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A Study on the Correlation of Bivariate And Trivariate Normal Models

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A STUDY ON THE CORRELATION OF BIVARIATE AND TRIVARIATE NORMAL MODELS

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Maria del Pilar Orjuela Garavito

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To: Dean Kenneth Furton
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This thesis, written by Maria del Pilar Orjuela Garavito, and entitled A Study on the Correlation of Bivariate and Trivariate Normal Models, having been approved in respect to style and intellectual content, is referred to you for judgment.

We have read this thesis and recommend that it be approved.

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The thesis of Maria del Pilar Orjuela Garavito is approved.

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DEDICATION

To my parents.

ACKNOWLEDGMENTS

Thanks, everyone!

ABSTRACT OF THE THESIS

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by

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Suppose two or more variables are jointly normally distributed. If there is a common relationship between these variables, it would be very important to quantify this relationship by a parameter called the correlation coefficient which measures its strength, and the use of it can develop an equation for predicting, and ultimately draw testable conclusion about the parent population.

My research focused on the correlation coefficient ρ for the bivariate and trivariate normal distribution when equal variances and equal covariances are considered. Particularly, we derived the Maximum Likelihood Estimators (MLE) of the distribution parameters assuming all of them are unknown, and we studied the properties and asymptotic distribution of $\hat{\rho}$. Showing the asymptotic normality of the distribution of $\hat{\rho}$, we were able to construct confidence intervals of the correlation coefficient ρ and test hypothesis about ρ. With a series of simulations, the performance of our new estimators were studied and were compared with those estimators that already exist in the literature. The results indicated that the MLE has a better or similar performance than the others.

Keywords: Bivariate Normal Distribution, Trivariate Normal Distribution, Correlation coefficient, MLE, Pearson Correlation Coefficient.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

The Normal Distribution is one of the most common distributions used in a variety of fields, not only because it appears in theoretical work as an approximation of the distribution of different kind of measurements, but also because there are so many phenomena in the nature that follows the normal distribution. For instance, in natural sciences, quantities such as height of people and intelligence coefficients are approximately normally distributed (Clauser [2007]); in physics velocities of the molecules in an ideal gas, the position of a particle which experiences diffusion and the ground state in a quantum harmonic oscillator, are also normally distributed.

The Multivariate case of the Normal Distribution and, in particular, the Bivariate and Trivariate cases, play a fundamental role in Multivariate Analysis. Bivariate normal distributions have shown to be important from the theoretical point of view and have been widely studied because its several useful and elegant properties. However, the Trivariate case, although also important, has received much less attention.

Sometimes there is a relationship and dependence among the variables under the Multivariate Normal Distribution. For this reason, would be very useful to quantify this relationship in a number called the correlation coefficient, measure its strength, develop an equation for predicting, and ultimately draw testable conclusion about the parent population. My thesis focuses on the measure of correlation, assuming that there is common relationship in hte Bivariate and Trivariate models when all the variables have common variance.

1.2 Background and Theory

In the first half of the nineteenth century, scientists such as Laplace, Plana, Gauss and Bravais studied the Bivariate Normal and Seal and Lancaster contributed to these developments as well. But it was until 1888 when Francis Galton, who first introduced the concept of a measure of correlation in a bivariate data, re-described and recognized the mathematical definition of it and found some applications.

Inference regarding the correlation coefficient of two random variables from the Bivariate Normal Distribution has been investigated by many authors. In particular, Fisher (1915) and Hotelling (1953) derived exact forms of the density for the sample correlation coefficient but obtaining confidence intervals was computationally intensive (1988). Fisher also derived his famous transformation for large samples for the limiting distribution of the bivariate correlation coefficient, and, in 1921, he obtained results for the correlation coefficient for small samples (1921). Other authors such as Ruben (1966) established the distribution of the correlation coefficient in the form of a normalization approximation, and Kraemer (1973) and Samiuddin (1970) in the form of a t approximations. More recently, Sun and Wong (2007) used a likelihood-based higher-order asymptotic method to obtain confidence intervals very accurate even when the sample size is small.

Some other authors, such as Goria (1980), Masry (2011) and Paul (1988) studied the correlation coefficient when specific assumptions are given to the Bivariate Normal distributions. For example, Goria (1980) deduced two test for testing the correlation coefficient in the bivariate normal population assuming that the ratio of the variances is known; Masry (2011) worked with the estimation of the correlation coefficient of bivariate data under dependence; and Paul (1988) proposed a method for testing the equality of several correlation coefficients, for more than two independent random samples from bivariate normal populations.

Since very little attempt, in the literature, has been made to develop and evaluate statistics for testing the validity of the assumption of a common correlation with equal variances, commonly called covariance permutation-symmetric for multivariate normal distribution, our task is then to explore and study the correlation coefficient under this assumption in the bivariate and trivariate normal distributions.

The idea of our study is that, on the basis of a random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ of n observations from a Bivariate Normal population with equal variance, we will find the estimators of the parameters assuming that all of them are unknown. Particularly, we will derive the Maximum Likelihood estimators of the distribution parameters, and we will study the asymptotic distribution of correlation coefficient. Showing the asymptotic normality of this distribution, we will then be able to construct confidence intervals of it and test hypothesis. The performance of our new estimator will be studied and will be compared with those estimators that already exist in the literature.

In the case of the Trivariate Normal distribution, we derive the maximum likelihood estimator for the correlation coefficient from a random sample of n observations $(X_{11}, X_{21}, X_{31}), \ldots, (X_{n1}, X_{n1}, X_{n1})$ as well assuming that this sample comes from a covariance permutation-symmetric Trivariate Normal population. We will compare it with that one obtained from the average of pairwise bivariate Pearson correlation coefficients and the average of pairwise MLE of ρ.

The thesis is organized as follows. The derivation of the Maximum Likelihood Estimators of the parameters of the Bivariate Normal distribution under the assumption of equal variances is reviewed in Chapter 2. Moreover, a discussion about the correlation coefficient is performed, whose properties are announced and sampling distribution is found. In Chapter 3 a similar procedure is made when a Trivariate Normal distribution is considered under the same assumption of equal variances. Chapter 4 and 5 focus on numerical computations using the MLE of the correlation coefficient for Bivariate and Trivariate cases respectively, comparing the performance of confidence regions, probability of Type I error and Power of testing for its true value.

CHAPTER 2

BIVARIATE NORMAL DISTRIBUTION

In Chapter 2, we study a particular case of the Bivariate Normal Distribution. Using a frequently used method, namely the method of the Maximum Likelihood Estimation, we find the estimators of the parameters of this special case. We discuss its properties and focus on the **MLE** estimator of its correlation coefficient.

The outline of the chapter is as follows. In Section 2.1 we introduce the probability density function of the Bivariate Normal Distribution and its MLEs. In Section 2.2 we discuss the probability density function of the Bivariate Normal Distributionn with equal variances. In particular, in Section 2.2.2 we find the likelihood function and in Section 2.2.3 we discuss the MLEs of the parameters. Later, in Section 2.2.4 we show properties of those MLEs found before and finally, in Section 2.2.6, we focus on the study of the estimator of the correlation coefficient $\hat{\rho}$.

