0342	-1.092	0.9422
7	Efron's Percentile g2	-1.5675	-0.6239	0.9436
8	Efron's Percentile B2	-1.6202	-0.4435	1.1767
9	Efron's Percentile b2	-1.6529	-0.7462	0.9067
10	Bias Corrected Percentile g2	-1.6252	-0.7786	0.8466
11	Bias Corrected Percentile G2	-1.6567	-0.6932	0.9635
12	Bias Corrected Percentile b2	-1.6646	-0.8843	0.7804
13	Classical g2	-2.5859	0.0445	2.6304
14	Classical G2	-2.8042	0.2447	3.0489
15	Classical b2	-2.6093	-0.127	2.4823

TABLE 5.3: Upper and lower confidence interval bounds trying to estimate the population parameter $-6/5$

CHAPTER 6

SUMMARY AND CONCLUSION

In this research, we explored the performance of three kurtosis estimators. Said performance was measured through the construction of confidence intervals. We first introduced what is kurtosis, its definition as well as misconceptions propagated over the years. Given the normal distribution as the basis of a zero kurtosis; a distribution with positive kurtosis tends to have a lot more outliers comparing to that of a negative kurtosis. We then introduced and derived the three main sample kurtosis that are wildly used in books and statistical software. While all three estimators g_2, G_2 and b_2 are generally biased, G_2 is an unbiased estimator for the normal distribution. Next, we introduced the different distributions that were studied in the paper and we derived many of their kurtosis parameters. After, we described our simulation process on how we generated our data for various sample sizes. We then analyzed with recommendations as to which kurtosis estimator and confidence interval construction method performed better on different distribution with positive and negative kurtosis. The criteria of performance were judged on which had shorter intervals with coverage probability of at least meeting the 95% confidence threshold. We saw that when dealing with a normal distribution with kurtosis 0 or a distribution that has kurtosis close to zero, then the classical method with parameter G_2 is always best to use in constructing confidence intervals. When dealing with negative kurtosis distribution, the classical method did perform well. But we noticed that Efron's Percentile Method performed equally as well, but with smaller intervals. For skewed distributions, it is best to use Efron's Percentile Method with sample estimator G_2. Last, for distributions with positive kurtosis, almost all methods had a hard time reaching that 95% threshold of confidence intervals. Efron's percentile method got close but rarely reached a 95%

confidence level. The construction of confidence intervals using the classical method for distributions with high positive kurtosis had poor performance of capturing the kurtosis parameter of a given distribution. We then gave two real life examples of how we can use the different confidence intervals for each of the three estimators in capturing the kurtosis parameter of a distribution. We used a sample data from the highest grossing documentaries and was successful in capturing the kurtosis parameter in 13 out of the 15 constructed intervals. We also explored data from the stock market, where we looked at daily returns of bonds in over a month. We were able to fit the data to a uniform distribution. When we attempted in estimating its kurtosis parameter, every confidence interval techniques with all three estimators succeeded in estimating the true parameter.

The kurtosis is one of the least used data characterization if compared to its sisters the mean and variance. But in the financial industry, the kurtosis plays an important role in calculating risks, also known as the "Kurtosis Risk". This is the risk when a statistical model assumes the normal distribution but most of its returns, or worst losses are not clustered around the mean as expected. This issue was extensively researched by Mandelbrot. In his book, he argued that relying too much on the normal distribution is a serious flaw on many models (Mandelbrot and Hudson, 2010). This nightmare came to reality when Long-Term Capital Management, a hedge fund that went bankrupt because they understated the kurtosis of many of their financial securities (Krugman, 2012) Yes, even when the kurtosis is not as wildly used, there are some important life situations when being able to estimate the kurtosis parameter is necessary. So, knowing which confidence interval methods to use as well as which estimator best estimates the true kurtosis parameter could be very useful.

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