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An Assessment of the Performances of Several Univariate Tests of Normality

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AN ASSESSMENT OF THE PERFORMANCES OF SEVERAL UNIVARIATE TESTS OF NORMALITY

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James Olusegun Adefisoye

2015

To: Dean Michael R. Heithaus
College of Arts and Sciences

This thesis, written by James Olusegun Adefisoye, and entitled An Assessment of the Performances of Several Univariate Tests of Normality, having been approved in respect to style and intellectual content, is referred to you for judgment.

We have read this thesis and recommend that it be approved.

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Date of Defense: March 24, 2015

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DEDICATION

I dedicate this work to my lovely wife Temitope Christiana James-Adefisoye and my yet unborn children.

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ABSTRACT OF THE THESIS

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by

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Florida International University, 2015

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Professor B.M. Golam Kibria, Co-Major Professor

Professor Florence George, Co-Major Professor

The importance of checking the normality assumption in most statistical procedures especially parametric tests cannot be over emphasized as the validity of the inferences drawn from such procedures usually depend on the validity of this assumption. Numerous methods have been proposed by different authors over the years, some popular and frequently used, others, not so much. This study addresses the performance of eighteen of the available tests for different sample sizes, significance levels, and for a number of symmetric and asymmetric distributions by conducting a Monte-Carlo simulation. The results showed that considerable power is not achieved for symmetric distributions when sample size is less than one hundred and for such distributions, the kurtosis test is most powerful provided the distribution is leptokurtic or platykurtic. The Shapiro-Wilk test remains the most powerful test for asymmetric distributions. We conclude that different tests are suitable under different characteristics of alternative distributions.

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CHAPTER ONE: INTRODUCTION

1.1 Why Test Normality?

In theoretical and empirical research, there are assumptions that are usually tested to ensure the validity of inferences from such research; one of such assumptions is the normality assumption. Data often approximates a normal “bell-shaped” curve; some distributions become normal asymptotically. The normality or (lack thereof) of an underlying data distribution can have an effect to a greater or lesser degree on the properties of estimation or inferential procedures used in the analysis of the data.

The standard errors and consequently, the test statistics computed from such standard errors in parametric statistics such as the t-test, tests for regression coefficients, analysis of variance, and the F-test of homogeneity of variance include the tests that have as an underlying assumption, the distribution of the population from which the sample data was generated to have be normal. The validity of inferences from such tests usually depends on validity of the normality assumption. Also, the probability associated with the test statistics are derived from distributions that are normal or asymptotic normal. Normality is an important requirement for the data with random independent variables which is often are used in everyday research. If the independent variables are random, distributions with high kurtosis tend to give liberal tests and excessively small standard errors, while low kurtosis tends to produce the opposite effects (Bollen, 1989). The normality assumption is therefore very important and this has caused the Gaussian or normal distribution to be a long focal point of much of statistical study.

Checking the validity of the normality assumption in a statistical procedure can be done in two ways: empirical procedure using graphical analysis and the goodness-of- fit tests methods. The goodness-of-fit tests which are formal statistical procedures for assessing the underlying distribution of a data set are our focus here. These tests usually provide more reliable results than graphical analysis.

1.2 The Normal Distribution and its Characteristics

The normal distribution is a probability model for continuous variables. The probability density function of the normal distribution with mean μ and variance σ^2 is defined as:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad -\infty < x < \infty \,,\; -\infty < \mu < \infty \,, \sigma > 0 \qquad (1.1)$$

The normal distribution is completely determined by its parameters μ and σ^2. The density curve of a normal random variable is shown in Figure 1.1.

The normal distribution is the only absolutely continuous distribution all of whose cumulants beyond the first two (i.e., other than the mean and variance) are zero. It is also the continuous distribution with the maximum entropy for a given mean and variance.

The normal distribution is a subclass of the elliptical distributions. The normal distribution is symmetric about its mean, and is non-zero over the entire real line. The value of the normal distribution is practically zero when the value x lies more than a few standard deviations away from the mean. Therefore, it may not be an appropriate model when one expects a significant fraction of outliers and least squares and other statistical inference methods that are optimal for normally distributed variables often become highly unreliable when applied to such data. In such cases, a more heavy-tailed distribution should be assumed and the appropriate robust statistical inference methods applied.

In characterizing the location and variability of a data set, a further characterization of the data includes skewness and kurtosis. Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. The skewness for a normal distribution is zero, and any symmetric data should have a skewness near zero. Negative values for the skewness indicate that data are left skewed and positive values for the skewness indicate that data are right skewed. Kurtosis on the other hand is a measure of whether the data are peaked or flat. The normal distribution is a reference point and has a kurtosis coefficient of zero. Thus, data sets with high kurtosis tend to have a distinct peak near the mean, decline rather rapidly, and have heavy tails. Data sets with low kurtosis tend to have a flat top near the mean rather than a sharp peak. A uniform distribution would be the extreme case.

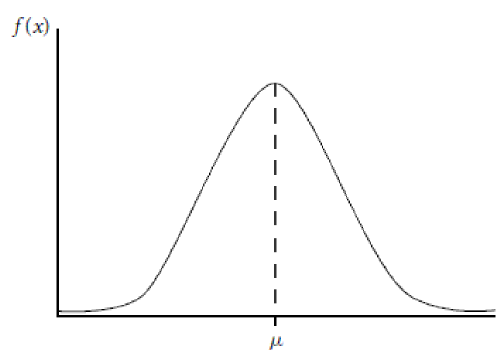

Figure 1.1 The Normal Distribution

The normal distribution can be rescaled through a process called standardization which allows us to obtain a dimensionless quantity by subtracting the population mean from an individual raw score and then dividing the difference by the population standard deviation. This transformation can be denoted as $z = \frac{x - \mu}{\sigma}$. The probability density function of z is then given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty \tag{1.2}$$

which has a mean of zero and a standard deviation of one. The density curve of the standard normal distribution is shown in Figure 1.2.

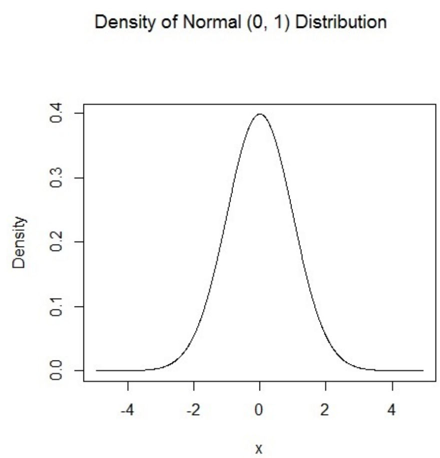

Figure 1.2 The Standard Normal Distribution

The Gaussian distribution belongs to the family of stable distributions which are the attractors of sums of independent, identically distributed distributions whether or not the mean or variance is finite.

The importance of the normal curve stems primarily from the fact that the distributions of many natural phenomena are at least approximately normally distributed. One of the first applications of the normal distribution was to the analysis of errors of measurement made in astronomical observations, errors that occurred because of imperfect instruments and imperfect observers. Galileo in the 17th century noted that these errors were symmetric and that small errors occurred more frequently than large errors. This led to several hypothesized distributions of errors, but it was not until the early 19th century that it was discovered that these errors followed a normal distribution. Independently, the mathematicians Adrian in 1808 and Gauss in 1809 developed the formula for the normal distribution and showed that errors were fit well by this distribution (Lane, *n.d.*).

This same normal distribution had been discovered by Laplace in 1778 when he derived the extremely important central limit theorem. Laplace showed that even if a distribution is not normally distributed, the means of repeated samples from the distribution would be very nearly normally distributed, and that the larger the sample size, the closer the distribution of means would be to a normal distribution (Lane, *n.d.*).

Most statistical procedures for testing differences between means assume normal distributions. Because the distribution of means is very close to normal, these tests work well even if the original distribution is only roughly normal. For more on normal distribution, readers are referred to Ahsanullah et al. (2014).

1.3 Alternative Distributions

For comparison purposes, data were generated from several alternative non-normal distributions to be able to examine the performances of the tests under consideration given different distributions of data. The alternative distributions are as highlighted below.

1.3.1 Symmetric Distributions

These include symmetric and short-tailed distributions such as Beta (1, 1), Beta (2, 2), Beta (3, 3), Uniform (0, 1) and T (10); and symmetric long-tailed distributions such as T (5) and Laplace (0, 1).

Beta (1, 1), Beta (2, 2), and Beta (3, 3)

The Beta family of distribution is a family of continuous probability distributions defined on the interval [0, 1] and parametrized by two positive shape parameters, denoted by α and β, that appear as exponents of the random variable and control the shape of the distribution. The distribution is often used as a prior distribution for binomial proportions in Bayesian analysis (Evans *et al.* 2000, p. 34). The probability distribution function (pdf) is given by:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \ , \qquad (1.3)$$

where α and β are the shape parameters.

With the parameters of the distribution set at $\alpha = 1, \beta = 1$; $\alpha = 2$, $\beta = 2$ and $\alpha = 3$, $\beta = 3$, we have three density curves that are symmetric and short-tailed but with varying length of the tails as can be seen in the figure below.

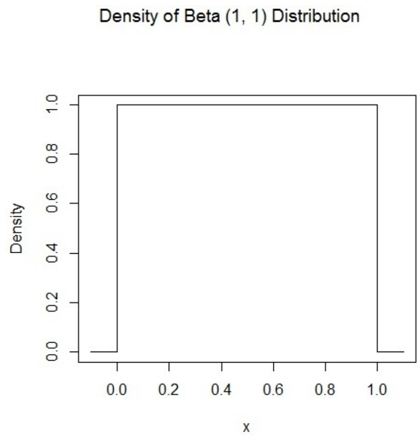

Figure 1.3(a) Density of Beta (1, 1)

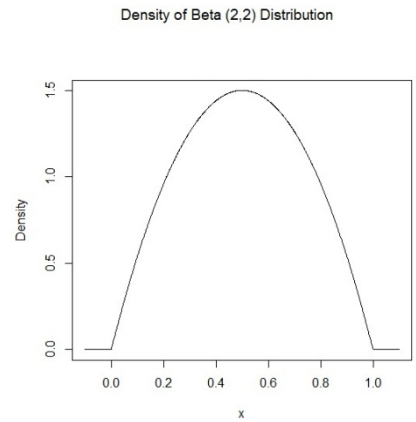

Figure 1.3(b)Density of Beta (2, 2)

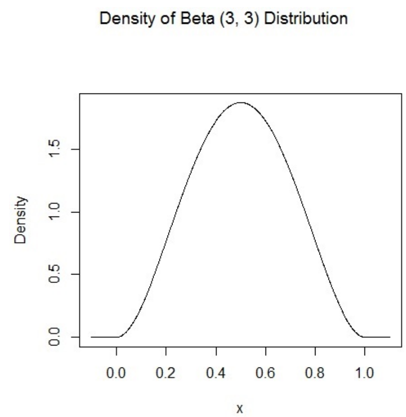

Figure 1.3(c) Density of Beta (3, 3)

Uniform (0, 1)

The uniform distribution, sometimes also known as a rectangular distribution, is a distribution that has constant probability. The probability density function (pdf) for a continuous uniform distribution on the interval $[\alpha, \beta]$ which are the parameters of the distribution is given by:

$$f(x) = \frac{1}{\beta - \alpha} \; . \qquad (1.4)$$

With the parameters of the distribution set at $\alpha = 0$, $\beta = 1$ we have the standard uniform distribution which is symmetric and short-tailed as shown in the figure below.

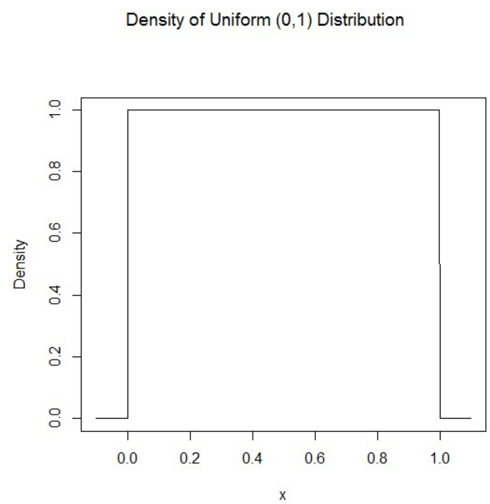

Figure 1.4 Density of a Uniform (0, 1) distribution

T (10) and T (5)

The t-distribution is any member of a family of continuous probability distributions that arises when estimating the mean of a normally distributed population in situations where the sample size is small and population standard deviation is unknown. The pdf of t is given by

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}, \tag{1.5}$$

where v is the number of degrees of freedom and Γ is the gamma function.

With the degree of freedom set at $v = 10$ and $v = 5$, we have a symmetric, short-tailed distribution and symmetric, long-tailed distribution respectively as shown in the figure below.

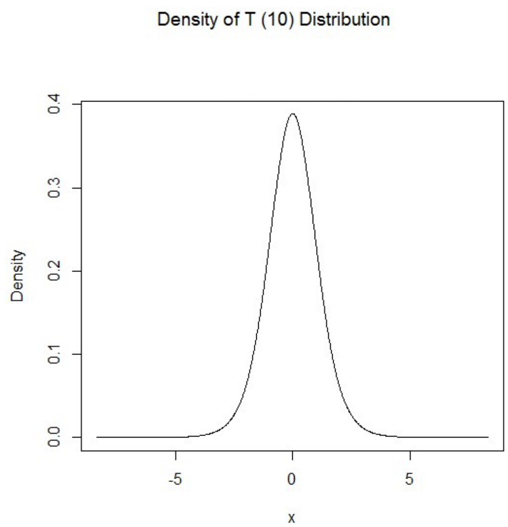

Fig 1.5 (a) Density of a T(10) distribution

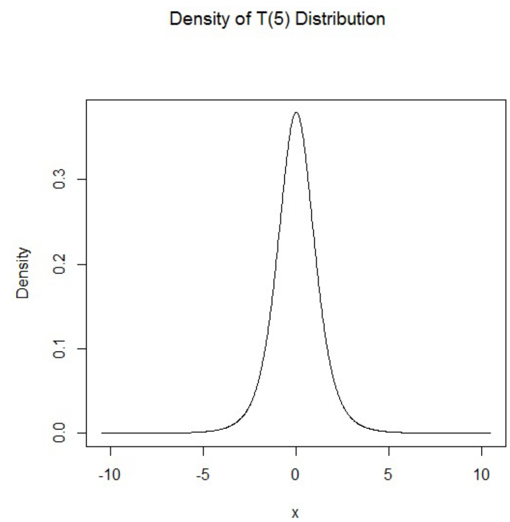

Fig 1.5 (b) Density of a T(5) distribution

Laplace (0, 1)

The Laplace distribution, also called the double exponential distribution, is the distribution of differences between two independent variates with identical exponential distributions (Abramowitz and Stegun, 1972). The probability density is given by

$$f(x) = \frac{1}{2b} e^{-|x-\mu|/b}, \tag{1.6}$$

where μ and b are the mean and rate respectively.