2.1 General Discussion

Let X and Y be two random variables. It is said that (X, Y) have a Bivariate Normal Distribution if the joint probability density function $f(x, y)$ is of the form

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\Big\{-\frac{1}{2}Q(x,y)\Big\} \qquad -\infty < x, y < \infty \tag{2.1}$$

where

$$Q(x,y) = \frac{1}{1-\rho^2}\left[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2 - 2\rho\frac{(x-\mu_1)}{\sigma_1}\frac{(y-\mu_2)}{\sigma_2} + \Big(\frac{y-\mu_1}{\sigma_2}\Big)^2\right] \tag{2.2}$$

It can be shown that

$$E(X) = \mu_1, E(Y) = \mu_2, Var(X) = \sigma_1^2, Var(Y) = \sigma_2^2$$

and ρ is the correlation between X and Y, where $-\infty < \mu_1, \mu_2 < \infty, \sigma_1^2, \sigma_2^2 > 0$ and $-1 \leq \rho \leq 1$.

Suppose that n observations are taken from this Bivariate Normal Distribution, let say $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The Maximum Likelihood Estimators, MLEs, of the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ are:

$$\hat{\mu}_1 = \frac{\sum_{i=1}^n x_i}{n} = \overline{x} \tag{2.3}$$

$$\hat{\mu}_2 = \frac{\sum_{i=1}^n y_i}{n} = \overline{y} \tag{2.4}$$

$$\hat{\sigma}_1^2 = \frac{\sum_{i=1}^n (x_i - \overline{x})^2}{n} \tag{2.5}$$

$$\hat{\sigma}_2^2 = \frac{\sum_{i=1}^n (y_i - \overline{y})^2}{n} \tag{2.6}$$

$$r_n = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^n (x_i - \overline{x}) \sum_{i=1}^n (y_i - \overline{y})}} \tag{2.7}$$

It can be shown that the properties of the estimator of the correlation coefficiente r_n for the general case of the Bivariate Normal distribution are the following:

Properties of r_n

1. $\sqrt{n}\big(r_n - \rho\big) \rightarrow N\big(0, (1-\rho^2)^2\big)$
2. $\sqrt{n}\Big(\frac{1}{2}\ln\big(\frac{1+r_n}{1-r_n}\big) - \frac{1}{2}\ln\big(\frac{1+\rho}{1-\rho}\big)\Big) \rightarrow N(0,1)$. This is called Fisher's z-Transformation

2.2 Case of Equal Variances

2.2.1 Data and Model

Suppose (X, Y) is a random vector following a Bivariate Normal distribution $N_2(\boldsymbol{\mu}, \Sigma)$ where

$$\boldsymbol{\mu} = (\mu_1, \mu_2)'$$

and Σ is the 2×2 variance-covariance matrix. In the present section we assume that $Var(X) = Var(Y) = \sigma^2$ and $Cov(X, Y) = \sigma^2\rho$. Then, the joint probability density function for (X, Y) is given by

$$f(x, y; \mathbf{u}, \sigma^2, \rho) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{ - \frac{W(x,y)}{2\sigma^2(1-\rho^2)} \right\} \tag{2.8}$$

where

$$W(x, y) = (x - \mu_1)^2 - 2\rho(x - \mu_1)(y - \mu_2) + (y - \mu_1)^2 \tag{2.9}$$

provided that $-1 < \rho < 1$.

2.2.2 Likelihood Function

We want to find the MLEs of the unkown parameters of the distribution μ_1, μ_2, τ and ρ where $\tau = \sigma^2$. For this purpose, let us assume $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is a random sample from this population. Since they are independents and identically distributed, we can write the likelihood function $L(\mathbf{u}, \sigma^2, \rho; (\boldsymbol{x}, \boldsymbol{y}))$ as

$$L(\mathbf{u}, \sigma^2, \rho; (\boldsymbol{x}, \boldsymbol{y})) = \prod_{i=1}^{n} f(x_i, y_i; \mathbf{u}, \sigma^2, \rho) \tag{2.10}$$

$$= \prod_{i=1}^{n} \frac{1}{2\pi\tau\sqrt{1-\rho^2}} \exp\left\{ - \frac{1}{2\tau(1-\rho^2)} Q(x_i, y_i) \right\} \tag{2.11}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and

$$Q(x_i, y_i) = (x_i - \mu_1)^2 - 2\rho(x_i - \mu_1)(y_i - \mu_2) + (y_i - \mu_2)^2. \tag{2.12}$$

The likelihood function can be written as

$$L(\mathbf{u}, \sigma^2, \rho; (\boldsymbol{x}, \boldsymbol{y})) = \frac{1}{(2\pi)^n \tau^n (1-\rho^2)^{\frac{n}{2}}} \exp\left\{ - \frac{1}{2\tau(1-\rho^2)} Q^*(x, y) \right\} \tag{2.13}$$

where

$$Q^*(x, y) = \sum_{i=1}^{n} \Big[(x_i - \mu_1)^2 - 2\rho(x_i - \mu_1)(y_i - \mu_2) + (y_i - \mu_2)^2 \Big]. \tag{2.14}$$

2.2.3 Derivation of MLEs of Covariance Matrix Symmetric Bivariate Normal Parameters

Using the likelihood function $L(\mathbf{u}, \sigma^2, \rho; (\boldsymbol{x}, \boldsymbol{y}))$ we want to find the MLEs of $\hat{\mu_1}, \hat{\mu_2}, \hat{\sigma_1}, \hat{\sigma_2}, \hat{\rho}$ that maximizes $L(\mathbf{u}, \sigma^2, \rho; (\boldsymbol{x}, \boldsymbol{y}))$.

For this purpose, it is convenient to work with the natural logarithm of the likelihood function (since it is a monotone function)

$$\begin{aligned} \ln L(\mathbf{u}, \sigma^2, \rho; (\boldsymbol{x}, \boldsymbol{y})) = & - n \ln(2\pi) - n \ln \tau - \frac{n}{2} \ln(1 - \rho^2) \\ & - \frac{1}{2\tau(1-\rho^2)} \Big[\sum_{i=1}^{n} [(x_i - \mu_1)^2 - 2\rho(x_i - \mu_1)(y_i - \mu_2) + (y_i - \mu_2)^2] \Big] \end{aligned} \tag{2.15}$$

and solving the system of equations

$$\frac{\partial \ln L}{\partial \mu_1} = \frac{\partial \ln L}{\partial \mu_2} = \frac{\partial \ln L}{\partial \tau} = \frac{\partial \ln L}{\partial \rho} = 0 \tag{2.16}$$

we can obtain the desired estimators.

The equations in (2.16 can be written as

$$\frac{\partial \ln L}{\partial \mu_1} = -\frac{1}{2\tau(1-\rho^2)}\sum_{i=1}^{n}\big[-2(x_i-\mu_1)+2\rho(y_i-\mu_2)\big]=0 \quad (2.17)$$

$$\frac{\partial \ln L}{\partial \mu_2} = -\frac{1}{2\tau(1-\rho^2)}\sum_{i=1}^{n}\big[2\rho(x_i-\mu_1)-2(y_i-\mu_2)\big]=0 \quad (2.18)$$

$$\frac{\partial \ln L}{\partial \tau} = -\frac{n}{\tau}+\frac{\sum_{i=1}^{n}\big[(x_i-\mu_1)^2+(y_i-\mu_2)^2-2\rho(x_i-\mu_1)(y_i-\mu_2)\big]}{2(1-\rho^2)\tau^2}=0 \quad (2.19)$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \rho} &= \frac{2n\rho}{2(1-\rho^2)}-\frac{[-2\sum_{i=1}^{n}(x_i-\mu_1)(y_i\mu_2)][2\tau(1-\rho^2)]}{4\tau^2(a-\rho^2)^2} \\ &-\frac{2\tau(-2\rho)\sum_{i=1}^{n}\big[(x_i-\mu_1)^2+(y_i-\mu_2)^2-2\rho(x_i-\mu_1)(y_i-\mu_2)\big]}{4\tau^2(a-\rho^2)^2}=0 \quad (2.20)\end{aligned}$$