For this research work, the mean was set 0 and the rate at 1 and we have a symmetric and long-tailed distribution. The figure below shows the shape of the specified distribution:

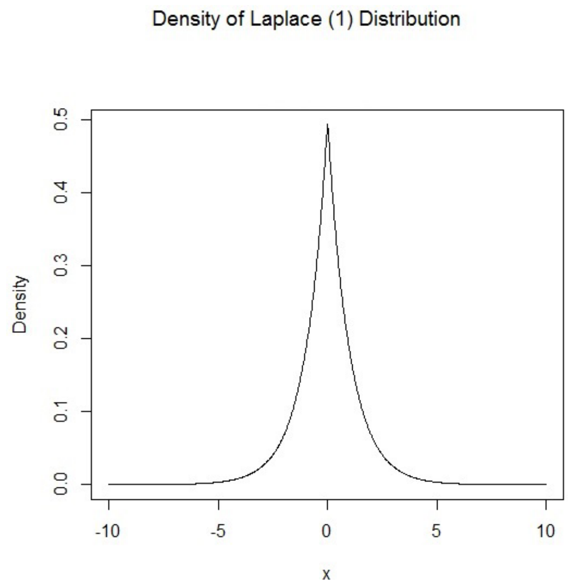

Figure 1.6 Density of a Laplace (0, 1) distribution

1.3.2 Asymmetric distributions

These include distributions such as Gamma (4,5), Chi-Square (3), Exponential (1), Log-Normal (0,1) which are asymmetric long-tailed; and Weibull (2,2) and Gompertz (10, 0.001) which are asymmetric short-tailed.

Gamma (4, 5)

The gamma distribution is a two-parameter family of continuous probability distributions. Gamma distributions have two free parameters, labeled α and θ, which are the shape and the scale parameter respectively. The pdf of the distribution is given by:

$$f(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^{\alpha}}, \tag{1.7}$$

with the parameters set at $\alpha = 4$ and $\theta = 5$, we have a right-skewed, long-tailed distribution as shown in the figure below.

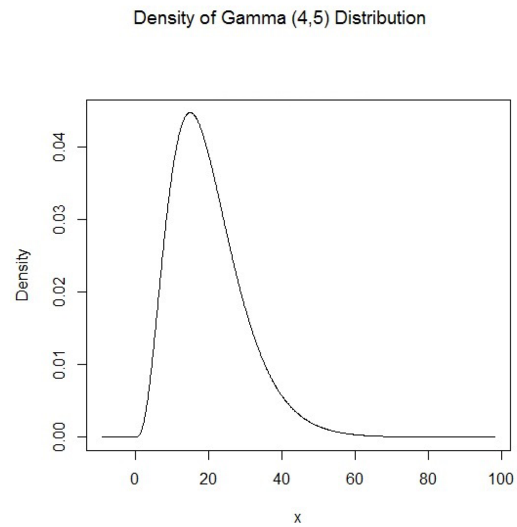

Figure 1.7 Density of a Gamma (4, 5) distribution

Chi-Square (3)

The chi-square distribution is one of the most widely used probability distributions in inferential statistics. It is a special case of the gamma distribution with $\alpha = v/2$ and $\theta = 2$. The chi-squared distribution with *v* degrees of freedom has a pdf given by:

$$f(x) = \frac{x^{(v/2-1)} e^{-x/2}}{2^{v/2}\Gamma(v/2)} \tag{1.8}$$

where $\Gamma(v/2)$ denotes the Gamma function, which has closed-form values for integer v. With k =3, we have a right-skewed, long-tailed distribution as shown below

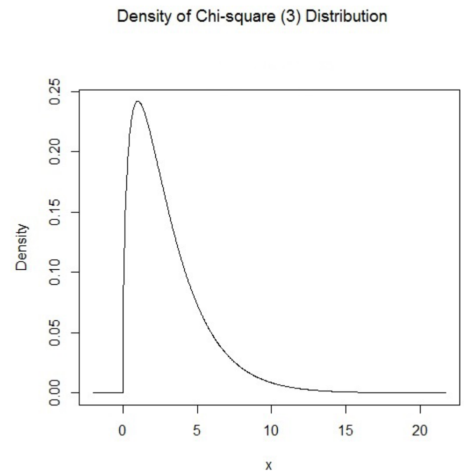

Figure 1.8 Density of a Chi-square (3) distribution

Exponential (1)

The exponential distribution is the probability distribution that describes the time between events in a Poisson process and as such, is commonly used for the analysis of Poisson processes. It is also special case of the gamma distribution with $\alpha = 1$ and $\theta = \lambda$.

The pdf is given by

$$f(x) = \lambda e^{-\lambda x}, \quad (1.9)$$

where $\lambda > 0$ is the parameter of the distribution, often called the rate parameter. The distribution is supported on the interval $[0, \infty)$.

With the rate set at $\lambda = 1$, we have a right-skewed, long-tailed distribution as shown below

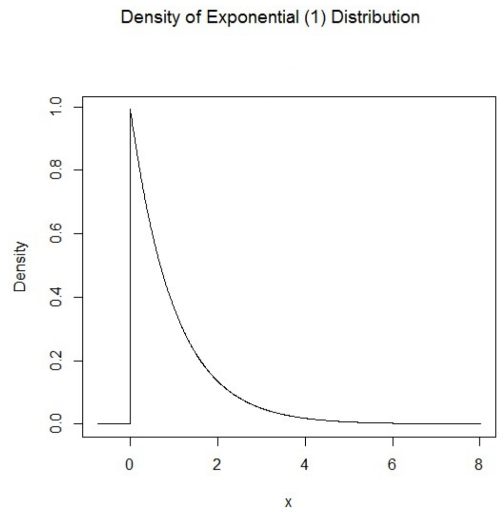

Figure 1.9 Density of an Exponential (1) distribution

Log-Normal (0, 1)

A log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed. Thus, if the random variable x is log-normally distributed, then $y = \log(x)$ has a normal distribution. Likewise, if y has a normal distribution, then $x = \exp(y)$ has a log-normal distribution. The log-normal distribution is the maximum entropy probability distribution for a random variate x for which the mean and variance of $\ln(x)$ are fixed. The distribution has the following pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \qquad (1.10)$$

where μ is the log-scale parameter and $\sigma > 0$ is the shape parameter.

With the parameters set at $\mu = 0$ and $\sigma = 1$, we have a left-skewed, long-tailed distribution as shown below.

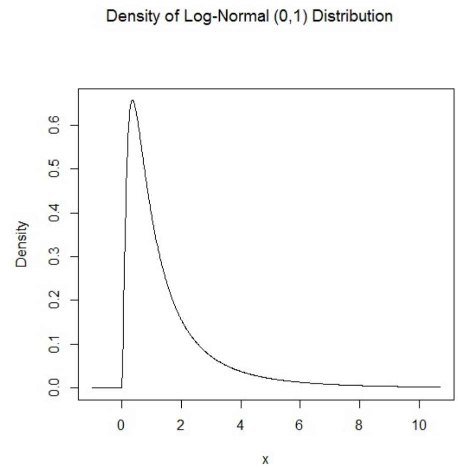

Figure 1.10 Density of a Log-Normal (0, 1) distribution

Gompertz (10, 0.001)

Gompertz distribution is a continuous probability distribution often applied to describe the distribution of adult lifespans. The pdf of the Gompertz distribution is:

$$f(x) = b\eta e^{bx} e^{\eta} \exp(-\eta e^{bx}), \tag{1.11}$$

where $b > 0$ is the scale parameter and $\eta > 0$ is the shape parameter of the distribution. With the parameters set at $b = 10$ and $\eta = 0.001$, we have a left-skewed, short-tailed distribution as shown below.

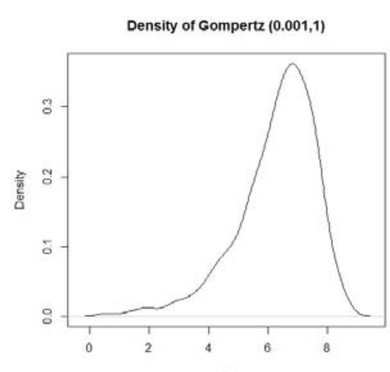

Figure 1.11 Density of a Gompertz (0.001, 1) distribution

Weibull (2, 2)

The Weibull distribution is a distribution used in the lifetimes of objects, life data analysis and reliability engineering. The pdf of the Weibull distribution is given by

$$f(x) = \alpha\beta^{-\alpha}x^{\alpha-1}e^{-(x/\beta)^{\alpha}}, \qquad (1.12)$$

where α is the shape parameter and β is the scale parameter. With the parameters set at $\alpha = 2$ and $\beta = 2$, we have a left-skewed, long-tailed distribution as shown below.

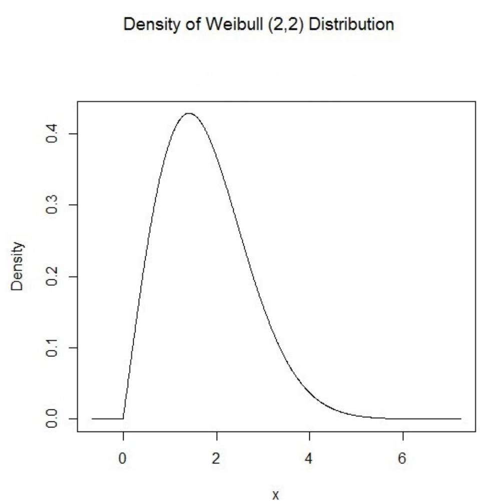

Figure 1.12 Density of a Weibull (2, 2) distribution

The organization of the thesis is as follows: Different statistical test for normality are presented in chapter 2. A simulation study has been conducted and results are presented in chapter 3. Applications to real life data to illustrate the findings of the thesis are presented in chapter 4, and concluding remarks are given in chapter 5.

CHAPTER TWO: TESTS OF NORMALITY

Since the normality assumption is an important aspect of most statistical procedures, it is necessary to device a highly robust and generally acceptable technique to perform this test. It is revealed that over forty (40) different test has been proposed over time to verify the normality or lack of normality in a population (Thode Jr., H.C. , 2002). The main goal of these researches have been to determine the performance of available test and/or propose an alternative to the previously existing. The performance of these test is usually measured in terms of the power of the test and the probability of type I error (α).

A test is said to be powerful when it has a high probability of rejecting the null hypothesis of normality when the sample under study is taken from a non-normal distribution. On the other hand, the type I error rate is the rate of rejection of the null hypothesis of normality when the distribution is truly normal. The best tests are those that have type I error rate around the specified significance level and have the higher power of detecting non-normality.

2.1 Lilliefors Test [LL]

Kolmogorov (1933) had introduced the famous Kolmogorov-Smirnov goodness-of-fit test used to test if a set of data fits a particular distribution for which Smirnov (1948) provided the table of critical values.

This Kolmogorov-Smirnov test required the specification of the parameters of the distribution being examined. Critical values were published in Smirnov (1948).

To test for normality, Lilliefors (1967) extended Kolmogorov's test for testing a composite hypothesis that the data came from a normal distribution with unknown location and scale parameter. The test statistic is defined as:

$$D = Sup_x \left| F^*(x) - S_n(x) \right| , \tag{2.1}$$

where $S_n(x)$ is the sample cumulative distribution function and $F^*(x)$ is the cumulative distribution function (CDF) of the null distribution.

The Lilliefor's test is similar to the Kolmogorov-Smirnov test but the distribution of the test statistic under H_0 is different and hence has a different critical value.

2.2 Anderson–Darling Test [AD]

The AD test was proposed by Anderson and Darling (1952). The test is used to test whether a given sample of data is drawn from a given probability distribution. It tests the hypothesis that a sample has been drawn from a population with a specified continuous distribution function $F(x)$. The AD test is of the form

$$AD = n\int_{-\infty}^{\infty} \left[F_n(x) - \Phi(x)\right]^2 \psi(x) dF(x) , \tag{2.2}$$

where $F_n(x)$ is the empirical distribution function (EDF), $\Phi(x)$ is the cumulative distribution function of the standard normal distribution and $\psi(x)$ is a weight function.

Let $x_1, x_2, ..., x_n$ be n sample observations under H_0, and let $x_{(1)} < x_{(2)} < ... < x_{(n)}$ be the n

ordered sample observations, then AD can be expressed as

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2j-1)[\ln \mu_i + \ln(1-\mu_{n-i+1})] , \tag{2.3}$$

Where $\mu_i = F(x_{(i)})$ and $x_{(i)}$ is the ith ordered statistic.

The null hypothesis is rejected for large values of the test statistic.

The AD test is one of the best empirical distribution function statistics for detecting most departures from normality (Stephens, 1974), (Petrovich, *n.d.*). Very large sample sizes may reject the assumption of normality with only slight imperfections, but industrial data with sample sizes of two hundred (200) and more have passed the Anderson–Darling test and may not produce a result (Petrovich, *n.d.*).

2.3 Chi-Square Test [CS]

The chi-square goodness-of-fit test (Snedecor and Cochran, 1989) is used to test if a sample of data came from a population with a specified distribution. The test statistic is defined as:

$$\chi^2 = \sum_{i=1}^{k}\frac{(O_i - E_i)^2}{E_i} , \tag{2.4}$$

where 'O_i' and 'E_i' refers to the *i*th observed and expected frequencies respectively and k is the number of bins/groups.

An attractive feature of the chi-square goodness-of-fit test is that it can be applied to any univariate distribution for which you can calculate the cumulative distribution function. The chi-square goodness-of-fit test is applied to binned data (i.e., data put into classes). This is actually not a restriction since for non-binned data you can simply calculate a histogram or frequency table before generating the chi-square test. However, the value of the chi-square test statistic is dependent on how the data is binned. To bin the data, the recommendation of Moore (1986) was adopted in this study.

2.4 Skewness Test [SK]

The skewness test is derived from the third sample moment. It is used to test the null hypothesis of normality versus non-normality associated with skewness. The coefficient of skewness of a set of data can be used to determine if it came from a population that is normally distributed (Bai and Ng, 2005). The skewness statistic is defined as:

$$g_1 = k_3 / \sqrt{(s^2)^3} \ ,$$

where $k_3 = \dfrac{n\sum_{i=1}^{n}(x_i - \bar{x})^3}{(n-1)(n-2)}$ and s is the sample standard deviation.

Under H_0, the test statistic $Z(g_1)$ is approximately normally distributed for $n > 8$ and is defined as :

$$Z(g_1) = \delta \ln\left(\frac{Y}{\alpha} + \sqrt{\left(\frac{Y}{\alpha}\right)^2 + 1}\right) \qquad (2.5)$$

where $\alpha = \sqrt{\frac{2}{W^2 - 1}}$, $\delta = \frac{1}{\sqrt{\ln W}}$, $W^2 = \sqrt{2(B-1)} - 1$,

$$B = \frac{a(n^2 + 27n - 70)(n+1)(n+3)}{(n-2)(n+5)(n+7)(n+9)}$$

$$\sqrt{b_1} = \frac{(n-2)g_1}{\sqrt{n(n-1)}} \text{ and } Y = \sqrt{b_1}\left(\frac{(n+1)(n+3)}{6(n-2)}\right)^{1/2}.$$

2.5 Kurtosis Test [KU]

The kurtosis test is derived from the fourth sample moment. The coefficient of kurtosis of a set of data can be used to test the null hypothesis of normality versus non-normality due to kurtosis (Bai and Ng, 2005). The kurtosis statistic is defined as:

$$g_2 = k_4 \sqrt{s^4} \tag{2.6}$$

where $k_4 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^4 n(n+1)/(n-1) - 3\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^2}{(n-2)(n-3)}$.

Under H_0, the test statistic $Z(g_2)$ is approximately normally distributed for n ≥ 20 and thus more suitable for this range of sample size. $Z(g_2)$ is given as

$$Z(g_2) = \left(1 - \frac{2}{9A} - \sqrt[3]{\frac{1 - 2/A}{1 + H\sqrt{2/(A-4)}}}\right) \Big/ \sqrt{2/9A} \tag{2.7}$$

where $A = 6 + \frac{8}{J}\left[\frac{2}{J} + \sqrt{1 + \frac{4}{J^2}}\right]$, $J = \frac{6(n^2 - 5n + 2)}{(n+7)(n+9)}\sqrt{\frac{6(n+3)(n+5)}{n(n-2)(n-3)}}$,

$$H = \frac{(n-2)(n-1)|g_2|}{(n+1)(n-1)\sqrt{G}} \text{ and } G = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}.$$

2.6 D'Agostino-Pearson K^2 Test [DK]

The sample skewness (g_1) and kurtosis (g_2) are used separately in the skewness and kurtosis tests in testing the hypothesis if random samples are taken from a normal population; g_1 and g_2 tests detects deviations due to skewness and kurtosis respectively. D'Agostino and Pearson proposed the test (also known as D'Agostino's K-Squared) in 1973. The test combines g_1 and g_2 to produce an omnibus test of normality. The test statistics is:

$$K^2 = (Z(g_1))^2 + (Z(g_2))^2, \tag{2.8}$$

where $(Z(g_1))^2$ and $(Z(g_2))^2$ are the normal approximations to g_1 and g_2 respectively.

The test statistic follows approximately a chi-square distribution with 2 degree of freedom when a population in normally distributed. The test is appropriate for a sample size of at least twenty and the algorithm available in R-software will only compute the SK, KU and DK for this range of sample size.

2.7 Shapiro–Wilk Test [SW]

Shapiro and Wilk (1965) utilizes the null hypothesis principle to check whether a sample $x_1, x_2, ..., x_n$ came from a normally distributed population, the Shapiro-Wilk's test statistic W is thus derived from the sample itself and the expected values of order statistics from a standard normal distribution. The W statistic is defined by:

$$W = \frac{1}{D}\left[\sum_{i=1}^{m} a_i (x_{(n-i+1)} - x_{(i)})\right]^2 , \quad (2.9)$$

where $m = n/2$ if n is even while $m = (n-1)/2$ if n is odd. $D = \sum_{i=1}^{n}(x_i - \bar{x})^2$ and $x_{(i)}$ represents the ith order statistic of the sample, the constants a_i are given by

$$(a_1, a_2, ..., a_n) = \frac{m'V^{-1}}{(m'V^{-1}V^{-1}m)^{1/2}} \text{ and}$$

$$m = (m_1, m_2, ..., m_n)'$$

where $m_1, m_2, ..., m_n$ are the expected values of the order statistics of independent and identically distributed random variables sampled from the standard normal distribution, and V is the covariance matrix of those order statistics.

The values of W lie between 0 and 1, and small values of the statistic indicate departure from normality under H_0, thus we reject the null hypothesis if W is less than the corresponding critical value. W has a distribution that is independent of s^2 and $\bar{x}$, and is both scale and origin invariant.

2.8 Shapiro-Francia Test [SF]

Shapiro and Francia (1972) suggested an approximation to the Shapiro-Wilk W-test called W'. Let $x_1, x_2, ... x_n$ be a random sample to be tested for departure from normality, ordered $x_{(1)} < x_{(2)} < ... < x_{(n)}$, and let m' denote the vector of expected values of standard normal order statistics. The test statistic is defined as:

$$W' = \frac{\left(\sum_{i=1}^{n} m_i x_{(i)}\right)^2}{\left(\sum_{i=1}^{n} m_i^2\right)\left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right)}. \quad (2.10)$$

The W' equals the product-moment correlation coefficient between the $x_{(i)}$ and the m_i, and therefore measures the straightness of the normal probability plot $x_{(i)}$; small values of W' indicate non-normality.

Shapiro-Francia test is particularly useful as against the Shapiro-Wilk test especially for large samples where explicit values of m and V utilized in the Shapiro-Wilk test are not readily available and the computation of V^{-1} is time consuming.

2.9 Jarque-Bera Test [JB]

Jarque-Bera test is based on the sample skewness and sample kurtosis, it was proposed by Jarque and Bera in 1987. The test uses the Lagrange multiplier procedure on the Pearson family of distributions to obtain tests for normality. The test statistic is given as:

$$JB = n\left(\frac{(\sqrt{b_1})^2}{6} + \frac{(b_2 - 3)^2}{24}\right), \quad (2.11)$$

where $\sqrt{b_1}$ and b_2 are the skewness and kurtosis measures and are given by $\frac{m_3}{(m_2)^{3/2}}$ and $\frac{m_4}{(m_2)^3}$ respectively; and m_2, m_3, m_4 are the second, third and fourth central moments respectively.

2.10 Robust Jarque-Bera Test [RJB]

Gel and Gastwirth (2008) proposed a robust modification to the Jarque-Bera test. Since sample moments utilized in the Jarque-Bera test are sensitive to outliers, the Robust Jarque-Bera uses a robust estimate of the dispersion in the skewness and kurtosis instead of the second order central moment m_2. Let $x_1, x_2, \ldots x_n$ be a sample of independent and identically distributed random variables. The robust sample estimates of skewness and kurtosis are $\frac{m_3}{J_n^3}$ and $\frac{m_4}{J_n^4}$ respectively, which leads to the new robust Jarque–Bera (RJB) test statistic:

$$RJB = \frac{n}{6}\left(\frac{m_3}{J_n^3}\right)^2 + \frac{n}{64}\left(\frac{m_4}{J_n^4} - 3\right)^2, \tag{2.12}$$

where $J_n = \frac{C}{n}\sum_{i=1}^{n}|X_i - M|$, $C = \sqrt{\pi/2}$ and M is the sample median.

Under the null hypothesis of normality, the RJB test statistic asymptotically follows the chi-square distribution with 2 degrees of freedom. The normality hypothesis of the data is rejected for large values of the test statistic.

2.11 Doornik-Hansen Test [DH]

In order to improve the efficiency of the Jarque-Bera test, Doornik and Hansen (1994) proposed a series of modification. The modification involved the use of the transformed skewness according to the following expression:

$$Z(\sqrt{b_1}) = \frac{\ln(Y/c + \sqrt{(Y/c)^2 + 1})}{\sqrt{\ln(w)}}, \tag{2.13}$$

where Y, c and w are obtained by

$$Y = \sqrt{b_1} \cdot \sqrt{\frac{(n+1)(n+3)}{6(n-2)}}, w^2 = -1 + \sqrt{2\beta_2 - 1}$$

$$\beta_2 = \frac{3(n^2 + 27n - 70)(n+1)(n+3)}{(n-2)(n+5)(n+7)(n+9)} \text{ and } c = \sqrt{\frac{2}{(w^2 - 1)}},$$

and the use of a transformed kurtosis according to the proposal by Bowman and Shenton (1977). Bowman and Shenton had proposed the transformed kurtosis z_2 obtained by

$$z_2 = \left[\left(\frac{\xi}{2a}\right)^{1/3} - 1 + \frac{1}{9a}\right](9a)^{1/2} \tag{2.14}$$

with ξ and a obtained by

$$\xi = (b_2 - 1 - b_1)2k; \qquad k = \frac{(n+5)(n+7)(n^3 + 37n^2 + 11n - 313)}{12(n-3)(n+1)(n^2 + 15n - 4)}$$

$$a = \frac{(n+5)(n+7)\left[(n-2)(n^2 + 27n - 70) + b_1 \cdot (n-7)(n^2 + 2n - 5)\right]}{6(n-3)(n+1)(n^2 + 15n - 4)}.$$

The test statistic proposed by Doornik and Hansen is given by

$$DH = \left[Z\left(\sqrt{b_1}\right)\right]^2 + \left[z_2\right]^2. \tag{2.15}$$

The normality hypothesis is rejected for large values of the test statistic. The test is approximately chi-squared distributed with two degrees of freedom.

2.12 Brys-Hubert-Struyf MC-MR test [BH]

Brys, et al. (2004, 2007) have proposed a goodness-of-fit test derived from robust measures of skewness and tail weight. The considered robust measure of skewness is the medcouple (MC) defined as

$$MC = \underset{x_{(i)} \leq m_F \leq x_{(j)}}{med} h(x_{(i)}, x_{(j)}),$$

where *med* stands for median. m_F is the sample median and h is a kernel function given by

$$h(x_{(i)}, x_{(j)}) = \frac{(x_{(j)} - m_F) - (m_F - x_{(i)})}{x_{(i)} - x_{(j)}}. \tag{2.16}$$

The left medcouple (LMC) and the right medcouple (RMC) are the considered robust measures of left and right tail weight respectively and are defined by

$$LMC = -MC(x < m_F) \qquad \text{and} \qquad RMC = MC(x > m_F).$$

The test statistic T_{MC-LR} is then defined by

$$T_{MC-LR} = n(w - \omega)' V^{-1} (w - \omega) \tag{2.17}$$

in which w is set as [MC, LMC, RMC]’, and ω and V are obtained based on the influence function of the estimators in ω. According to Brys, et al. (2004), for the case of normal distribution, ω and V are defined as

$$\omega = [0, 0.199, 0.199]'; \qquad V = \begin{bmatrix} 1.25 & 0.323 & -0.323 \\ 0.323 & 2.62 & -0.0123 \\ -0.323 & -0.0123 & 2.62 \end{bmatrix}.$$

The normality hypothesis of the data is rejected for large values of the test statistic which approximately follows the chi-square distribution with three degrees of freedom.

2.13 Bonett-Seier Test [BS]

Bonett and Seier (2002) have suggested a modified measure of kurtosis for testing normality, which is based on a modification of proposal by Geary (1936). The test statistic of the new kurtosis measure T_w is thus given by:

$$T_w = \frac{\sqrt{n+2} \cdot (\omega - 3)}{3.54} \tag{2.18}$$

where $\omega = 13.29\left[\ln \sqrt{m_2} - \ln\left(n^{-1}\sum_{i=1}^{n}|x_i - \bar{x}|\right)\right]$.

The normality hypothesis is rejected for both small and large values of T_w using a two-sided and, according to Bonett-Seier (2002), it is suggested that T_w approximately follows a standard normal distribution.

2.14 Brys-Hubert-Struyf-Bonett-Seier Joint test [BHBS]

Considering that the Brys–Hubert–Struyf MC–LR test is, mainly, a skewness associated test and that the Bonett–Seier's proposal is a kurtosis based test, a test considering both these measures was proposed by Romao et al. (2010) for testing normality. The joint test attempts to make use of the two referred focused tests in order to increase the power to detect different kinds of departure from normality. This joint test is proposed based on the assumption that the individual tests can be considered independent based on a simulation study yielded a correlation coefficient of approximately −0.06. The normality hypothesis of the data is rejected for the joint test when rejection is obtained for either one of the two individual tests for a significance level of $\alpha/2$.

2.15 Bontemps-Meddahi tests [BM(1) and BM(2)]

Bontemps and Meddahi (2005) have proposed a family of normality tests based on moment conditions known as Stein equations and their relation with Hermite polynomials. The test statistics are developed using the generalized method of moments approach associated with Hermite polynomials, which leads to test statistics that are robust against parameter uncertainty (Hansen, 1982). The general expression of the test family is thus given by

$$BM_{3-p} = \sum_{k=3}^{p} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} H_k(z_i) \right)^2 , \qquad (2.19)$$

where $z_i = (x_i - \bar{x}) / s$ and $H_k(\cdot)$ represents the *k*th order normalized Hermite polynomial.

Different tests can be obtained by assigning different values of *p,* which represents the maximum order of the considered normalized Hermite polynomials in the expression above. Two different tests are considered in this work with $p = 4$ and $p = 6$; these tests are termed BM_{3-4} and BM_{3-6} respectively. The hypothesis of normality is rejected for large values of the test statistic and according to Bontemps and Meddahi (2005); the general BM_{3-p} family of tests asymptotically follows the chi-square distribution with $p-2$ degree of freedom.

2.16 Gel-Miao-Gastwirth test [GMG]

Gel, et al. (2007) have proposed a directed normality test, which focuses on detecting heavier tails and outliers of symmetric distributions. The test is based on the ratio of the standard deviation and the robust measure of dispersion J_n as defined in the expression

$$J_n = \frac{\sqrt{\pi / 2}}{n} \sum_{i=1}^{n} \left| x_i - M \right|, \tag{2.20}$$

where M is the sample median.

The test statistic is thus given by

$$R_{sJ} = \frac{s}{J_n},$$

and should tend to one under a normal distribution. The normality hypothesis is rejected for large values of the R_{sJ}, and the statistic $\sqrt{n}(R_{sJ} - 1)$ is asymptotically normally distributed (Gel et al., 2007).

2.17 G Test [G]

Chen (2014) indicated that Chen and Ye (2009) proposed a new test called the G test statistics. The test is used to test if an underlying population distribution is a uniform distribution. Suppose $x_1, x_2, ..., x_n$ are the observations of a random sample from a population distribution with distribution function *F(x)*. Suppose also that $x_{(1)}, x_{(2)}, ..., x_{(n)}$ are the corresponding order statistics. The test statistic has the following form:

$$G(x_1, x_2, ..., x_n) = \frac{(n+1)\sum_{i=1}^{n+1}\left(x_{(i)} - x_{(i-1)} - \frac{1}{n+1}\right)^2}{n}, \qquad (2.21)$$

where $x_{(0)}$ is defined as 0, and $x_{(n+1)}$ is defined as 1.

We can observe that $F(x_{(1)}), F(x_{(2)}), ..., F(x_{(n)})$ are the ordered observations of a random sample from the $U(0,1)$ distribution and thus the G Statistic can be expressed as

$$G(x_{(1)}, x_{(2)}, ..., x_{(n)}) = \frac{(n+1)\sum_{i=1}^{n+1}\left(F_0(x_{(i)}) - F_0(x_{(i-1)}) - \frac{1}{n+1}\right)^2}{n}. \qquad (2.22)$$

When the population distribution is the same as the specified distribution, the value of the test statistic should be close to zero. On the other hand, when the population distribution is far away from the specified distribution, the value should be pretty close to one.

In order to use the test for normality, we can assume *F(x)* to be a normal distribution. Considering the case where the parameters of the distribution are not known, Lilliefor's idea is adopted by calculating $\bar{x}$ and s^2 from the sample and using them as estimates for

μ and σ^2 respectively, and thus *F(x)* is the cumulative distribution function of the $N(\bar{x}, s^2)$ distribution. By using the transformation

$$z = \frac{x-\mu}{\sigma}$$

the test statistic becomes

$$G(x_{(1)}, x_{(2)}, \ldots, x_{(x)}) = \frac{(n+1)\sum_{i=1}^{n+1}\left(\int_{-\infty}^{z(i)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - \int_{-\infty}^{z(i-1)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - \frac{1}{n+1}\right)^2}{n}. \qquad (2.23)$$

The hypothesis of normality should be rejected at significant level α if the test statistic is bigger that its $1-\alpha$ critical value. A table of critical values is available in Chen and ye (2009) and Chen (2014) for sample sizes 2 to 50. For the purpose of this work, the table of critical values was extended for some sample sizes greater than 50 up to 1000.

2.18 Other Test Statistics in Literature

One can find a variety of goodness-of-fit tests in literature and an attempt of giving a complete overview would not be successful at this point. A few other references are available, where the reader can find some other approaches than those presented above.

An extensive survey of goodness-of-fit testing is given by D'Agostino and Stephens (1986) and also Marhuenda et al. (2005). Miller and Quesenberry (1979) as well as Quesenberry and Miller (1977) collected various statistics for testing uniformity, too.