Simplifying equations (2.17) and (2.18), we have

$$-2\sum_{i=1}^{n}(x_i-\mu_1)+2\rho\sum_{i=1}^{n}(y_i-\mu_2)=0 \quad (2.21)$$

$$2\rho\sum_{i=1}^{n}(x_i-\mu_1)-2\sum_{i=1}^{n}(y_i-\mu_2)=0 \quad (2.22)$$

multiplying equation (2.21) by ρ and adding it to equation (2.22), we obtain

$$2(\rho^2-1)\sum_{i=1}^{2}(y_1-\mu_2)=0 \quad (2.23)$$

from where

$$\sum_{i=1}^{n}(y_i-\mu_2)=0 \quad (2.24)$$

since $\rho \neq \pm 1$, which implies that

$$\hat{\mu}_2=\frac{\sum_{i=1}^{n} y_i}{n}=\overline{y} \quad (2.25)$$

In similar way, we obtain

$$\hat{\mu}_1=\frac{\sum_{i=1}^{n} x_i}{n}=\overline{x} \quad (2.26)$$

Now, in order to obtain $\hat{\rho}$ and $\hat{\tau}$, let us use equations (2.19) and (2.20). Finding the common denominator in (2.19) and replacing μ_1 by $\overline{x}$ and μ_2 by $\overline{y}$, we get

$$\frac{-2n\tau(1-\rho^2) + \sum_{i=1}^{n}\left[(x_i-\overline{x})^2 + (y_i-\overline{y})^2\right] - 2\rho\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{2\tau^2(1-\rho^2)} = 0 \quad (2.27)$$

and multiplying it by $2\tau^2(1-\rho^2)$ we can obtain

$$-2n\tau(1-\rho^2) + \sum_{i=1}^{n}\left[(x_i-\overline{x})^2 + (y_i-\overline{y})^2\right] - 2\rho\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) = 0 \quad (2.28)$$

Similarly, simplifying equation (2.20) and replacing μ_1 and μ_2 by their estimators we have

$$\frac{n\rho}{(1-\rho^2)} + \frac{\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\tau(1-\rho^2)} - \frac{\rho\sum_{i=1}^{n}\left[(x_i-\overline{x})^2 + (y_i-\overline{y})^2\right]}{\tau(1-\rho^2)^2} + \frac{2\rho^2\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\tau(1-\rho^2)^2} = 0 \quad (2.29)$$

and multiplying it by $\tau(1-\rho^2)^2$ and combining like terms , equation (2.29) becomes

$$n\tau\rho(1-\rho^2) + (1-\rho^2+2\rho^2)\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) - \rho\sum_{i=1}^{n}\left[(x_i-\overline{x})^2 + (y_i-\overline{y})^2\right] = 0 \quad (2.30)$$

which is the same as

$$n\tau\rho(1-\rho^2) - \rho\sum_{i=1}^{n}\left[(x_i-\overline{x})^2 + (y_i-\overline{y})^2\right] + (1+\rho^2)\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) = 0 \quad (2.31)$$

Multiplying equation (2.28) by ρ and equation (2.31) by 2 and adding them up, we get

$$-\rho\sum_{i=1}^{n}\left[(x_i-\overline{x})^2 + (y_i-\overline{y})^2\right] + 2\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) = 0 \quad (2.32)$$

from where

$$\hat{\rho} = \frac{2\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}\left[(x_i - \overline{x})^2 + (y_i - \overline{y})^2\right]} \tag{2.33}$$

which is different from $\hat{r}_n$ obtained in (2.7).

Finally, solving for τ we obtain that

$$\hat{\tau} = \frac{\sum_{i=1}^{n}\Big[(x_i - \overline{x})^2 + (y_i - \overline{y})^2 - 2\hat{\rho}(x_i - \overline{x})(y_i - \overline{y})\Big]}{2n(1-\hat{\rho}^2)} \tag{2.34}$$

which is equivalent to

$$\hat{\tau} = \frac{\sum_{i=1}^{n}\left[(x_i - \overline{x})^2 + (y_i - \overline{y})^2\right]}{2n} \tag{2.35}$$

and thus

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}\left[(x_i - \overline{x})^2 + (y_i - \overline{y})^2\right]}{2n}} \tag{2.36}$$

2.2.4 Properties of the MLEs of the Covariance Matrix Symmetric Bivariate Normal Parameters

1. Mean of $\hat{\mu}_1, \hat{\mu}_2$:

   The estimators of $\hat{\mu}_1$ and $\hat{\mu}_2$ in our model are unbiased estimators. It can be easily proven:

   Proof.

$$\begin{aligned} E\left[\hat{\mu}_1\right] &= E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] \\ &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right] \end{aligned}$$

and using the fact that $E\left[X_i\right] = \mu_1$ then

$$\begin{aligned} E\left[\hat{\mu}_1\right] &= \frac{1}{n}\sum_{i=1}^{n}\mu_1 \\ &= \frac{1}{n}n\mu_1 \\ &= \mu_1 \end{aligned}$$

Therefore $E\left[\hat{\mu}_1\right] = \mu_1$ which means that $\hat{\mu}_1$ is an unbiased estimator of μ_1 $\square$

Similarly, $\hat{\mu}_2$ is an unbiased estimator of μ_2.

2. Mean of $\hat{\mu}_1 - \hat{\mu}_2$:

   It is easy to show that the mean of $\hat{\mu}_1 - \hat{\mu}_2$ is just $\mu_1 - \mu_2$.

3. Variance of $\hat{\mu}_1, \hat{\mu}_2$:

   The variance of $\hat{\mu}_1$ and $\hat{\mu}_1$ is $\frac{\sigma^2}{n}$

Proof.

$$\begin{aligned} Var\big(\hat{\mu}_1\big) &= Var\Big(\frac{\sum_{i=1}^{n} X_i}{n}\Big) \\ &= \frac{1}{n^2}Var\Big(\sum_{i=1}^{n} X_i\Big) \end{aligned}$$

Since $X_i, i = 1, \ldots, n$ are independent, then

$$\begin{aligned} Var\big(\hat{\mu}_1\big) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 \\ &= \frac{1}{n^2}n\sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Therefore $Var(\hat{\mu}_1) = \frac{\sigma^2}{n}$ $\square$

4. Variance of $\hat{\mu}_1 - \hat{\mu}_2$:

   The variance of $\hat{\mu}_1 - \hat{\mu}_2$ is given by $\frac{2\sigma^2}{n}(1-\rho)$

   Proof.

$$
\begin{aligned}
Var(\hat{\mu}_1 - \hat{\mu}_2) &= Var(\hat{\mu}_1) + Var(\hat{\mu}_2) - 2Cov(\hat{\mu}_1, \hat{\mu}_2) \\
&= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} - 2Cov\left(\frac{\sum_{i=1}^n X_i}{n}, \frac{\sum_{j=1}^n Y_j}{n} \right) \\
&= \frac{2\sigma^2}{n} - \frac{2}{n^2} \sum_{i=1}^n \sum_{j=1}^n Cov(X_i, Y_j)
\end{aligned}
$$

   since $Cov(X_i, Y_j) = \sigma^2\rho$ if $i = j$ and $Cov(X_i, Y_j) = 0$ if $i \neq j$, then

$$
\begin{aligned}
Var(\hat{\mu}_1 - \hat{\mu}_2) &= \frac{2\sigma^2}{n} - \frac{2}{n^2} \sum_{i=1}^n \sigma^2\rho \\
&= \frac{2\sigma^2}{n} - \frac{2}{n^2} \frac{\sigma^2\rho}{n} \\
&= \frac{2\sigma^2}{n} - \frac{2\sigma^2\rho}{n}\rho \\
&= \frac{2\sigma^2}{n}(1-\rho)
\end{aligned}
$$

   □

2.2.5 Sampling Distributions of MLEs of the Covariance Matrix Symmetric Bivariate Normal Parameters

1. The distribution of $\hat{\mu}_1 - \hat{\mu}_2$

   Since $\hat{\mu}_1 - \hat{\mu}_2$ is a linear combinations of $(X_i, Y_i), i = 1, \ldots, n$ and those come from a Bivariate Normal populations, the linear combination is also a normal population. From the properties seen in Section 2.2.4, then we can conclude that $\hat{\mu}_1 - \hat{\mu}_2 \sim N\left(\mu_1 - \mu_2, \frac{2\sigma^2}{n}(1-\rho) \right)$.

2.2.6 Discussion about the correlation coefficient $\hat{\rho}$

In subsection we will discuss some properties and the limiting distribution of the estimator r_n of the correlation coefficient ρ that was found in Section 2.2.3.

Recall that for the general case of the Bivariate Normal Distribution, the estimator $\hat{r}_n$ for the correlation coefficient ρ is given by

$$\hat{r}_n = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2\right)\left(\sum_{i=1}^{n}(y_i - \overline{y})^2\right)}} \tag{2.37}$$

while the estimator for the correlation coefficient in case of equal variances $\hat{\rho}$ was found to be

$$\hat{\rho}_n = \frac{2\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}\left[(x_i - \overline{x})^2 + (y_i - \overline{y})^2\right]} \tag{2.38}$$

Properties of $\hat{\rho}_n$

1. $-1 \leq \hat{\rho}_n \leq 1$

   Proof.

$$\begin{aligned}
\hat{\rho}_n \leq 1 &\Leftrightarrow \frac{2\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}\left[(x_i - \overline{x})^2 + (y_i - \overline{y})^2\right]} \leq 1 \\
&\Leftrightarrow 2\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) \leq \sum_{i=1}^{n}[(x_i - \overline{x})^2 + (y_i - \overline{y})^2] \\
&\Leftrightarrow \sum_{i=1}^{n}[(x_i - \overline{x})^2 + (y_i - \overline{y})^2] - 2\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) \geq 0 \\
&\Leftrightarrow \sum_{i=1}^{n}[(x_i - \overline{x})^2 + (y_i - \overline{y})^2 - 2(x_i - \overline{x})(y_i - \overline{y})] \geq 0 \\
&\Leftrightarrow \sum_{i=1}^{n}[(x_i - \overline{x}) + (y_i - \overline{y})]^2 \geq 0
\end{aligned}$$

   In similar way, $\hat{\rho}_n \geq -1$. □

2. $P(-1 \leq \hat{\rho}_n \leq 1) = 1$

Proof.

$$
\begin{aligned}
P\big(\hat{\rho}_n = 1\big) &= P\Big(\sum_{i=1}^{n}[(X_i - \overline{X})^2 + (Y_i - \overline{Y})^2] = 0\Big) \\
&= P\Big(\bigcap_{i=1}^{n}\big[(X_i - \overline{X})^2 + (Y_i - \overline{Y})^2\big] = 0\Big) \\
&\leq P\Big((X_i - \overline{X})^2 + (Y_i - \overline{Y})^2 = 0\Big) = 0
\end{aligned}
$$

□

3. $\hat{\rho}_n$ and $\hat{r}_n$ have the same sign and $|\hat{\rho}_n| \leq |r_n|$

Proof. The desired inequality is equivalent to

$$
\frac{2\Big|\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})\Big|}{\sum_{i=1}^{n}(x_i - \overline{x})^2 + \sum_{i=1}^{n}(y_i - \overline{y})^2} \leq \frac{\Big|\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})\Big|}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2 \sum_{i=1}^{n}(y_i - \overline{y})^2}}
$$

$$
\Leftrightarrow 4\Big(\sum_{i=1}^{n}(x_i - \overline{x})^2\Big)\Big(\sum_{i=1}^{n}(y_i - \overline{y})^2\Big) \leq \Big[\sum_{i=1}^{n}(x_i - \overline{x})^2 + \sum_{i=1}^{n}(y_i - \overline{y})^2\Big]^2
$$

$$
\Leftrightarrow \Big(\sum_{i=1}^{n}(x_i - \overline{x})^2\Big)^2 + \Big(\sum_{i=1}^{n}(y_i - \overline{y})^2\Big)^2 - 2\Big(\sum_{i=1}^{n}(x_i - \overline{x})^2\Big)\Big(\sum_{i=1}^{n}(y_i - \overline{y})^2\Big) \geq 0
$$

which is certainly true. Therefore, $|\hat{\rho}_n| \leq |r_n|$ □

The limiting distribution of $\hat{\rho}_n$

In the present subsection we will show that as the sample size goes to infinity ($n \to \infty$) it holds that $\sqrt{n}\big(\hat{\rho}_n - \rho\big) \to N\big(0, \gamma^2\big)$ in distribution where the constant γ^2 will be determined later.

To show this asymptotic normality the following results are needed.

Result 2.2.1. *$\sqrt{n}\big(\overline{X}_n - \mu_1\big)^2 \to$ 0 in probability as $n \to \infty$*

Proof. For any fixed $\epsilon > 0$, by Chebychev's inequality, it holds that

$$P\left(\left|\sqrt{n}(\overline{X}_n - \mu_1)^2\right| > \epsilon\right) \leq \frac{E\left[\sqrt{n}\left(\overline{X}_n - \mu_1\right)^2\right]}{\epsilon}$$
$$= \frac{\sqrt{n}Var(\overline{X}_n)}{n\epsilon}$$
$$= \frac{\sqrt{n}\sigma^2}{n\epsilon}$$
$$= \frac{\sigma^2}{\sqrt{n}\epsilon} \to 0$$

as $n \to \infty$ $\square$

Result 2.2.2. $\sqrt{n}\left(\overline{Y}_n - \mu_2\right)^2 \to$ *0 in probability as* $n \to \infty$

It can be shown in the similar way as in 1.

Result 2.2.3. $\sqrt{n}\left(\overline{X}_n - \mu_1\right)\left(\overline{Y}_n - \mu_2\right) \to$ *0 in probability as* $n \to \infty$

Proof. For any given $\epsilon > 0$ we have

$$P\left(\left|\sqrt{n}\left(\overline{X}_n - \mu_1\right)\left(\overline{Y}_n - \mu_2\right)\right| > \epsilon\right) \leq \frac{\sqrt{n}}{\epsilon}E\left|(\overline{X}_n - \mu_1)(\overline{Y}_n - \mu_2)\right|$$
$$\leq \frac{\sqrt{n}}{\epsilon}\sqrt{E(\overline{X}_n - \mu_1)^2 E(\overline{Y}_n - \mu_2)^2}$$
$$= \frac{\sqrt{n}}{\epsilon}\frac{\sigma^2}{n}$$
$$= \frac{\sigma^2}{\epsilon\sqrt{n}} \to 0$$

as $n \to \infty$, where the first inequality follows from Chebychev's inequality and the second one is from Cauchy-Schwarz inequality. $\square$

Lemma 2.2.4. *As $n \to \infty$ the limit distribution of*

$$\Bigg(\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})-Cov(X,Y)\right],$$
$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2-\sigma^2\right],$$
$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\overline{Y})^2-\sigma^2\right]\Bigg)$$

is the same as

$$\Bigg(\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)-Cov(X,Y)\right],$$
$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_1)^2-\sigma^2\right],$$
$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu_2)^2-\sigma^2\right]\Bigg)$$