Some Kolmogorov-Smirnov type statistics, are considered in Rényi (1953) and Birnbaum and Lientz (1969). Some other recent ideas of constructing goodnessof-fit tests can be found in Glen et al. (2001), Goegebeur and Guillou (2010), Meintanis (2009), Steele and Chaseling (2006), Sürücü (2008) and Zhao et al. (2009).

Since goodness-of-fit tests are always related to characterizations of distributions, in the sense that they are constructed to detect significant deviation of the data from characterizing properties of the hypothetical distribution, other references include Ghurye (1960), O'Reilly and Stephens (1982), Paul (2003) and their references. The performance of these test vary and have also been widely discussed.

For instance, Razali and Wah (2011) suggested that Shapiro-Wilk test has the highest power of the four tests they compared. The four tests were the Shapiro-Wilk, Kolmogorov-Smirnov, Anderson-Darling and the Lilliefors test. They concluded however that the power of Shapiro-Wilk test is low for small sample size.

A study carried out by Yap and Sim (2011) to compare the Shapiro-Wilk, Kolmogorov-Smirniv, Lilliefors, Anderson-Darling, Cramer von Mises, D'Agostino and Pearson, Jarque-Bera and the Chi-Square test revealed that both the D'Agostino and Pearson, and Shapiro-Wilk have better power compared with the other tests. For asymmetric distributions, they concluded that Shapiro-Wilk is the most powerful followed by the Anderson-Darling test. Their results also showed that Kolmogorov-Smirnov and the Chi-Square test performed poorly. Some authors have actually suggested that the chi-square test should not be used to perform a test of normality.

Although, D'Agostino et al. (1990) pointed out that g1 and g2 as well as the Shapiro-Wilk and D'Agostino tests are excellent and powerful tests, Keskin (2006) found that their performance was not adequate in certain conditions especially for a beta (3, 1.5) distribution. This was consistent with the findings of Filliben (1975), and, Mendes and Pala (2003).

The goal of this research work is to compare the performance of several of the tests of normality available and to find out which of them is more powerful in detecting normality or lack of it, in a set of data and in specific situations in which each is powerful.

In this work, we have considered mainly the commonly used methods such as CS, AD, SW, and LL along with some of the methods that have been proposed in recent years. The LL was considered in place of the popular Kolmogorov-Smirnov (KS) test since the mean and the variance are estimated from the simulated data.

CHAPTER THREE: SIMULATION STUDY

The performance of a test statistic can be evaluated primarily by conducting a power study and by examining the type I error rates associated with the test. Since a theoretical comparison among the proposed test statistics is not feasible, a Monte Carlo simulation was conducted to compare the performance of the test statistics in this Chapter. The R programming software version 3.1.2 was used to carry out the study and the R packages used are "lawstat", "nortest", "normtest", "tseries", "moments", "fBasics", "PoweR" and "distr".

The first part of the simulation study involved the generation of random samples from the Standard normal distribution for the different sample sizes. Each sample generated was then tested for normality and the type I error rate, that is, the rate of rejection of the hypothesis of normality of the data, was then recorded at specified significance levels.

In the second part of the simulation study, data were generated from several alternative non-normal distributions as highlighted in section 1.3. These include symmetric, short-tailed distributions such as Beta (1, 1), Beta (2, 2), Beta (3, 3), Uniform (0,1) and T (10); symmetric long-tailed distributions such as T (5) and Laplace (0, 1); asymmetric distributions such as Gamma (4,5), Chi-Square (3), Exponential (1), Log-Normal (0,1) which are long-tailed; and Weibull (2,2) and Gompertz (10, 0.001) which are asymmetric short-tailed.

3.1 Simulation Procedure

For the sample sizes considered in this study which are 10, 20, 30, 40, 50, 100, 200, 500 and 1000; the following steps were performed.

1. Generate a random sample $x_1, x_2, ..., x_n$ of size n from a specified alternative distribution.
2. Test the generated data for normality simultaneously using the all the tests of normality considered herein.
3. Compare the value of each of the test statistics with their corresponding critical values at the indicated significance levels of 0.01, 0.05, and 0.10; and decide whether to reject the null hypothesis of normality at the specified significance level.
4. Perform steps 1 to 3 a total of 10,000 times
5. Calculate the rejection rates for each of the tests.

The rejection rate for the data from normal distribution is the type I error rate while that from an alternative distribution represents the power of the test.

3.2 Simulated Results and Discussion

The results of the simulation vary across different levels of significance, sample size and alternative distributions. The results for the 0.05 significance level for the different distribution considered are as presented in the table 3.1 to table 3.5 while those for the 0.01 significance level are given in table 3.6 to table 3.10. Results are discussed at the 5% level without loss of generality.

Table 3.1: Simulated Type I error rate at 5% significance level

Normal (0, 1) – Skewness = 0 , Kurtosis = 0

N	LL*	AD*	CS*	DK*	SK*	KU*	SW*	SF*	JB	RJB*	DH*	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	4.90	5.16	6.39	-	-	-	4.93	5.30	0.93	5.27	4.63	0.14	4.07	15.96	0.24	1.73	7.99	1.34
20	4.67	4.87	4.86	5.80	4.81	4.60	4.68	4.98	2.32	6.03	4.84	1.09	4.67	13.65	2.69	3.60	8.44	4.73
30	4.94	5.21	5.75	5.90	5.38	5.10	5.41	5.58	3.27	6.38	5.16	4.64	4.76	14.12	2.67	5.15	8.90	5.37
40	4.67	5.37	5.95	5.89	5.37	5.13	5.31	5.61	3.67	6.45	5.00	2.58	4.84	13.77	3.27	6.37	9.87	4.94
50	4.72	4.87	5.04	5.75	4.90	5.03	4.96	5.17	3.74	5.87	4.81	4.09	4.41	14.02	3.30	6.67	9.05	4.41
100	5.22	5.42	5.01	5.76	4.94	5.47	5.03	5.46	4.49	5.84	5.30	3.76	5.17	14.42	4.23	9.30	10.03	5.55
200	5.20	5.05	5.10	5.31	5.14	5.09	5.31	5.44	4.59	5.33	5.06	4.25	5.04	14.90	4.45	11.35	9.70	4.32
500	4.77	4.54	5.12	4.83	4.43	5.18	4.56	4.55	4.22	4.17	4.33	4.85	4.83	15.51	4.16	12.23	10.18	4.90
1000	4.69	4.90	5.03	4.82	4.76	5.06	4.94	5.34	4.78	4.55	4.84	4.85	5.13	15.02	4.73	14.39	10.29	5.69

*Tests with acceptable Type I error rates

KEY

S/N	NORMALITY TEST	ABBREVIATION
1.	Lilliefor's	LL
2.	Anderson-Darling	AD
3.	Chi-Square	CS
4.	D'Agostino's K Squared	DK
5.	Skewness	SK
6.	Kurtosis	KU
7.	Shapiro-Wilk	SW
8.	Shapiro-Francia	SF
9.	Jarque-Bera	JB
10.	Robust Jarque-Bera	RJB
11.	Doornik-Hansen	DH
12.	Brys-Hubert-Struyf MC-MR	BH
13.	Bonett-Seier	BS
14.	Brys-Hubert-Struyf-Bonett-Seier Joint	BHBS
15.	Bontemps-Meddahi (1)	BM(1)
16.	Bontemps-Meddahi (2)	BM(2)
17.	Gel-Miao-Gastwirth	GMG
18.	G	G

Table 3.2: Simulated power for symmetric short-tailed distributions at 5% significance level

Beta (1, 1) – Skewness = 0 , Kurtosis = -1.20

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	6.35	7.73	8.74	-	-	-	7.97	5.01	0.38	1.72	5.93	0.61	8.88	16.00*	0.10	0.68	7.66	3.65
20	9.39	17.23	8.35	15.51	0.54	30.27*	20.07	8.20	0.06	0.22	9.28	3.77	21.72	24.49	0.05	1.16	18.83	6.60
30	13.88	30.02	10.96	40.25	0.39	57.15*	39.14	17.55	0.00	0.06	18.17	11.66	37.18	40.43	0.00	13.03	36.31	8.63
40	19.84	43.75	16.27	63.22	0.35	76.88*	58.57	31.32	0.03	0.06	30.38	11.45	49.18	49.07	0.01	36.35	50.49	11.16
50	26.02	58.13	19.75	80.80	0.21	89.09*	75.15	48.13	0.01	0.04	45.63	17.02	63.13	62.30	0.01	59.85	65.56	14.91
100	59.19	94.80	46.27	99.65	0.12	99.88*	99.52	96.70	56.71	3.94	94.81	28.58	93.48	91.46	48.03	98.30	95.09	48.77
200	94.50	100.00	90.24	100.00	0.07	100.00	100.00	100.00	100.00	98.74	100.00	52.55	99.92	99.92	100.00	100.00	99.97	98.41
500	100.00	100.00	100.00	100.00	0.12	100.00	100.00	100.00	100.00	100.00	100.00	92.28	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	100.00	100.00	100.00	0.11	100.00	100.00	100.00	100.00	100.00	100.00	99.87	100.00	100.00	100.00	100.00	100.00	100.00

Beta (2, 2) – Skewness = 0 , Kurtosis = -0.86

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	4.40	4.30	6.25	-	-	-	4.17	3.45	0.27	2.25	3.32	0.11	5.02	14.89*	0.07	0.59	6.53	1.91
20	5.13	6.00	5.26	3.61	0.78	9.48	5.60	2.72	0.11	0.73	2.55	1.46	8.30	14.93*	0.14	0.58	7.99	3.15
30	6.31	7.86	6.04	8.31	0.38	18.32	7.84	3.23	0.05	0.18	3.13	6.31	12.36	19.83*	0.07	2.06	13.07	3.99
40	7.02	10.41	7.38	16.04	0.26	28.76*	11.11	3.76	0.00	0.06	4.06	4.58	17.42	21.86	0.00	5.26	19.61	4.14
50	8.25	13.47	7.35	23.54	0.25	38.86*	15.19	5.62	0.03	0.08	6.10	7.56	21.28	26.97	0.08	10.80	25.41	4.41
100	15.06	31.17	11.25	64.97	0.18	78.79*	44.79	20.82	1.35	0.01	25.31	7.69	46.33	46.10	11.76	47.16	54.35	6.96
200	33.46	70.79	23.12	97.45	0.14	99.21*	92.32	75.40	61.04	25.32	82.26	12.04	81.34	77.99	86.39	92.60	87.82	13.75
500	83.33	99.82	69.55	100.00	0.06	100.00	100.00	100.00	100.00	99.72	100.00	25.03	99.73	99.49	100.00	100.00	99.91	70.53
1000	99.74	100.00	98.85	100.00	0.05	100.00	100.00	100.00	100.00	100.00	100.00	46.76	100.00	100.00	100.00	100.00	100.00	99.99

Beta (3, 3) – Skewness = 0 , Kurtosis = -0.67

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	4.34	4.30	6.20	-	-	-	4.02	3.34	0.31	2.55	3.40	0.10	4.60	15.21*	0.11	0.70	6.29	1.84
20	4.56	4.66	4.77	2.58	1.02	5.65	4.11	2.16	0.17	1.11	1.99	1.35	5.74	14.31*	0.08	0.77	6.49	2.80
30	4.78	5.29	5.07	4.10	0.74	9.50	4.51	2.02	0.14	0.59	1.59	5.59	7.20	16.14*	0.08	1.33	8.49	2.96
40	5.02	6.27	6.23	7.20	0.60	14.85	5.88	2.13	0.10	0.35	1.98	3.74	9.83	16.74*	0.08	2.73	11.90	3.33
50	5.88	6.98	5.61	9.80	0.50	18.93	6.59	2.56	0.05	0.20	2.68	5.82	11.36	19.15*	0.03	4.41	14.51	3.42
100	8.16	13.13	7.05	27.73	0.21	42.81*	15.34	5.97	0.23	0.06	7.73	5.98	23.21	28.10	0.13	17.74	29.80	4.39
200	14.49	29.97	11.19	66.11	0.18	79.72*	44.09	22.79	13.83	3.32	33.20	6.96	48.17	47.86	12.49	53.18	58.22	5.48
500	40.98	80.44	26.38	99.26	0.25	99.88*	97.53	90.57	94.20	78.87	96.26	12.81	89.92	87.52	93.88	97.75	94.21	14.16
1000	79.60	99.51	60.57	100.00	0.19	100.00	100.00	99.99	99.99	99.95	100.00	20.61	99.73	99.55	99.99	100.00	99.92	47.51

Uniform (0, 1) – Skewness = 0, Kurtosis = -1.20

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	6.47	7.89	8.76	-	-	-	8.19	4.81	0.20	1.73	6.03	0.38	8.53	15.54*	0.09	0.56	7.57	3.51
20	9.34	16.72	8.09	15.57	0.64	30.12*	19.84	8.14	0.07	0.36	9.31	3.33	21.35	24.14	0.05	1.24	18.77	6.84
30	13.62	29.81	11.65	39.28	0.29	56.82*	38.00	17.67	0.01	0.06	17.93	12.20	35.17	39.54	0.01	12.89	34.32	8.93
40	19.65	43.77	16.34	62.32	0.27	76.33*	58.22	30.65	0.03	0.06	30.24	11.21	49.57	49.24	0.02	36.03	50.88	11.19
50	25.43	56.68	19.57	79.77	0.17	88.59*	74.76	46.87	0.01	0.01	44.40	16.89	61.46	60.62	0.00	58.41	64.67	14.73
100	58.64	94.78	45.61	99.74	0.13	99.90*	99.59	96.74	55.78	4.24	95.06	28.35	93.39	91.45	47.50	98.63	95.17	48.08
200	94.57	99.98	90.42	100.00	0.08	100.00	100.00	100.00	99.96	98.73	100.00	53.80	99.87	99.86	99.96	100.00	99.93	98.38

500	100.00	100.00	100.00	100.00	0.09	100.00	100.00	100.00	100.00	100.00	100.00	91.55	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	100.00	100.00	100.00	0.07	100.00	100.00	100.00	100.00	100.00	100.00	99.84	100.00	100.00	100.00	100.00	100.00	100.00

T (10) – Skewness = 0 , Kurtosis = 1

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	6.32	7.00	7.78	-	-	-	7.18	8.38	1.97	9.10	7.24	0.06	5.57	16.81*	0.92	3.39	11.43	1.70
20	7.40	8.75	6.29	12.58	11.07	9.84	9.94	11.63	7.65	14.54	11.97	0.97	8.80	17.07*	8.51	9.87	16.02	6.90
30	7.93	9.87	6.23	15.43	13.31	12.15	11.79	14.33	11.84	18.07	14.82	4.80	10.61	18.91*	13.35	14.98	18.86	6.54
40	7.81	11.15	7.33	17.02	14.26	13.72	13.64	17.01	14.36	20.30	17.00	2.42	12.90	19.98	16.76	18.93	20.66*	6.61
50	8.50	11.59	6.51	18.92	15.23	15.53	14.54	18.72	17.15	22.80	19.01	4.42	14.00	21.55	20.44	22.53	23.35*	6.88
100	11.14	16.35	7.54	27.39	19.78	24.95	23.37	28.61	28.78	33.79	29.17	3.86	21.92	27.86	33.51	37.82*	32.84	7.50
200	14.60	-	8.30	39.79	21.97	40.68	35.20	41.93	44.35	47.88	44.16	4.45	34.70	38.09	50.45	55.74*	46.90	8.28
500	28.45	-	12.28	69.96	26.19	74.37	65.55	71.84	75.16	76.42	74.97	6.50	65.98	65.81	80.12	84.05*	75.54	10.72
1000	48.79	-	19.70	92.51	28.12	94.93	90.29	92.62	94.48	94.53	94.48	8.03	89.68	88.77	96.37	97.44*	94.19	15.00

* The most powerful test for each sample size.

Table 3.3: Simulated power for symmetric long-tailed distributions at 5% significance level

T (5) – Skewness = 0 , Kurtosis = 6

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	9.50	10.59	9.55	-	-	-	10.64	12.84	4.68	14.92	11.71	0.09	8.08	18.92*	2.38	7.21	17.06	2.39
20	12.55	16.87	9.18	23.39	21.03	18.34	18.83	22.57	16.73	26.19*	22.28	0.94	16.71	23.50	14.46	20.44	26.62	4.52
30	15.19	21.63	10.93	29.57	24.84	24.98	24.22	29.19	24.94	34.43*	29.67	5.02	23.17	30.08	23.35	30.23	34.46	6.02
40	18.88	27.02	13.86	35.65	29.60	31.45	31.31	36.67	33.33	42.38*	36.84	3.07	30.61	35.06	32.00	39.62	42.13	6.81
50	21.25	30.39	12.89	41.15	31.92	37.81	36.07	41.96	40.33	47.95*	42.87	4.46	35.79	39.58	39.05	47.00	47.26	7.85
100	33.42	-	18.34	60.04	40.00	59.63	56.37	62.89	62.90	69.02*	64.10	4.47	58.54	59.73	62.22	71.35	68.79	11.61
200	54.81	-	29.62	82.72	48.32	84.95	81.31	85.70	85.97	88.95	86.59	6.29	83.88	82.98	85.75	91.48*	89.48	16.27
500	89.27	-	57.90	99.01	56.30	99.41	98.93	99.22	99.38	99.56	99.41	12.21	99.22	99.15	99.37	99.81*	99.60	26.45
1000	99.54	-	90.50	100.00	62.33	100.00	100.00	100.00	100.00	100.00	100.00	22.21	99.99	99.99	100.00	100.00	99.99	43.70

Laplace (0, 1) – Skewness = 0, Kurtosis = 3

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	13.82	15.67	12.87	-	-	-	15.18	18.35	6.38	20.72	17.28	0.14	11.60	21.35	3.24	10.04	24.24*	3.53
20	21.40	26.64	14.63	30.23	25.62	23.82	25.59	31.62	22.13	38.28	30.58	1.04	28.34	32.25	18.50	28.11	43.64*	7.46
30	29.55	37.34	19.77	38.19	30.23	33.37	35.85	42.89	33.47	50.95	40.87	5.06	41.86	44.26	31.25	42.85	57.87*	10.21
40	35.98	45.85	25.23	44.30	32.87	40.93	44.08	51.59	42.22	60.17	48.54	3.52	53.63	54.14	40.09	54.16	68.45*	12.93
50	43.15	54.46	28.04	51.89	35.52	49.66	52.77	60.12	51.44	68.84	56.87	5.96	64.55	64.71	49.88	64.29	76.65*	14.46
100	70.97	83.08	47.55	73.76	40.91	76.35	80.08	84.99	78.09	89.49	80.48	10.14	90.25	89.63	77.33	89.05	94.86*	23.09
200	94.33	98.30	78.05	94.24	46.68	96.43	97.45	98.12	96.45	99.13*	96.88	22.97	99.53	99.45	96.34	99.19	99.85	34.59
500	99.99	-	99.42	99.97	50.17	99.99	100.00	100.00	99.99	100.00	99.99	56.45	100.00	100.00	99.99	100.00	100.00	65.01
1000	100.00	-	100.00	100.00	51.04	100.00	100.00	100.00	100.00	100.00	100.00	88.25	100.00	100.00	100.00	100.00	100.00	88.63

* The most powerful test for each sample size.

Table 3.4: Simulated power for asymmetric long-tailed distributions at 5% significance level

Gamma (4, 5) – Skewness = 1, Kurtosis = 4

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	10.80	13.35	11.75	-	-	-	13.90	14.60*	4.20	12.91	10.82	0.18	6.33	15.29	2.14	6.74	13.94	3.04
20	17.99	25.04	12.80	25.33	29.06	15.23	29.35*	28.72	16.66	25.01	23.36	1.86	8.98	15.93	13.95	22.88	19.18	5.10
30	25.46	37.27	18.12	36.61	44.25	19.50	44.73*	42.62	28.64	36.00	37.66	8.95	10.79	21.10	26.06	38.64	23.41	7.21
40	33.01	48.66	24.17	45.78	55.37	22.80	57.83*	54.69	39.27	44.23	51.24	8.28	12.44	22.26	36.91	52.37	26.68	9.13
50	41.17	59.08	27.29	55.34	67.21	27.16	69.45*	65.96	49.68	53.87	63.88	14.18	14.10	26.52	47.69	65.19	29.40	11.56
100	70.50	89.38	51.49	88.10	94.15	39.31	95.81*	94.38	86.83	84.90	94.74	25.54	17.19	37.93	85.79	94.71	37.72	28.81
200	95.30	99.83	87.32	99.85	99.92	59.49	99.97	99.93	99.85	99.70	99.98*	54.21	23.67	63.53	99.84	99.95	52.12	68.82
500	100.00	-	99.98	100.00	100.00	89.89	100.00	100.00	100.00	100.00	100.00	93.18	40.14	95.54	100.00	100.00	78.55	99.84
1000	100.00	-	100.00	100.00	100.00	99.08	100.00	100.00	100.00	100.00	100.00	99.94	61.99	99.96	100.00	100.00	94.91	100.00

Chi-square (3) – Skewness = 1.63, Kurtosis = 4

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	21.12	28.35	25.57	-	-	-	31.10*	31.17	10.13	24.03	23.18	0.53	8.65	15.06	5.73	15.52	22.60	6.00
20	41.37	58.54	41.52	48.03	56.67	27.68	65.81*	62.43	35.86	46.78	54.81	6.68	15.33	22.63	38.74	49.12	35.69	15.01
30	58.93	79.93	60.50	65.81	77.03	36.56	87.19*	83.50	57.69	64.33	79.42	22.99	19.90	37.94	63.35	75.69	44.56	26.46
40	72.69	91.13	75.15	78.97	88.94	44.94	95.81*	93.85	74.43	77.60	92.01	27.79	23.82	44.42	80.93	89.92	51.43	45.11
50	82.14	96.42	84.91	88.63	95.33	52.12	98.87*	98.05	86.37	86.50	97.17	41.66	27.07	56.44	91.32	96.17	57.83	59.96
100	99.11	99.99	99.56	99.97	99.93	75.37	100.00	100.00	99.92	99.71	100.00	75.32	42.01	85.26	100.00	100.00	76.92	98.64
200	100.00	-	100.00	100.00	100.00	94.23	100.00	100.00	100.00	100.00	100.00	97.66	61.11	99.06	100.00	100.00	93.09	100.00
500	100.00	-	100.00	100.00	100.00	99.95	100.00	100.00	100.00	100.00	100.00	100.00	89.71	100.00	100.00	100.00	99.79	100.00
1000	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.11	100.00	100.00	100.00	100.00	100.00

Exponential (1) – Skewness = 2, Kurtosis = 6

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	29.66	40.68	39.63	-	-	-	43.77*	42.43	15.09	30.94	33.13	1.35	11.69	16.84	9.60	21.91	29.45	10.08
20	58.02	77.82	66.21	60.58	70.32	36.38	83.73*	80.15	48.38	59.55	73.27	14.74	20.46	32.04	43.54	65.46	46.95	28.66
30	78.46	93.46	85.56	78.72	88.64	48.70	96.72*	94.97	72.85	77.59	92.18	40.45	27.77	56.07	69.75	89.21	59.27	53.56
40	90.23	98.23	95.02	89.73	95.70	56.62	99.51*	98.81	87.47	88.33	97.89	51.34	33.04	66.37	85.41	97.08	66.39	76.19
50	96.32	99.70	98.44	96.42	98.82	66.25	99.95*	99.84	95.63	95.02	99.63	66.17	38.29	77.95	94.79	99.42	74.15	90.73
100	100.00	100.00	100.00	100.00	100.00	88.86	100.00	100.00	100.00	99.99	100.00	94.51	57.70	97.51	100.00	100.00	91.64	99.99
200	100.00	-	100.00	100.00	100.00	99.06	100.00	100.00	100.00	100.00	100.00	99.91	81.54	99.99	100.00	100.00	99.31	100.00
500	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	98.55	100.00	100.00	100.00	100.00	100.00
1000	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.98	100.00	100.00	100.00	100.00	100.00

Log-Normal (0, 1) – Skewness = 6.18, Kurtosis = 113.94

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	47.33	58.45	54.74	-	-	-	61.25*	60.53	29.19	48.72	51.10	2.23	20.24	24.68	22.18	38.88	46.20	21.62
20	78.59	90.30	82.36	79.82	86.91	59.63	93.10*	91.67	71.53	80.43	88.43	21.71	41.48	54.33	67.68	84.36	71.66	52.75
30	93.07	98.29	95.05	93.79	97.19	75.69	99.12*	98.75	91.68	93.79	98.23	50.65	58.46	78.84	90.33	97.26	85.58	79.10
40	98.23	99.75	98.97	98.29	99.55	85.18	99.89*	99.86	97.85	98.09	99.76	65.76	69.87	89.52	97.47	99.67	92.04	92.83
50	99.52	99.99	99.75	99.70	99.91	90.88	100.00	100.00	99.59	99.48	100.00	80.16	77.62	95.05	99.48	99.98	95.52	97.92
100	100.00	-	100.00	100.00	100.00	99.62	100.00	100.00	100.00	100.00	100.00	98.41	96.05	99.87	100.00	100.00	99.81	100.00
200	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.88	100.00	100.00	100.00	100.00	100.00

500	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

* The most powerful test for each sample size.

Table 3.5: Simulated power for asymmetric short-tailed distributions at 5% significance level

Weibull (2, 2) – Skewness = 0.63, Kurtosis = 0.25

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	6.80	7.54	8.30	-	-	-	7.79	8.12	2.00	7.14	6.27	0.18	5.01	14.33*	0.77	3.44	9.58	1.97
20	9.72	12.55	8.01	13.06	14.44	8.64	15.13*	14.10	6.63	12.14	10.41	1.48	5.81	13.24	5.35	9.97	11.04	3.60
30	13.20	18.62	10.10	17.88	21.16	10.78	23.49*	20.55	11.14	15.85	17.03	7.12	6.73	17.09	9.71	17.56	12.29	4.33
40	16.54	24.45	12.44	22.81	28.70	11.90	31.85*	27.62	15.87	19.89	24.37	5.24	7.41	16.06	14.17	25.89	13.25	5.41
50	20.75	31.11	13.71	28.29	37.62	13.10	41.47*	36.02	21.25	24.84	33.37	8.77	7.14	18.10	19.78	35.24	13.78	6.20
100	38.78	60.46	25.45	56.37	69.28	15.57	79.33*	71.64	50.06	47.50	72.64	13.80	8.58	22.05	48.25	72.93	14.79	12.64
200	70.35	92.84	55.70	94.84	95.73	17.67	99.28*	98.30	93.57	90.14	98.41	28.82	9.30	34.33	93.18	97.91	14.80	40.73
500	98.71	100.00	98.30	100.00	100.00	24.71	100.00	100.00	100.00	100.00	100.00	67.76	12.27	69.02	100.00	100.00	15.72	99.07
1000	100.00	100.00	100.00	100.00	100.00	33.66	100.00	100.00	100.00	100.00	100.00	94.19	16.94	93.45	100.00	100.00	15.98	100.00

Gompertz (0.001, 1) – Skewness = -1, Kurtosis = 1.5

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	11.70	14.07	12.44	-	-	-	14.68	15.71	4.37	14.19	11.60	0.18	6.74	16.51	2.19	7.55	15.39*	3.13
20	18.57	26.21	12.68	28.62	31.71*	16.71	30.00	30.45	19.23	28.67	25.44	2.20	9.92	17.49	16.24	25.56	22.06	5.03
30	26.17	37.73	16.94	38.98	45.13*	21.26	44.52	43.37	31.63	39.06	38.66	9.08	12.67	22.61	29.21	40.97	27.22	6.74
40	34.63	48.94	22.99	49.94	59.04*	25.44	57.90	56.48	43.47	50.32	52.19	9.19	14.45	24.10	40.99	55.88	31.16	8.69
50	41.23	57.81	24.89	58.39	68.09*	29.77	66.80	65.27	53.30	58.08	62.51	13.42	16.46	29.14	51.39	65.75	34.79	10.47
100	70.46	87.48	45.79	87.97	94.22*	46.68	93.49	92.40	87.28	86.98	92.56	25.69	23.46	42.58	86.46	93.54	48.16	22.62
200	95.23	99.35	81.31	99.67	99.90*	71.15	99.86	99.82	99.63	99.50	99.86	51.98	36.73	68.55	99.63	99.84	67.80	50.14
500	100.00	100.00	99.90	100.00	100.00	96.22	100.00	100.00	100.00	100.00	100.00	92.55	65.88	97.63	100.00	100.00	92.56	94.33
1000	100.00	100.00	100.00	100.00	100.00	99.91	100.00	100.00	100.00	100.00	100.00	99.76	89.28	99.99	100.00	100.00	99.53	99.97

* The most powerful test for each sample size.