Proof. It is easy to see that

$$\begin{aligned}
&\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})\\
&\quad=\sum_{i=1}^{n}\Big[(X_i-\mu_1)-(\overline{X}-\mu_1)\Big]\Big[(Y_i-\mu_2)-(\overline{Y}-\mu_2)\Big]\\
&\quad=\sum_{i=1}^{n}\Big[(X_i-\mu_1)(Y_i-\mu_2)-(X_i-\mu_1)(\overline{Y}-\mu_2)\\
&\qquad-(\overline{X}-\mu_1)(Y_i-\mu_2)+(\overline{X}-\mu_1)(\overline{Y}-\mu_2)\Big]\\
&\quad=\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)-n(\overline{X}-\mu_1)(\overline{Y}-\mu_2).
\end{aligned}\tag{2.39}$$

It yields

$$\begin{aligned}
&\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})-Cov(X,Y)\right]\\
&=\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)-Cov(X,Y)-(\overline{X}-\mu_1)(\overline{Y}-\mu_2)\right]\\
&=\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)-Cov(X,Y)\right]-\sqrt{n}(\overline{X}-\mu_1)(\overline{Y}-\mu_2)
\end{aligned}$$

In the same manner we can obtain

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2-\sigma^2\right]=\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_1)^2-\sigma^2\right]-\sqrt{n}(\overline{X}-\mu_1)^2$$

and

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\overline{Y})^2-\sigma^2\right]=\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu_2)^2-\sigma^2\right]-\sqrt{n}(\overline{Y}-\mu_2)^2$$

Using the Results 2.2.1 - 2.2.3 obtained above, we can conclude that the desired result is true. □

Lemma 2.2.5. *The limiting distribution of*

$$\sqrt{n}\big(\hat{\rho}_n-\rho\big)$$

is the same as

$$\sqrt{n}\big(\widetilde{\rho}_n-\rho\big)$$

where

$$\widetilde{\rho}_n=\frac{2\sum_{i=1}^{n}(X_1-\mu_1)(Y_1-\mu_2)}{\sum_{i=1}^{n}(X_i-\mu_1)^2+\sum_{i=1}^{n}(Y_i-\mu_2)^2} \tag{2.40}$$

Proof. Obviously

$$\begin{aligned}
&\frac{\sqrt{n}}{2}\big(\hat{\rho}_n-\widetilde{\rho}_n\big)\\
&=\frac{1}{2}\sqrt{n}\left(\frac{2\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2}-\frac{2\sum_{i=1}^{n}(x_1-\mu_1)(y_1-\mu_2)}{\sum_{i=1}^{n}(x_i-\mu_1)^2+\sum_{i=1}^{n}(y_i-\mu_2)^2}\right)\\
&=\frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2}-\frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\mu_1)^2+\sum_{i=1}^{n}(Y_i-\mu_2)^2}
\end{aligned}$$

adding and subtracting

$$\sqrt{n}\frac{\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2},$$

then we get

$$\begin{aligned}
&\frac{\sqrt{n}}{2}\big(\hat{\rho}_n - \widetilde{\rho}_n\big)\\
&= \frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2} - \frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\mu_1)^2+\sum_{i=1}^{n}(Y_i-\mu_2)^2}\\
&\quad \sqrt{n}\frac{\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2} - \sqrt{n}\frac{\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2}
\end{aligned}$$

grouping them

$$\begin{aligned}
&\frac{\sqrt{n}}{2}\big(\hat{\rho}_n - \widetilde{\rho}_n\big)\\
&=\Big[g\frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2} - \frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2}\Big]\\
&\quad+\Big[\frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2} - \frac{\sqrt{n}\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\mu_1)^2+\sum_{i=1}^{n}(Y_i-\mu_2)^2}\Big]
\end{aligned}$$

and factoring, we have

$$\begin{aligned}
&\frac{\sqrt{n}}{2}\big(\hat{\rho}_n - \widetilde{\rho}_n\big)\\
&= \sqrt{n}\left[\frac{\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})-\sum_{i=1}^{n}(X_i-\mu_1)(Y_i-\mu_2)}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2}\right]\\
&\quad+\sqrt{n}\sum_{i=1}^{n}\Big[(X_i-\mu_1)(Y_i-\mu_2)\Big]\\
&\quad\left[\frac{1}{\sum_{i=1}^{n}(X_i-\overline{X})^2+\sum_{i=1}^{n}(Y_i-\overline{Y})^2} - \frac{1}{\sum_{i=1}^{n}(x_i-\mu_1)^2+\sum_{i=1}^{n}(y_i-\mu_2)^2}\right]
\end{aligned}$$

thus, applying the equality proved above in Equation 2.39

$$\begin{aligned}
&\frac{\sqrt{n}}{2}\left(\hat{\rho}_n - \widetilde{\rho}_n\right) \\
&= \frac{-n\sqrt{n}(\overline{X}-\mu_1)(\overline{Y}-\mu_2)}{\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{Y})^2} \\
&\quad - \frac{\sqrt{n}\sum_{i=1}^n (X_i-\mu_1)(Y_i-\mu_2)}{\left[\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{Y})^2\right]\left[\sum_{i=1}^n (X_i-\mu_1)^2 + \sum_{i=1}^n (Y_i-\mu_2)^2\right]} \\
&\quad \left\{\left[\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{Y})^2\right] - \left[\sum_{i=1}^n (X_i-\mu_1)^2 + \sum_{i=1}^n (Y_i-\mu_2)^2\right]\right\} \\
&= \frac{-n\sqrt{n}(\overline{X}-\mu_1)(\overline{Y}-\mu_2)}{\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{y})^2} \\
&\quad + \frac{\sqrt{n}\sum_{i=1}^n (X_i-\mu_1)(Y_i-\mu_2)}{\left[\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{Y})^2\right]\left[\sum_{i=1}^n (X_i-\mu_1)^2 + \sum_{i=1}^n (Y_i-\mu_2)^2\right]} \\
&\quad \left[\sqrt{n}\left(\overline{X}-\mu_1\right)^2 + \sqrt{n}\left(\overline{Y}-\mu_2\right)^2\right]
\end{aligned} \tag{2.41}$$

Note that as $n \to \infty$

$$\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}) = \frac{1}{n}\sum_{i=1}^n (X_i-\mu_1)^2 - (\overline{X}-\mu_1)^2 \to \sigma^2$$

with probability one, and likewise

$$\frac{1}{n}\sum_{i=1}^n (Y_i - \overline{Y}) = \frac{1}{n}\sum_{i=1}^n (Y_i-\mu_1)^2 - (\overline{Y}-\mu_1)^2 \to \sigma^2$$

with probability one.

Thus the first term in (2.41)

$$\frac{-n\sqrt{n}(\overline{X}-\mu_1)(\overline{Y}-\mu_2)}{\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{Y})^2} = -\frac{\sqrt{n}(\overline{X}-\mu_1)(\overline{Y}-\mu_2)}{\frac{\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{i=1}^n (Y_i-\overline{Y})^2}{n}} \to 0$$

in probability by Result 2.2.3.

In the same way, it can be shown that as $n \to \infty$, the second term in (2.41) converges to zero in probability.