Table 3.6: Simulated Type I error rate at 1% significance level

Normal (0, 1) – Skewness = 0 , Kurtosis = 0

N	LL*	AD*	CS*	DK	SK*	KU*	SW*	SF*	JB*	RJB	DH*	BH	BS*	BHBS	BM(1)*	BM(2)	GMG	G
10	0.90	0.86	1.16	-	-	-	0.88	0.85	0.18	2.97	0.57	0.00	0.82	2.87	0.03	0.40	2.55	0.13
20	0.94	0.97	1.21	1.89	0.97	0.92	1.00	1.05	0.99	3.59	1.13	0.10	0.99	2.76	2.69	1.15	2.60	0.92
30	0.94	0.89	0.98	1.80	0.99	1.04	0.96	0.98	1.36	3.73	1.30	1.06	0.99	3.45	1.14	1.96	2.56	0.82
40	0.93	0.79	1.13	2.10	1.13	1.21	0.91	1.09	1.76	3.69	1.25	0.38	0.94	2.86	1.47	2.49	2.47	0.98
50	0.97	0.99	1.06	1.97	1.07	1.29	1.04	1.10	1.89	3.41	1.30	1.03	1.14	3.20	1.61	2.90	2.22	1.06
100	1.03	0.96	0.93	1.72	0.93	1.41	0.91	1.02	1.72	2.59	1.21	0.81	1.07	3.45	1.63	3.38	2.39	1.02
200	1.01	0.87	1.03	1.58	0.95	1.36	0.95	1.09	1.81	2.18	1.24	0.96	0.95	3.09	1.79	4.72	1.84	0.99
500	0.82	1.04	0.93	1.36	0.96	1.37	1.11	1.05	1.46	1.48	1.21	1.14	1.00	3.57	1.45	5.06	1.94	0.61
1000	0.71	1.03	0.96	1.47	1.02	1.25	1.15	1.23	1.44	1.30	1.27	0.89	1.06	3.42	1.43	5.93	2.17	0.30

*Tests with acceptable Type I error rates

Table 3.7: Simulated power for symmetric short-tailed distributions at 1% significance level

Beta (1, 1) – Skewness = 0 , Kurtosis = -1.20

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	1.40	1.36	2.42	-	-	-	1.33	0.66	0.06	1.01	1.22	0.02	0.40	2.48*	0.02	0.19	0.79	0.44
20	1.97	3.87	2.71	4.24	0.04	11.36*	2.86	0.66	0.02	0.11	1.54	0.39	3.36	4.82	0.02	0.04	1.39	1.53
30	3.25	9.11	3.75	18.82	0.00	32.78*	9.29	1.90	0.00	0.01	2.79	2.98	9.77	14.19	0.00	0.12	6.24	2.17
40	5.05	16.94	5.65	38.32	0.01	54.17*	20.35	5.82	0.00	0.02	6.73	2.80	18.00	19.99	0.00	2.99	14.05	2.67
50	7.19	26.96	7.83	59.69	0.00	73.70*	36.02	12.02	0.00	0.00	12.34	5.41	29.19	31.51	0.00	12.33	25.96	3.89
100	25.39	79.05	27.17	98.06	0.00	99.14*	94.52	76.80	0.17	0.00	70.60	11.94	75.53	73.21	0.04	86.23	76.66	19.48
200	73.39	99.83	79.40	100.00	0.00	100.00	100.00	100.00	96.19	47.20	99.94	30.27	99.19	98.62	95.18	100.00	99.39	91.42
500	99.98	100.00	100.00	100.00	0.00	100.00	100.00	100.00	100.00	100.00	100.00	79.24	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	100.00	100.00	100.00	0.00	100.00	100.00	100.00	100.00	100.00	100.00	99.04	100.00	100.00	100.00	100.00	100.00	100.00

Beta (2, 2) – Skewness = 0 , Kurtosis = -0.86

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	0.70	0.60	1.21	-	-	-	0.58	0.33	0.05	1.35	0.45	0.00	0.31	2.27*	0.00	0.10	0.91	0.14
20	0.97	1.03	1.37	0.69	0.02	2.13	0.47	0.15	0.02	0.31	0.16	0.13	0.72	2.57*	0.14	0.06	0.50	0.49
30	1.17	1.66	1.34	2.45	0.01	5.72*	0.87	0.16	0.02	0.07	0.32	1.61	1.65	4.84	0.07	0.08	1.05	0.79
40	1.14	1.86	1.64	5.62	0.00	11.97*	1.29	0.24	0.00	0.00	0.37	0.66	2.99	5.21	0.00	0.11	2.33	0.79
50	1.43	2.96	1.80	9.75	0.01	18.14*	2.38	0.40	0.00	0.02	0.60	1.87	5.16	7.70	0.08	0.49	4.72	0.99
100	3.29	9.86	3.59	40.73	0.00	56.26*	12.90	3.26	0.00	0.00	4.16	2.11	17.80	18.11	11.76	11.80	19.70	1.45
200	10.72	39.54	8.94	89.44	0.00	95.39*	65.37	35.72	5.51	0.18	43.25	3.85	54.38	50.02	86.39	67.05	60.43	3.86
500	50.28	97.60	46.84	100.00	0.00	100.00	100.00	99.89	99.61	90.78	99.90	10.01	98.01	96.53	100.00	99.95	98.77	36.77
1000	95.17	100.00	95.56	100.00	0.00	100.00	100.00	100.00	100.00	100.00	100.00	24.62	100.00	99.98	100.00	100.00	100.00	98.28

Beta (3, 3) – Skewness = 0 , Kurtosis = -0.67

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	0.79	0.68	1.31	-	-	-	0.67	0.48	0.07	1.57	0.54	0.00	0.40	2.50*	0.01	0.16	1.33	0.14
20	0.88	0.78	1.16	0.41	0.04	1.21	0.39	0.16	0.06	0.48	0.22	0.03	0.66	2.68*	0.03	0.10	0.64	0.38
30	0.86	0.86	0.96	0.93	0.04	2.52	0.45	0.13	0.01	0.25	0.19	1.31	0.76	3.54*	0.01	0.07	0.67	0.59
40	0.99	1.18	1.22	1.91	0.05	4.85*	0.61	0.20	0.02	0.12	0.30	0.49	1.30	3.49	0.02	0.17	1.14	0.73
50	0.98	1.44	1.21	3.11	0.03	6.62*	0.88	0.21	0.02	0.08	0.28	1.29	1.74	4.63	0.02	0.26	1.43	0.67
100	1.61	3.11	1.73	11.88	0.00	21.06*	2.64	0.57	0.00	0.00	0.95	1.38	6.15	8.39	0.00	2.42	7.26	0.88
200	3.27	9.92	3.02	40.92	0.00	56.82*	14.27	4.26	0.18	0.00	7.12	1.77	20.56	19.69	0.13	19.70	24.96	1.18
500	13.88	52.63	10.50	95.96	0.00	98.55*	84.28	62.98	58.72	26.44	77.89	4.07	70.21	64.37	57.69	87.03	76.87	2.99
1000	45.11	95.59	36.27	99.98	0.00	100.00*	99.98	99.80	99.91	97.58	99.97	7.33	97.80	96.24	99.91	99.97	98.88	9.87

Uniform (0, 1) – Skewness = 0, Kurtosis = -1.20

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	1.29	1.19	2.19	-	-	-	1.17	0.53	0.04	0.96	1.16	0.00	0.35	2.42*	0.01	0.11	0.65	0.43
20	1.96	3.74	2.78	4.18	0.06	10.84*	2.77	0.63	0.02	0.15	1.37	0.29	3.15	4.62	0.01	0.08	1.41	1.47
30	3.03	8.90	3.66	18.06	0.01	31.22*	8.79	1.95	0.00	0.02	2.93	3.00	8.96	13.79	0.00	0.15	6.18	2.40
40	5.05	16.65	5.20	38.34	0.02	53.56*	19.63	5.25	0.00	0.03	5.95	2.79	17.53	19.83	0.00	2.69	14.24	2.86
50	6.72	25.90	7.77	58.83	0.00	72.22*	35.07	11.71	0.00	0.00	11.94	5.23	28.25	31.03	0.00	11.90	25.13	4.30
100	25.52	78.07	26.70	98.27	0.00	99.18*	94.69	76.15	0.16	0.00	69.62	11.65	75.47	72.76	0.05	86.33	76.65	19.28
200	73.03	99.80	79.96	100.00	0.00	100.00	100.00	99.97	96.48	45.83	99.91	29.85	99.11	98.47	95.45	99.96	99.33	91.21

500	99.99	100.00	100.00	100.00	0.01	100.00	100.00	100.00	100.00	100.00	100.00	78.38	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	100.00	100.00	100.00	0.00	100.00	100.00	100.00	100.00	100.00	100.00	99.11	100.00	100.00	100.00	100.00	100.00	100.00

T (10) – Skewness = 0 , Kurtosis = 1

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	1.55	1.70	1.54	-	-	-	1.94	2.03	0.67	6.10*	1.38	0.00	1.50	3.52	0.23	1.20	4.67	0.15
20	2.00	2.86	1.82	6.69	4.24	3.33	3.70	4.34	4.85	10.41*	4.46	0.08	3.70	5.34	8.51	5.37	7.55	1.48
30	1.81	3.17	1.39	8.42	5.28	4.58	4.85	5.98	7.76	13.09	6.39	1.14	4.87	6.61	13.35*	9.06	8.72	1.47
40	2.06	3.48	1.50	9.62	6.11	5.84	6.13	7.42	10.05	14.86	8.30	0.34	6.05	6.85	16.76*	12.11	10.55	1.34
50	2.26	4.04	1.41	10.54	6.58	6.53	6.69	7.97	11.60	16.87	9.56	0.92	6.85	7.91	20.44*	14.60	11.56	1.38
100	3.24	6.17	1.81	16.54	9.50	12.58	12.13	14.98	20.71	24.80	17.60	2 0.82	11.48	11.91	33.51*	27.14	18.10	1.66
200	4.16	-	1.78	25.51	11.50	24.02	21.16	25.52	33.67	36.76	30.20	1 0.75	19.97	19.52	50.45*	43.42	27.97	2.04
500	10.00	-	3.26	53.67	15.13	56.44	48.97	54.52	63.52	65.37	60.90	1 1.39	47.91	45.63	80.12*	74.10	57.61	1.61
1000	22.03	-	6.35	83.25	15.87	86.96	79.96	83.63	88.83	89.40	87.70	4 1.92	78.11	74.76	96.37*	93.53	84.52	0.87

* The most powerful test for each sample size.

Table 3.8: Simulated power for symmetric long-tailed distributions at 1% significance level

T (5) – Skewness = 0 , Kurtosis = 6

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	3.16	4.03	2.10	-	-	-	4.57	4.93	1.99	11.14*	3.21	0.00	3.34	4.91	0.68	3.18	8.95	0.33
20	4.69	7.40	3.22	14.76	10.29	8.52	9.24	11.00	11.51	21.01*	10.61	0.07	9.15	10.23	9.84	12.91	16.18	1.02
30	5.99	10.46	3.48	19.81	13.80	13.01	13.73	16.16	18.99	27.99*	17.26	1.08	14.40	15.23	17.48	21.61	21.83	1.14
40	7.89	13.97	4.17	25.20	18.00	18.64	19.49	22.31	26.49	34.93*	24.03	0.57	19.90	19.66	25.03	30.27	28.34	1.70
50	9.09	16.68	4.30	28.99	20.25	23.08	23.34	27.42	32.04	40.64*	29.16	0.86	24.17	23.87	30.81	37.44	32.96	2.14
100	16.63	-	7.36	45.81	27.27	42.48	41.45	47.02	53.74	61.06	50.28	0.89	44.37	42.97	52.94	61.27*	54.06	3.64
200	32.20	-	13.51	70.98	36.10	71.73	70.12	74.32	79.68	83.78	77.71	1.23	72.25	70.23	79.34	85.84*	79.82	5.74
500	72.92	-	36.12	97.32	45.27	98.05	97.55	98.13	98.61	98.99	98.50	3.38	97.96	97.36	98.60	99.29*	98.73	8.30
1000	97.61	-	76.67	99.97	52.40	100.00	100.00	100.00	100.00	100.00	100.00	8.18	99.98	99.96	100.00	100.00	99.98	9.66

Laplace (0, 1) – Skewness = 0, Kurtosis = 3

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	4.91	5.93	3.09	-	-	-	6.46	6.82	2.74	15.69*	4.53	0.00	5.28	6.89	0.82	4.25	13.94	0.46
20	8.99	12.88	5.31	18.81	13.10	10.67	13.45	15.79	14.84	31.04*	15.36	0.11	16.76	16.18	12.49	18.04	28.68	1.53
30	13.08	19.83	7.69	25.03	16.67	17.05	20.51	24.76	25.00	42.75*	24.73	0.86	28.41	27.36	22.81	30.67	41.25	2.61
40	16.56	27.37	9.41	29.51	18.99	22.03	26.91	31.97	32.30	51.73*	31.11	0.65	38.12	35.45	30.39	40.64	51.64	3.56
50	22.13	34.33	11.74	35.28	21.46	28.57	34.25	39.86	41.11	60.17	38.90	0.98	49.27	46.52	39.38	51.09	62.53*	4.21
100	46.11	65.91	26.33	55.98	27.26	55.37	63.68	69.07	68.45	84.23	65.88	2.20	81.55	79.14	67.54	80.74	88.49*	8.15
200	81.95	94.71	57.44	85.13	33.28	87.64	93.21	94.65	93.03	98.01	92.04	7.28	98.53	98.19	92.85	97.43	99.33*	15.15
500	99.88	-	97.55	99.87	37.66	99.92	99.99	99.99	99.96	100.00	99.96	32.76	100.00	100.00	99.96	100.00	100.00	33.87
1000	100.00	-	100.00	100.00	38.86	100.00	100.00	100.00	100.00	100.00	100.00	72.38	100.00	100.00	100.00	100.00	100.00	51.43

Table 3.9: Simulated power for asymmetric long-tailed distributions at 1% significance level

Gamma (4, 5) – Skewness = 1, Kurtosis = 4

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	3.15	4.27	2.68	-	-	-	4.91	4.63	1.67	9.49*	2.88	0.00	1.71	3.19	0.39	2.72	6.54	0.28
20	6.17	10.15	4.71	15.02	12.61	6.06	12.75	12.50	10.55	18.80*	9.55	0.11	3.62	5.17	8.71	12.13	9.86	0.97
30	9.80	18.23	6.51	23.09	22.31	9.07	24.02	22.59	19.67	27.50*	19.25	2.09	4.81	7.27	17.73	23.42	12.78	1.61
40	14.15	26.90	8.38	30.06	32.23	11.47	36.10*	32.67	27.15	34.45	30.36	1.73	5.84	7.99	25.32	33.97	14.60	2.05
50	19.24	36.22	10.83	38.23	42.19	13.96	47.03*	43.01	36.20	42.53	41.89	3.69	6.38	10.06	34.19	44.95	16.68	2.72
100	43.85	73.75	28.30	70.14	81.54	23.69	86.69*	82.59	70.64	72.09	84.31	9.74	8.57	17.05	69.31	83.34	22.83	10.07
200	82.73	98.31	69.51	98.02	99.31	42.61	99.81*	99.62	98.22	97.61	99.75	29.12	13.28	38.05	98.09	99.56	35.38	45.73
500	99.94	-	99.86	100.00	100.00	80.39	100.00	100.00	100.00	100.00	100.00	81.74	25.58	86.28	100.00	100.00	63.16	98.82
1000	100.00	-	100.00	100.00	100.00	97.75	100.00	100.00	100.00	100.00	100.00	99.42	45.40	99.76	100.00	100.00	87.71	100.00

Chi-square (3) – Skewness = 1.63, Kurtosis = 4

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	7.96	12.43	7.50	-	-	-	14.32	13.09	4.81	19.07*	9.18	0.01	3.14	3.87	1.74	7.07	13.04	0.98
20	20.45	36.49	22.22	33.10	32.09	14.67	41.29*	38.16	25.38	38.84	33.25	0.83	7.98	8.66	38.74	31.52	23.99	3.89
30	33.96	59.88	37.78	48.83	52.83	21.31	68.78*	62.84	43.55	54.74	59.58	7.41	11.50	18.27	63.35	54.74	31.13	9.15
40	47.67	76.60	49.38	63.20	70.43	28.38	85.28*	80.18	59.91	67.60	78.77	9.34	14.74	22.79	80.93	74.25	37.74	18.91
50	60.74	87.72	62.31	74.48	83.08	33.98	94.19*	91.11	73.30	77.77	89.96	18.91	17.45	33.00	91.32	86.59	43.45	32.26
100	94.65	99.86	96.31	98.37	99.49	60.34	100.00	99.98	98.68	98.07	99.97	52.36	29.91	67.07	100.00	99.92	65.35	93.49
200	99.99	-	100.00	100.00	100.00	87.39	100.00	100.00	100.00	100.00	100.00	91.70	48.43	95.88	100.00	100.00	87.03	100.00
500	100.00	-	100.00	100.00	100.00	99.77	100.00	100.00	100.00	100.00	100.00	99.99	82.19	100.00	100.00	100.00	99.35	100.00
1000	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	98.00	100.00	100.00	100.00	100.00	100.00

Exponential (1) – Skewness = 2, Kurtosis = 6

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	13.17	20.92	12.94	-	-	-	23.65*	21.51	8.09	25.35	16.02	0.05	4.75	5.16	3.73	11.54	19.00	1.96
20	33.61	57.08	42.34	45.10	45.68	21.35	63.16*	58.15	35.84	51.79	52.05	2.53	11.87	14.40	31.96	45.13	34.78	9.26
30	55.15	82.18	65.89	64.62	69.74	32.00	87.80*	83.42	59.88	69.75	80.16	16.84	18.26	32.40	56.44	73.18	46.73	25.66
40	72.37	93.30	78.71	77.25	84.79	39.58	96.69*	94.60	75.00	80.95	92.89	25.58	22.85	42.73	72.54	88.69	54.46	48.64
50	85.13	98.23	86.90	88.09	93.80	48.37	99.42*	98.87	87.73	90.10	98.32	41.02	27.12	57.83	86.32	96.72	62.92	72.03
100	99.91	100.00	99.84	99.91	99.98	77.97	100.00	100.00	99.91	99.83	100.00	83.78	45.36	91.50	99.89	100.00	85.16	99.97
200	100.00	-	100.00	100.00	100.00	97.08	100.00	100.00	100.00	100.00	100.00	99.37	71.65	99.89	100.00	100.00	98.15	100.00
500	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	96.61	100.00	100.00	100.00	99.99	100.00
1000	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.95	100.00	100.00	100.00	100.00	100.00

Log-Normal (0, 1) – Skewness = 6.18, Kurtosis = 113.94

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	27.78	38.97	23.22	-	-	-	42.12*	39.36	20.04	42.67	32.51	0.17	12.28	12.02	12.47	24.90	36.10	7.63
20	60.17	79.68	61.95	68.55	70.03	43.89	82.51*	79.58	61.17	74.74	76.08	5.17	32.52	36.41	56.97	70.43	62.51	29.33
30	82.07	94.87	83.26	87.39	90.33	62.36	96.85*	95.57	84.70	90.25	94.44	24.72	49.37	62.42	82.44	92.05	78.81	57.95
40	93.16	98.97	93.14	94.80	97.30	74.19	99.56*	99.30	94.20	96.47	99.12	38.77	61.44	77.45	93.23	98.29	87.59	81.04
50	97.61	99.86	97.08	97.94	99.31	82.70	99.94*	99.90	97.99	98.57	99.84	57.50	70.35	88.41	97.70	99.70	92.49	92.81
100	100.00	-	99.99	100.00	100.00	98.45	100.00	100.00	100.00	100.00	100.00	93.88	93.13	99.69	100.00	100.00	99.70	100.00
200	100.00	-	100.00	100.00	100.00	99.98	100.00	100.00	100.00	100.00	100.00	99.96	99.78	100.00	100.00	100.00	100.00	100.00

500	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	-	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

* The most powerful test for each sample size.

Table 3.10: Simulated power for asymmetric short-tailed distributions at 1% significance level

Weibull (2, 2) – Skewness = 0.63, Kurtosis = 0.25

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	1.57	1.84	1.75	-	-	-	2.23	2.08	0.51	5.09*	1.18	0.00	0.90	2.50	0.12	1.15	3.64	0.19
20	2.79	3.70	2.36	5.91	4.32	2.14	4.44	4.17	3.68	8.20*	2.88	0.15	1.27	3.03	2.74	4.23	3.62	0.59
30	3.69	6.35	3.02	8.35	6.95	3.35	7.98	6.95	6.17	10.38*	5.73	1.60	1.42	3.94	5.23	7.82	3.75	0.68
40	5.40	9.24	3.56	10.89	10.21	4.29	12.38	10.29	8.85	12.81*	9.12	0.96	1.73	3.71	7.82	11.64	4.19	1.23
50	7.19	12.23	4.44	13.68	14.96	4.67	18.08*	14.86	11.70	15.84	13.87	2.00	1.56	4.40	10.73	16.71	4.06	1.28
100	17.14	34.04	10.36	29.19	40.10	6.25	51.64*	42.07	27.86	29.33	45.42	3.98	2.03	6.70	26.71	44.27	4.29	2.93
200	43.08	77.93	31.00	73.01	83.45	7.48	94.85*	89.99	72.58	67.31	91.59	11.22	2.42	13.21	71.30	89.15	4.32	17.75
500	92.25	99.97	92.60	100.00	99.99	12.29	100.00	100.00	100.00	99.99	100.00	43.36	3.59	42.86	100.00	100.00	4.73	94.17
1000	99.98	100.00	100.00	100.00	100.00	18.93	100.00	100.00	100.00	100.00	100.00	83.69	6.09	81.91	100.00	100.00	4.44	100.00

Gompertz (0.001, 1) – Skewness = -1, Kurtosis = 1.5

N	LL	AD	CS	DK	SK	KU	SW	SF	JB	RJB	DH	BH	BS	BHBS	BM(1)	BM(2)	GMG	G
10	3.67	4.41	2.99	-	-	-	4.97	5.03	1.84	10.82*	3.08	0.02	2.05	3.52	0.52	2.79	7.59	0.38
20	6.72	11.12	4.42	17.36	14.31	6.75	14.02	14.18	12.16	22.29*	10.74	0.27	4.24	5.29	10.13	14.24	12.40	0.95
30	10.37	19.21	5.93	25.90	24.77	10.07	25.27	24.47	22.13	31.22*	20.51	2.32	5.72	8.15	20.10	26.29	15.68	1.59
40	15.54	27.73	7.76	34.08	34.65	12.19	36.24	34.42	31.08	39.85*	30.63	2.00	6.93	9.10	28.80	37.57	18.72	1.98
50	20.09	36.44	9.38	41.85	45.15	15.43	46.05	43.52	39.82	46.95	41.30	3.87	8.13	11.34	37.87	47.97*	20.93	2.83
100	45.49	71.54	23.22	74.59	82.73	28.12	83.32	80.71	74.81	76.82	81.43	9.44	12.19	20.54	73.80	83.67*	31.65	7.80
200	83.47	97.62	60.04	97.98	99.28*	52.85	99.20	98.92	98.10	98.00	99.17	28.35	22.13	45.62	98.02	99.20	51.22	28.26
500	99.92	100.00	99.13	100.00	100.00	90.52	100.00	100.00	100.00	100.00	100.00	79.65	48.43	91.18	100.00	100.00	83.76	81.39
1000	100.00	100.00	100.00	100.00	100.00	99.60	100.00	100.00	100.00	100.00	100.00	99.16	78.23	99.90	100.00	100.00	98.42	99.07

* The most powerful test for each sample size.

Table 3.1 gives the type I error rate while Table 3.2 to Table 3.5 give the power of the tests for the several alternative distributions.

An examination of the performance of the tests in terms of type I error rate shows that the LL, AD, CS, DK, SK, KU, SW, SF, RJB, DH tests were found better than the other tests; these tests have Type I error rates that were around the 5% level specified. The RJB test also have generally acceptable type I error rate but these rate were slightly higher than specified when the sample size was less than 50. The JB, BH, BS, BM (1) and G statistic all have Type I error rates lower than 5% and tend to under-reject while the BHBS, BM (2) and the GMG have Type I error rates higher than 5% and tend to over-reject. These results are generally consistent when compared with those obtained at the 0.01 and 0.10 significance levels.

A consideration of the results of power of the tests showed that different tests performed differently under different combinations of the sample size and the significance level. A general and expected pattern was observed that as sample size increase the power of the test also increase.

With Beta (1, 1), Beta (2, 2) and Beta (3, 3) as the alternative distributions, we have symmetric distributions with short tails, where Beta (1, 1) has the longest tail and the length of the tail reduces as the size of the parameters increase. The kurtosis test was the most powerful test for these distribution with the test achieving a 76.88% power at n=40. With Beta (2, 2), only the KU at 78.79% exhibited significant power when the sample size was less than 100, followed by the CS at 64.97%. However, with the sample size of 200, all the test reached at least 80% except for BHBS at 77.99, SF at 75.40, AD at

70.79% and JB at 61.04%. All other tests do not exhibit significant power especially the SK and BH which had 0.05% and 46.74 % power respectively, even at n=1000, and are clearly not suitable for these conditions. It is noticed that as the value of the parameter increases, the tail of the distribution reduces and consequently the coefficient of kurtosis resulting in a loss of power. In fact, for Beta (3, 3), considerable power was not achieved until when the sample size was 200; the kurtosis test was able to achieve a 79.72% power at this point.

In the case of a Uniform (0, 1) as the alternative distribution, the KU test had a power 88.59% at n=50 to prove being the most powerful under this condition, followed closely by DK (79.77%). With n=100, all tests excepts the LL, CS, SK, JB, RJB, BH, BM (1) and the G had power greater than 80%; the CS, SK, JB, RJB, BH, BM(1) and G particularly proved to be very bad test with $n \leq 50$ in this situation with the SK only achieving a power of 0.07% even at n=1000.

For a T (10) distribution, all the test were poor in detecting non-normality; even at n=500, only the BM(1) and BM(2) achieved a power of 80%, followed closely by the RJB (76.42%), GMG (75.54%), JB (75.16%), DH (74.97%), KU (74.37%) and SF (71.84%). All other test had power below 70% at n=500 or less. However, BM (2) is not acceptable as it has unacceptable type I error rate.

For a T(5) distribution that is symmetric and long-tailed, none of the tests was able to achieve a power of 80% even at n=100 with those that achieved closest to this cut-off point being the BM(2) (71.35%), RJB (69.02%), GMG (68.79%), DH (64.10%), JB (62.90%), SF (62.89%) and DK (60.04%).

Considering a Laplace (0, 1) with a mean of zero, the GMG is the most powerful for all sample sizes and achieved a power of 94.86% with n=100, with the AD, SW, SF, RJB, DH, BS, BHBS and BM(2) all achieving power above the 80% threshold. The SK and the G tests are the least powerful under this alternative distribution.

In the situation where the alternative distribution is a Gamma (4, 5), the most powerful test was the SW reaching a power of 95.81% at n=100, it was followed closely by the DH, BM(2), SF, and SK all achieving more than 90% power at n=100. The least powerful under the situation are the G, KU and BS. Both G and KU that did not achieve 80% power until n=500; the BS only achieved a power of 61.99% even at n=1000.

The chi-square (3) distribution proved to be one that was easily identified as being non-normal by all tests with SW(87.19%), SF (83.50%), AD(79.93%) and DH(79.42%) all achieving adequate power even at sample size as small as 30. At n=50, all eighteen tests considered had reached at least the 80% threshold except for the KU, BH, BS, BHBS, GMG and G. The least powerful was the BS test never achieving 100% power at n=1000 whereas all other tests have.

Exponential (1) also proved to be a distribution that was easy for the tests to identify as non-normal with the SW and SF having power above 80% at only n=20. All tests were able to achieve more than 80% power at only n=50 except for the KU, BH, BS, BHBS, and GMG. All tests however surpassed the 80% threshold at n=100 except for the BS which only achieved a 57.70% power at this sample size and proved the least powerful never achieving 100% power at n=1000 whereas all other tests have.

The SW test proved to be the most powerful under the Log-normal alternative distribution achieving a power of 83.73% at n=20, followed closely by its modified form the SF (80.13%). All tests surpassed the 80% threshold at n=40 except for the BH and BS which only achieved power of 65.76% and 69.87% respectively. BHBS, a joint test of the BH and BS however proved more powerful than the individual tests by achieving a power of 89.52 at n=40. However, BHBS is not recommended as it has unacceptable type I error rate.

The result of power on a weibull (2, 2) alternative distribution showed that the SW is the most powerful under this distribution. The test achieved a power of 79.33% at n= 100 which is just a little below the 80% rate that is usually described as acceptable. The SW is closely followed by the DH (72.64%) and SF (71.64). The AD, DK, SK, JB, RJB, BM(1) and BM(2) were also able to achieve at least 80% power at n=200. The BS once again proved to be the least powerful among the tests under this distribution by only achieving a power of 16.94%.

An asymmetric, short-tailed Gompertz distribution as an alternative distribution showed the SK test to be powerful, and a strong rival to the popular SW test, however, none of the test was able to achieve 80% power until the sample size was increased to 100 at which point all of the tests except the LL, CS, KU, BH, BS, GMG and G had surpassed the threshold. The BS once more was the least powerful under this distribution; despite most of the tests achieving the 80% threshold and a significant number of them achieving 100% at n=500, the test was only able to achieve 65.88% power.

As expected, the power of all the tests reduced at the 1% level of significance in contrast to those at the 5% level as follows. This is because we have a wider range of critical values for non-rejection of the hypothesis of normality thus leading to a higher level of confidence in the results from the tests. Little variations were observed in the results at the 1% level and these include the RJB and BM (1) being the most powerful test for a T (10) distribution as against BHBS, BM (2) and GMG at the 5% level. Also, RJB was more powerful for sample sizes of 40 or less for a laplace (0, 1) and then GMG for higher sample sizes. In contrast, GMG was ultimately the most powerful for laplace (0,1) at the 5% level.

A weibull (2, 2) distribution also showed RJB as the most powerful for sample sizes of 40 or less and SW for larger sample sizes as against BHBS for a sample size of 10 and SW for larger sample sizes at the 5% level. There is however, the most drastic change in the case of the Gompertz (0.001, 1) distribution at 1% level, where the GMG was the most powerful for sample size on 10 and SK for other sample sizes. The SK will probably be the most powerful for a sample size of 10 but for the unavailability of the SK along with the KU and DK for sample sizes less than 20. At the 5% level on the other hand, the RJB was the most powerful for sample sizes of 40 or less and BM (2) for larger sample sizes.

All these tests behave differently depending on the alternative distribution under consideration. Even though the BHBS, BM(2) and GMG show proved powerful in certain situations, they are not recommended for testing for normality as they do not effectively control for type I error rate.

CHAPTER FOUR: SOME APPLICATIONS

To illustrate the findings of this research, three real life medical dataset were analyzed as follows.

4.1 Non-sudden infant death syndrome (SIDS) Example

Several population studies have demonstrated an inverse correlation of sudden infant death syndrome (SIDS) rate with birth weight. The occurrence of SIDS in one of a pair of twins provides an opportunity to test the hypothesis that birth weight is a major determinant of SIDS. The data below consist of the birth weights (in grams) of one of each pair of 22 dizygous twins:

2098, 3119, 3515, 2126, 2211, 2750, 3402, 3232, 1701, 2410, 2892, 2608,

2693, 3232, 3005, 2325, 3686, 2778, 2552, 2693, 1899, 3714.

Data Source: Peterson et al., 1979.