Summarizing the above, we have shown that $\sqrt{n}\big(\rho_n - \widetilde{\rho}_n\big) \to 0$ in probability as $n \to \infty$. This further implies that the limiting distribution of $\sqrt{n}\big(\hat{\rho}_n - \rho\big)$ is the same as that of

$$\sqrt{n}\big(\widetilde{\rho}_n - \rho\big).$$

$\square$

Lemma 2.2.6. *As $n \to \infty$ it is true that*

$$\Big(\sqrt{n}(\xi_n - \rho\sigma^2), \sqrt{n}(\eta_n - \sigma^2), \sqrt{n}(\zeta_n - \sigma^2)\Big) \to N(0, \Sigma)$$

in distribution, where

$$\Sigma = \sigma^4 \begin{pmatrix} 1+\rho^2 & 2\rho & 2\rho \\ 2\rho & 2 & 2\rho^2 \\ 2\rho & 2\rho^2 & 2 \end{pmatrix}$$

and ξ_n, η_n and ζ_n are three random variables define as follows:

$$\xi_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_1)(Y_i - \mu_2)$$
$$\eta_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_1)^2$$
$$\zeta_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_2)^2$$

Proof. The Multivariate Central Limit Theorem implies

$$(\sqrt{n}(\xi_n - \rho\sigma^2), \sqrt{n}(\eta_n - \sigma^2), \sqrt{n}(\zeta_n - \sigma^2)) \to N(0, \Sigma)$$

in distribution where Σ is the covariance matrix of the random vector

$$\big((X_i - \mu_1)(Y_i - \mu_2), (X_i - \mu_1)^2, (Y_i - \mu_2)^2\big).$$

Therefore it is sufficient to obtain the matrix $\Sigma = \Big(\sigma_{ij}\Big)_{1\le i,j\le 3}$.

It can be shown that

$$\sigma_{11} = (1+\rho^2)\sigma^4 \quad \sigma_{12} = 2\rho^2\sigma^4 \quad \sigma_{13} = 2\rho^2\sigma^4$$
$$\sigma_{22} = 2\sigma^4 \quad \sigma_{23} = 2\rho^2\sigma^4$$
$$\sigma_{33} = 2\sigma^4$$

and so,

$$\Sigma = \sigma^4 \begin{pmatrix} 1+\rho^2 & 2\rho & 2\rho \\ 2\rho & 2 & 2\rho^2 \\ 2\rho & 2\rho^2 & 2 \end{pmatrix}$$

□

Theorem 2.2.7. *As $n \to \infty$, it holds that*

$$\sqrt{n}(\hat{\rho}_n - \rho) \to N(0, \gamma^2)$$

in distribution where $\gamma^2 = (1-\rho^2)^2$.

Proof. According to Lemma 2.2.5, it suffices to show that as $n \to \infty$

$$\sqrt{n}\big(\widetilde{\rho}_n - \rho\big) \to N\big(0, \gamma^2\big)$$

in distribution.

Clearly $\widetilde{\rho}_n$ can be expressed in terms of ξ_n, η_n, ζ_n as

$$\widetilde{\rho}_n = \frac{2\sum_{i=1}^n (x_i - \mu_1)(y_i - \mu_2)}{\sum_{i=1}^n (x_i - \mu_1)^2 + \sum_{i=1}^n (y_i - \mu_1)} = \frac{2\xi_n}{\eta_n + \zeta_n}$$

Define the function $g(u, v, w)$ by

$$g(u, v, w) = \frac{2u}{v + w}$$

then obviously

$$\widetilde{\rho}_n = g(u, v, w)$$

It is easy to see that

$$\left(\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial g}{\partial w}\right) = \left(\frac{2}{v+w}, \frac{-2u}{(v+w)^2}, \frac{-2u}{(v+w)^2}\right)$$

Evaluating it at $u = \rho\sigma^2,\ v = \sigma^2,\ w = \sigma^2$ we have

$$\begin{aligned}\Big(\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial g}{\partial w}\Big)_{(\rho\sigma^2,\sigma^2,\sigma^2)} &= \Big(\frac{1}{\sigma^2}, \frac{-\rho}{2\sigma^2}, \frac{-\rho}{2\sigma^2}\Big) \\ &= \frac{1}{\sigma^2}\Big(1, -\frac{\rho}{2}, -\frac{\rho}{2}\Big)\end{aligned}$$

By the delta method it follows that

$$\sqrt{n}\Big(g\big(\xi_n, \eta_n, \zeta_n\big) - g\big(\rho\sigma^2, \sigma^2, \sigma^2\big)\Big) \to N\big(0, \gamma^2\big)$$

in distribution where

$$\begin{aligned}\gamma^2 &= \frac{1}{\sigma^4} \times \Big(1, -\frac{\rho}{2}, -\frac{\rho}{2}\Big)\Sigma \begin{pmatrix} 1 \\ -\frac{\rho}{2} \\ -\frac{\rho}{2} \end{pmatrix} \\ &= \Big(1, -\frac{\rho}{2}, -\frac{\rho}{2}\Big) \begin{pmatrix} 1+\rho^2 & 2\rho & 2\rho \\ & 2 & 2\rho^2 \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\rho}{2} \\ -\frac{\rho}{2} \end{pmatrix} \\ &= (1-\rho^2)^2.\end{aligned}$$

Therefore, as $n \to \infty$

$$\sqrt{n}\big(\widetilde{\rho}_n - \rho\big) \to N\big(0, \gamma^2\big)$$

and consequently

$$\sqrt{n}\big(\hat{\rho}_n - \rho\big) \to N\big(0, \gamma^2\big)$$

in distribution $\square$

The result of Theorem 2.2.7 can be expressed as

$$\hat{\rho}_n \sim N\Big(\rho, \frac{(1-\rho^2)^2}{\sqrt{n}}\Big)$$

Notice that the variance of the distribution involves the parameter ρ. How can we find a distribution whose variance is parameter - free? In the next section we address this problem.

2.2.7 Variance Stabilizing transformation: The limiting distribution of the z- transformation of $\hat{\rho}$

The Variance Stabilizing Transformation is used when the constant variance assumption is violated; that is, when $Var(W)$, where W is a random variable, is a function of $E[W] = \mu$. In particular, the transformation assumes that $Var(W) = g^2(\mu)$, where the function $g(\cdot)$ is known. The idea of it is to find a (differentiable) function $h(\cdot)$ so that the transformed random variable $h(W)$ will have a variance which is approximately independent of its mean $E[h(W)]$.

The delta method yields the following approximation:

$$Var(h(W)) \approx [h'(\mu)]^2 g^2(\mu) \tag{2.42}$$

Then, the desired function $h(w)$ is the solution of the differential equation above. For our case, we need to find a function h such that

$$\sqrt{n}\Big(h(\hat{\rho}_n) - h(\rho)\Big) \sim N\Big(0, 1\Big)$$

therefore, applying the Variance Stabilizing transformation we have

$$(h'(\rho))^2(1-\rho^2)^2 = 1 \tag{2.43}$$

from where, solving for h

$$\begin{aligned}
\int h'(\rho))^2 d\rho &= \int \frac{1}{1-\rho^2} d\rho \\
h(\rho) &= \int \frac{1}{2(1-\rho)} + \frac{1}{2(1+\rho)} d\rho \\
h(\rho) &= -\frac{1}{2}\ln(1-\rho) + \frac{1}{2}\ln(1+\rho) \\
h(\rho) &= \frac{1}{2}\ln\Big(\frac{1+\rho}{1-\rho}\Big) \qquad (2.44)
\end{aligned}$$

Thus, we can conclude that

$$\frac{\sqrt{n}}{2}\left(\ln\Big(\frac{1+\hat{\rho}}{1-\hat{\rho}}\Big)-\ln\Big(\frac{1+\rho}{1-\rho}\Big)\right)\sim N\big(0,1\big) \tag{2.45}$$

The transformation express in Equation (2.44) is commonly called the "Fisher Transformation".