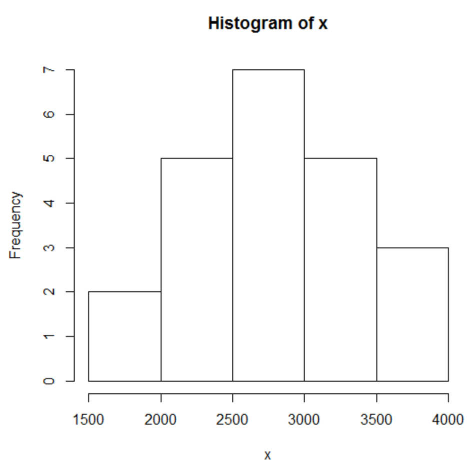

Figure 4.1 (a) Histogram of SIDS data

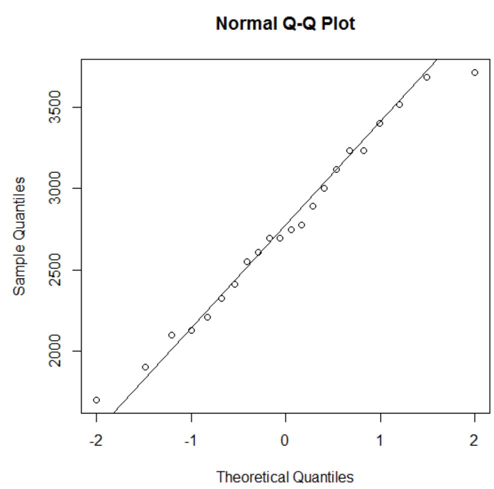

Figure 4.1 (b) QQplot of SIDS data

Table 4.1 Test results for non-sudden infant death syndrome (SIDS) data

Normality Test	Value of test statistic	P-value (or Critical Value)	Reject normality at α = 5%?
LL	0.0758	0.9854	Do not reject
AD	0.1417	0.9665	Do not reject
CS	1.4545	0.6928	Do not reject
DK	0.7598	0.6839	Do not reject
SK	0.0181	0.9856	Do not reject
KU	-0.8715	0.3835	Do not reject
SW	0.9778	0.8784	Do not reject
SF	0.9883	0.9772	Do not reject
JB	0.6827	0.7108	Do not reject
RJB	0.3954	0.8206	Do not reject
DH	0.2060	0.9021	Do not reject
BH	0.9322	0.8177	Do not reject
BS	-0.4967	0.6194	Do not reject
BHBS	5.3355	0.2546	Do not reject
BM(1)	0.5579	0.7566	Do not reject
BM(2)	0.9743	0.6144	Do not reject
GMG	0.9745	0.3268	Do not reject
G	0.0147	(0.0714)	Do not reject

The sample is positively skewed with skewness = 0.007 and short-tailed with kurtosis = -1.05, mean = 2756.409, standard deviation (SD) = 568.762 and sample size is 22. From the histogram and QQ plot above together with the summary of the data, we can see that the data can satisfactorily be modeled by the normal distribution as revealed by all the tests of normality.

4.2 Triglyceride Level Example

A study of changes in serum cholesterol and triglyceride levels of subjects following the Stillman diet was conducted. The diet consists primarily of protein and animal fats, restricting carbohydrate intake. The subjects followed the diet with length of time varying from 3 to 17 days. The mean cholesterol level increased significantly from 215 mg per/100 mL at baseline to 248 mg per/100 mL at the end of the diet. Below are the Baseline triglyceride level measurements:

159, 93, 130, 174, 148, 148, 85, 180, 92, 89, 204, 182, 110, 88, 134, 84.

Data Source: Rickman et al. (1974)

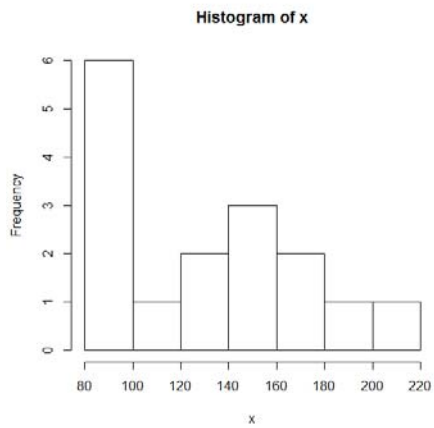

Figure 4.2(a) Histogram of triglyceride level data

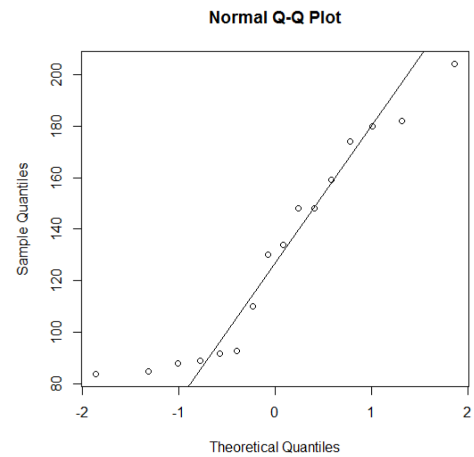

Figure 4.2 (b) QQplot of triglyceride level data

Table 4.2 Test results for triglyceride level data

Normality Test	Value of test statistic	P-value (or Critical Value)	Reject normality at α = 5%?
LL	0.2011	0.0830	Do not reject
AD	0.5826	0.1096	Do not reject
CS	3.5000	0.3208	Do not reject
DK	-	-	-
SK	-	-	-
KU	-	-	-
SW	0.9016	0.0853	Do not reject
SF	0.9213	0.1543	Do not reject
JB	0.6827	0.7108	Do not reject
RJB	0.3954	0.8206	Do not reject
DH	2.7390	0.2542	Do not reject
BH	1.7458	0.6268	Do not reject
BS	-1.6340	0.1023	Do not reject
BHBS	6.5317	0.1628	Do not reject
BM(1)	1.0166	0.6015	Do not reject
BM(2)	2.6699	0.2632	Do not reject
GMG	0.9024	0.0713	Do not reject
G	0.0565	(0.1007)	Do not reject

The sample is positively skewed with skewness = 0.23 and short-tailed with kurtosis = -1.52, mean = 131.25, SD = 40.74 and sample size is 16. There seems to be some concerns with the normality assumption about this set of data both from the histogram and the QQ plot. It can also be observed that the LL, AD, SW, BS and GMG only marginally failed to reject the normality assumption and in particular, the LL, SW and GMG would have rejected normality at 10% significance level. As the sample size is small at only sixteen, we can attribute some of the effect to sample size as these tests have small power at small sample sizes especially when the coefficient of skewness is close to zero. The RJB did not identify and serious problem of normality from the data while the DK, SK and KU will not produce results for sample size less than 20.

4.3 Postmortem Interval Example

The postmortem interval (PMI) is defined as the elapsed time between death and an autopsy. Knowledge of PMI is considered essential when conducting medical research on human cadavers. The following data are PMIs of 22 human brain specimens obtained at autopsy in a recent study:

5.5, 14.5, 6.0, 5.5, 5.3, 5.8, 11.0, 6.1, 7.0, 14.5, 10.4, 4.6, 4.3, 7.2, 10.5, 6.5, 3.3, 7.0, 4.1, 6.2, 10.4, 4.9.

Data Source: Hayes and Lewis, 1995

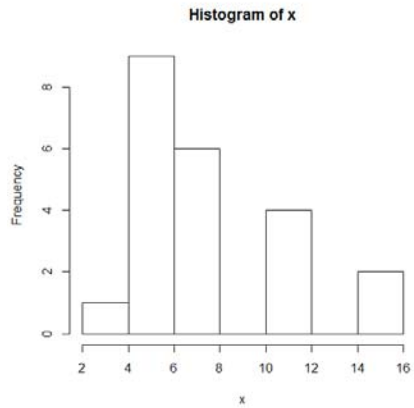

Figure 4.3(a) Histogram for postmortem interval data

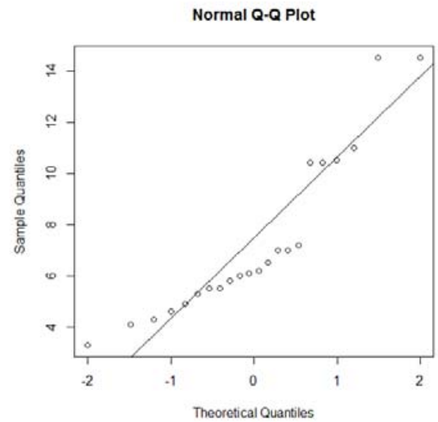

Figure 4.3(b) QQplot for postmortem interval data

Table 4.3 Test results for postmortem interval data

Normality Test	Value of test statistic	P-value (or Critical Value)	Reject normality at α = 5%?
LL	0.2398	0.0019	Reject
AD	1.2453	0.0023	Reject
CS	15.6364	0.0013	Reject
DK	5.5303	0.0630	Do not reject
SK	2.2380	0.0252	Reject
KU	0.7222	0.4702	Do not reject
SW	0.9091	0.2378	Do not reject
SF	0.9129	0.2244	Do not reject
JB	4.1410	0.1261	Do not reject
RJB	7.9721	0.0186	Reject
DH	8.9722	0.0113	Reject
BH	8.8778	0.0310	Reject
BS	-0.1260	0.8997	Do not reject
BHBS	11.5494	0.0210	Reject
BM(1)	3.6023	0.1651	Do not reject
BM(2)	7.5541	0.0229	Reject
GMG	1.0968	0.0439	Reject
G	0.1210	(0.0714)	Reject

The sample is positively skewed with skewness = 0.99 and short-tailed with kurtosis = -0.16, mean = 7.30, SD = 3.18 and sample size is 22. This dataset was originally modeled by a gamma distribution with shape parameter α = 5.25 and scale parameter β = 1.39 (see

Banik and Kibria, 2010), so we can assume that the hypothesis of normality will be rejected, however, seven of the eighteen test considered failed to reject this hypothesis including the popular DK, SW and SF tests. It can be noted that the coefficient of kurtosis of the data is 0.16 and close enough to that of a normal distribution at zero.

CHAPTER FIVE: SUMMARY AND CONCLUSION

It is necessary to carry out this research work because of the necessity to check the normality assumption in most statistical test. Parametric tests in particular are usually more powerful than their non-parametric counterparts but do require the validity of some assumptions of which the normality is one. Numerous methods have been proposed by different authors over the years and particularly, empirical tests which result in conclusive decision have become popular among the users of statistical methods.

In this work, eighteen different tests of normality comprising the most popular and frequently used such as the SW, LL, and CS among others along with some of the recently proposed tests were compared simultaneously in order to scrutinize their performance. The performance was measured in terms of type I error rate and power of the test. The type I error rate is the rate of rejection of the hypothesis of normality for data from the normal distribution while the power of the test is the rate of rejection of normality hypothesis for data generated from a non-normal distribution. Different samples sizes, each at significance levels 1%, 5% and 10%.

Type I error rates for the LL, AD, CS, DK, SK, KU, SW, SF, RJB, DH tests were around the 5% level specified and thus adequately control for error rates in data. The RJB test was also found out to have generally acceptable type I error rate but these rate were slightly higher than specified when the sample size was less than 50 thereby increasing the rate of false-positive for non-normality. The JB, BH, BS, BM (1) and G statistic all have Type I error rates lower than 5% and tend to under-reject while the BHBS, BM (2) and the GMG have Type I error rates higher than 5% and tend to over-reject.

Regarding the power of the test, symmetric distributions shows the influence of the coefficient of skewness on the tests; none of the tests produces significant power result as symmetric distribution is characterized by a skewness coefficient of zero which is similar to that of the normal distribution. The tests only yield a better performance when the distribution is long-tailed and have a co-efficient of kurtosis significantly different from zero. Generally, the tests did not achieve significant power with sample size below one hundred but the BM(2) seem to have a slight advantage over the others followed closely by the KU test provided the coefficient of kurtosis is significantly different from zero.

For asymmetric distributions, the coefficients of skewness appears to have a major influence on the power of the test, with the power rapidly increasing as the coefficient tends away from zero, the coefficients of kurtosis also tend to influence the power of the test with the power of the test increasing as the coefficient of kurtosis increases, but this measure does not exert much influence as the coefficient of skewness. The SW test is the best for asymmetric distribution followed closely by its modified form, the SF. The tests are adequate in detecting non-normality under this condition provided the sample size is at least fairly large.

From the section on application to real data, we see that not all the tests are powerful at detecting non-normality with small sample size, however, when the data is approximately normal, all the tests seems to agree with normality irrespective of the sample size.

The conclusions here are limited to the simulation conditions of this thesis. For a definite statement about the performance of the test statistics one might need more data and more simulation conditions.

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APPENDIX

Table A1: Extended table of critical values for the G-test

N	$G_{0.900}$	$G_{0.950}$	$G_{0.990}$	$G_{0.995}$	$G_{0.999}$
2	0.5360	0.6543	0.8349	0.8815	0.9459
3	0.3932	0.4895	0.6751	0.7358	0.8397
4	0.3126	0.3862	0.5498	0.6109	0.7280
5	0.2595	0.3185	0.4568	0.5135	0.6278
6	0.2212	0.2702	0.3874	0.4371	0.5448
7	0.1924	0.2337	0.3337	0.3777	0.4762
8	0.1699	0.2053	0.2922	0.3310	0.4208
9	0.1520	0.1829	0.2591	0.2935	0.3741
10	0.1373	0.1644	0.2316	0.2625	0.3361
11	0.1251	0.1492	0.2090	0.2363	0.3033
12	0.1149	0.1364	0.1903	0.2153	0.2763
13	0.1060	0.1255	0.1739	0.1963	0.2511
14	0.0984	0.1160	0.1599	0.1805	0.2311
15	0.0918	0.1079	0.1482	0.1669	0.2130
16	0.0859	0.1007	0.1375	0.1546	0.1971
17	0.0808	0.0943	0.1282	0.1440	0.1834
18	0.0761	0.0887	0.1200	0.1347	0.1718
19	0.0720	0.0837	0.1128	0.1264	0.1605
20	0.0683	0.0791	0.1062	0.1188	0.1506
21	0.0649	0.0751	0.1003	0.1121	0.1418
22	0.0618	0.0714	0.0950	0.1060	0.1336
23	0.0590	0.0680	0.0901	0.1005	0.1266
24	0.0564	0.0649	0.0856	0.0952	0.1197
25	0.0541	0.0620	0.0816	0.0907	0.1139
26	0.0519	0.0594	0.0778	0.0864	0.1084
27	0.0499	0.0570	0.0745	0.0826	0.1032
28	0.0480	0.0547	0.0713	0.0790	0.0986
29	0.0462	0.0527	0.0684	0.0757	0.0943
30	0.0446	0.0507	0.0656	0.0726	0.0902
31	0.0431	0.0489	0.0631	0.0698	0.0865
32	0.0416	0.0472	0.0608	0.0671	0.0830
33	0.0403	0.0456	0.0586	0.0646	0.0799
34	0.0390	0.0441	0.0565	0.0623	0.0767
35	0.0379	0.0427	0.0545	0.0600	0.0740
36	0.0367	0.0414	0.0528	0.0580	0.0715
37	0.0357	0.0402	0.0510	0.0560	0.0689
38	0.0347	0.0390	0.0494	0.0543	0.0666
39	0.0337	0.0379	0.0479	0.0526	0.0644
40	0.0328	0.0368	0.0465	0.0509	0.0621

41	0.0320	0.0358	0.0450	0.0493	0.0601
42	0.0312	0.0349	0.0438	0.0479	0.0583
43	0.0304	0.0340	0.0426	0.0466	0.0567
44	0.0297	0.0331	0.0414	0.0452	0.0548
45	0.0289	0.0323	0.0403	0.0440	0.0532
46	0.0283	0.0315	0.0393	0.0428	0.0518
47	0.0276	0.0308	0.0383	0.0417	0.0504
48	0.0270	0.0301	0.0373	0.0406	0.0490
49	0.0264	0.0294	0.0364	0.0396	0.0477
50	0.0259	0.0287	0.0355	0.0386	0.0465
60	0.0213	0.0234	0.0285	0.031	0.0368
70	0.0180	0.0197	0.0238	0.0257	0.0302
80	0.0156	0.017	0.0203	0.0218	0.0256
90	0.0138	0.0149	0.0176	0.0187	0.0216
100	0.0123	0.0132	0.0155	0.0165	0.0187
200	0.0058	0.0062	0.0069	0.0072	0.0080
300	0.0038	0.0040	0.0044	0.0045	0.0049
400	0.0028	0.0029	0.0032	0.0032	0.0035
500	0.0022	0.0023	0.0025	0.0025	0.0027
1000	0.0011	0.0011	0.0012	0.0012	0.0012