CHAPTER 3

TRIVARIATE NORMAL DISTRIBUTION

In this chapter we introduce the Trivariate Normal Distribution and describe its probability density function. In particular, making the assumptions of common variances and common correlation coefficient, we derive its Maximum Likelihood Estimators and focus our study on the estimation of the correlation coefficient ρ. Chapter 3 is developed as follows. In Section 3.1 we introduce the Trivariate Normal Distribution and its probability density function. In Section 3.2 we describe our particular case of the Covariance Matrix Symmetric Trivariate Normal Distribution, its probability density function in Section 3.2.1 and derive its estimators in Section 3.2.3 using the method of the Maximum Likelihood.

3.1 General Discussion

A three dimensional random variable (Y_1, Y_2, Y_3) is said to have a Trivariate Normal Distribution if the joint probability distribution function can be represented in matrix form as

$$f(\boldsymbol{y}) = \frac{1}{(2\pi)^{\frac{3}{2}} \left|\Sigma\right|^{\frac{1}{2}}} \exp\left\{ -\frac{(\boldsymbol{y}-\boldsymbol{u})^T \Sigma^{-1} (\boldsymbol{y}-\boldsymbol{u})}{2} \right\} \tag{3.1}$$

where $\boldsymbol{y} = (y_1, y_2, y_3)'$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)'$, Σ is a 3×3, positive definite and symmetric variance-covariance matrix of the form

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \tag{3.2}$$

and $|\Sigma| > 0$ is the determinant of Σ.

The density function of a Trivariate Normal Distribution can be written also as

$$f(y_1, y_2, y_3) = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2\sigma_{33} - \sigma_{13}^2\sigma_{22} - \sigma_{23}^2\sigma_{11}}}$$
$$\exp\left\{ - \frac{1}{2(\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2\sigma_{33} - \sigma_{13}^2\sigma_{22} - \sigma_{23}^2\sigma_{11})} W(y_1, y_2, y_3)\right\} \quad (3.3)$$

where

$$\begin{aligned} W(y_1, y_2, y_3) = (y_1 - \mu_1)^2(\sigma_{22}\sigma_{33} - \sigma_{23}^2) + (y_2 - \mu_2)^2(\sigma_{11}\sigma_{33} - \sigma_{13}^2) \\ +(y_3 - \mu_3)^2(\sigma_{11}\sigma_{22} - \sigma_{12}^2) + 2[(y_1 - \mu_1)(y_2 - \mu_2) \\ (\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_{33}) + (y_1 - \mu_1)(y_2 - \mu_2)(\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_{33}) \\ +(y_1 - \mu_1)(y_2 - \mu_2)(\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_{33})] \end{aligned} \quad (3.4)$$

and provided that $\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2\sigma_{33} - \sigma_{13}^2\sigma_{22} - \sigma_{23}^2\sigma_{11} \neq 0$.

3.2 Covariance Matrix Symmetric

We focus our study on a special form of the variance-covariance matrix Σ, a symmetric matrix where all variances are assumed to be equal and correlation coefficients too. It will be discussed below.

3.2.1 Model

For our study let us consider the case of equal variances, i.e. $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma^2$ and equal correlation coefficient for each pair of variables, that is $\rho_{12} = \rho_{13} = \rho_{23} = \rho$, thus the variance-covariance matrix Σ has the form

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma^2\rho & \sigma^2\rho \\ \sigma^2\rho & \sigma^2 & \sigma^2\rho \\ \sigma^2\rho & \sigma^2\rho & \sigma^2 \end{pmatrix} = \sigma^2 \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix} \quad (3.5)$$

It can be shown that $|\Sigma| = (\sigma^2)^3(1 + 2\rho^3 - 3\rho^2)$ and thus

$$\Sigma^{-1} = \frac{1}{\sigma^2(2\rho^3 - 3\rho^2 + 1)} \begin{pmatrix} 1-\rho^2 & \rho^2-\rho & \rho^2-\rho \\ \rho^2-\rho & 1-\rho^2 & \rho^2-\rho \\ \rho^2-\rho & \rho^2-\rho & 1-\rho^2 \end{pmatrix} \tag{3.6}$$

provided that $|\Sigma| = (\sigma^2)^3(2\rho^3 - 3\rho^2 + 1) = (\sigma^2)^3(2\rho + 1)(\rho - 1)^2 \neq 0$.

Therefore, the probability density function for this Trivaritate Normal Distribution is written as

$$f(y_1, y_2, y_3) = \frac{1}{(2\pi)^{\frac{3}{2}}(\sigma^2)^{\frac{3}{2}}\sqrt{2\rho^3 - 3\rho^2 + 1}} \exp\left\{ - \frac{w^*}{2\sigma^2(2\rho^2 - \rho - 1)} \right\} \tag{3.7}$$

where

$$\begin{aligned} w^* = 2\rho[(y_1 - \mu_1)(y_2 - \mu_2) + (y_1 - \mu_1)(y_3 - \mu_3) + (y_2 - \mu_2)(y_3 - \mu_3)] \\ -(1 + \rho)[(y_1 - \mu_1)^2 + (y_2 - \mu_2)^2 + (y_3 - \mu_3)^2] \end{aligned} \tag{3.8}$$

and $-\frac{1}{2} < \rho < 1$.

3.2.2 Likelihood Function

To obtain the likelihood function of the Covariance matrix Symmetric Trivariate Normal Distribution let us take a random sample of n 3×1-vectors $\boldsymbol{y}_1$, $\boldsymbol{y}_2$,. . .,$\boldsymbol{y}_n$, where $\boldsymbol{y}_k = (y_{k1}, y_{k2}, y_{k3})$ for $1 \leq k \leq n$.

Thus, the likelihood function $L(\boldsymbol{y}; \boldsymbol{\mu}, \sigma^2, \rho)$ is

$$L(\boldsymbol{\mu}, \sigma^2, \rho; \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n) = \prod_{i=1}^{n} f(\boldsymbol{y}_i; \boldsymbol{\mu}, \sigma^2, \rho) \tag{3.9}$$

$$= \prod_{i=1}^{n} \frac{1}{(2\pi)^{\frac{3}{2}}(\sigma^2)^{\frac{3}{2}}\sqrt{2\rho^3 - 3\rho^2 + 1}} \exp\left\{ \frac{-w_i^*}{2\sigma^2(2\rho^2 - \rho - 1)} \right\} \tag{3.10}$$

where

$$
\begin{aligned}
w_i^* =& 2\rho[(y_{i1}-\mu_1)(y_{i2}-\mu_2)+(y_{i1}-\mu_1)(y_{i3}-\mu_3)+(y_{i2}-\mu_2)(y_{i3}-\mu_3)] \\
&-(1+\rho)[(y_{i1}-\mu_1)^2+(y_{i2}-\mu_2)^2+(y_{i3}-\mu_3)^2]
\end{aligned}
\tag{3.11}
$$

Taking $\tau = \sigma^2$, the likelihood function $L(\boldsymbol{\mu}, \sigma^2, \rho; \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n)$ is

$$
L(\boldsymbol{\mu}, \sigma^2, \rho; \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n) = \frac{1}{(2\pi)^{\frac{3n}{2}}\tau^{\frac{3n}{2}}(2\rho^3-3\rho^2+1)^{\frac{n}{2}}} \exp\left\{\frac{-w^{**}}{2\tau(2\rho^2-\rho-1)}\right\}
\tag{3.12}
$$

where $\mu = (\mu_1, \mu_2, \mu_3)$ and

$$
\begin{aligned}
w^{**} =& 2\rho\sum_{i=1}^{n}[(y_{i1}-\mu_1)(y_{i2}-\mu_2)+(y_{i1}-\mu_1)(y_{i3}-\mu_3)+(y_{i2}-\mu_2)(y_{i3}-\mu_3)] \\
&-(1+\rho)\sum_{i=1}^{n}[(y_{i1}-\mu_1)^2+(y_{i2}-\mu_2)^2+(y_{i3}-\mu_3)^2]
\end{aligned}
\tag{3.13}
$$

and the natural logarithm of the same function

$$
\begin{aligned}
\ln L(\boldsymbol{\mu}, \sigma^2, \rho; \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n) =& \frac{(1+\rho)\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{3}(y_{ij}-\mu_j)^2}{2\tau(2\rho^2-\rho-1)} \\
&-\frac{2\rho\sum\limits_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\mu_j)(y_{ik}-\mu_k)}{2\tau(2\rho^2-\rho-1)} \\
&-\ln(2\pi)^{\frac{3n}{2}}-\frac{3n}{2}\ln\tau-\frac{n}{2}\ln(2\rho^3-3\rho^2+1)
\end{aligned}
\tag{3.14}
$$

3.2.3 Derivation of Maximum Likelihood Estimators

Since $\mu_1, \mu_2, \mu_3, \tau$ and ρ are unknown, we want to find the MLE's for these parameters, solving the system of equations

$$
\frac{\partial \ln L}{\partial \mu_i} = 0,\ i = 1, 2, 3; \quad \frac{\partial \ln L}{\partial \tau} = 0,\ \frac{\partial \ln L}{\partial \rho} = 0
$$

that is:

$$\frac{\partial \ln L}{\partial \mu_1} = \frac{-2(1+\rho)\sum_{i=1}^{n}(y_{i1}-\mu_1) + 2\rho\sum_{i=1}^{n}[(y_{i2}-\mu_2)+(y_{i3}-\mu_3)]}{2\tau(2\rho^2-\rho-1)} = 0 \tag{3.15}$$

$$\frac{\partial \ln L}{\partial \mu_2} = \frac{-2(1+\rho)\sum_{i=1}^{n}(y_{i2}-\mu_2) + 2\rho\sum_{i=1}^{n}[(y_{i1}-\mu_1)+(y_{i3}-\mu_3)]}{2\tau(2\rho^2-\rho-1)} = 0 \tag{3.16}$$

$$\frac{\partial \ln L}{\partial \mu_3} = \frac{-2(1+\rho)\sum_{i=1}^{n}(y_{i3}-\mu_3) + 2\rho\sum_{i=1}^{n}[(y_{i1}-\mu_1)+(y_{i2}-\mu_2)]}{2\tau(2\rho^2-\rho-1)} = 0 \tag{3.17}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \tau} =& -\frac{(1+\rho)\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\mu_j)^2 - 2\rho\sum_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\mu_j)(y_{ik}-\mu_k)}{2\tau^2(2\rho^2-\rho-1)} \\ &- \frac{3n}{2\tau} = 0\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \rho} =& (2\rho^2-\rho-1)\psi\Big[\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\mu_j)^2 - 2\sum_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\mu_j)(y_{ik}-\mu_k)\Big] \\ &- (4\rho-1)\psi\Big[(1+\rho)\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\mu_j)^2 - 2\rho\sum_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\mu_j)(y_{ik}-\mu_k)\Big] \\ &- \frac{n(6\rho^2-6\rho)}{2(2\rho^3-3\rho^2+1)} = 0\end{aligned} \tag{3.19}$$

where

$$\psi = \frac{2\tau}{4\tau^2(2\rho^2-\rho-1)^2}$$

In order to solve (3.15) - (3.18) for μ_1, μ_2 and μ_3 we begin by multiplying equations (3.15),(3.16) and (3.17) by $-\tau(2\rho^2-\rho-1)$ and simplifying, we get

$$(1+\rho)\sum_{i=1}^{n}(y_{i1}-\mu_1) - \rho\sum_{i=1}^{n}[(y_{i2}-\mu_2)+(y_{i3}-\mu_3)] = 0 \tag{3.20}$$

$$(1+\rho)\sum_{i=1}^{n}(y_{i2}-\mu_2) - \rho\sum_{i=1}^{n}[(y_{i1}-\mu_1)+(y_{i3}-\mu_3)] = 0 \tag{3.21}$$

$$(1+\rho)\sum_{i=1}^{n}(y_{i3}-\mu_3) - \rho\sum_{i=1}^{n}[(y_{i1}-\mu_1)+(y_{i2}-\mu_2)] = 0 \tag{3.22}$$

Now, adding equation (3.21) to equation (3.22) and simplifying the sum we get

$$\sum_{i=1}^{n}[(y_{i2} - \mu_2) + (y_{i3} - \mu_3)]) - 2\rho \sum_{i=1}^{n}(y_{i1} - \mu_1) = 0 \tag{3.23}$$

Multiplying (3.23) by ρ, adding it to equation (3.20) the we obtain

$$(-2\rho^2 + \rho + 1) \sum_{i=1}^{n}(y_{1i} - \mu_1) = 0$$

$$(2\rho + 1)(\rho - 1) \sum_{i=1}^{n}(y_{1i} - \mu_1) = 0$$

recall that $\rho \neq 1$ and $\rho \neq -\frac{1}{2}$, therefore it follows that

$$\sum_{i=1}^{n}(y_{i1} - \mu_1) = 0$$

which implies that the MLE $\hat{\mu}_1$ for μ_1 is

$$\hat{\mu}_1 = \frac{\sum\limits_{i=1}^{n} y_{i1}}{n} = \overline{y}_1 \tag{3.24}$$

In similar way, we can obtain

$$\hat{\mu}_2 = \frac{\sum\limits_{i=1}^{n} y_{i2}}{n} = \overline{y}_2 \tag{3.25}$$

and

$$\hat{\mu}_3 = \frac{\sum\limits_{i=1}^{n} y_{i3}}{n} = \overline{y}_3 \tag{3.26}$$

Now, in order to find the estimators for ρ and τ, let us simplify equations (3.18) and equation (3.19) and substitute the estimators of μ_1, μ_2, and μ_3 found in equations (3.24),(3.25) and (3.26):

Multiplying equation (3.18) by $-2\tau^2(2\rho^2 - \rho - 1)$ we obtain

$$(1+\rho)\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij} - \overline{y}_j)^2 - 2\rho \sum_{i=1}^{n}\sum_{\substack{j,k=1 \\ j\neq k}}^{3}(y_{ij} - \overline{y}_j)(y_{ik} - \overline{y}_k)$$
$$+3n\tau(2\rho^2 - \rho - 1) = 0 \tag{3.27}$$

and multiplying equation (3.19) by $2\tau(2\rho^2-\rho-1)^2$ and symplifying it we get

$$(2\rho^2-\rho-1)\Big[\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2-2\sum_{i=1}^{n}\sum_{\substack{j,k=1\\j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)\Big]$$
$$-(4\rho-1)\Big[(1+\rho)\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2-2\rho\sum_{i=1}^{n}\sum_{\substack{j,k=1\\j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)\Big]$$
$$-6n\rho\tau(2\rho^2-\rho-1)=0. \quad (3.28)$$

Grouping commom terms, the equation (3.28) will become

$$\big[(2\rho^2-\rho-1)-(4\rho-1)(1+\rho)\big]\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2$$
$$+\big[-2(2\rho^2-\rho-1)+2\rho(4\rho-1)\big]\sum_{i=1}^{n}\sum_{\substack{j,k=1\\j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)$$
$$-6n\rho\tau(2\rho^2-\rho-1)=0 \qquad (3.29)$$

and finally combining like terms we get

$$-2\rho(\rho+2)\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2+2(2\rho^2+1)\sum_{i=1}^{n}\sum_{\substack{j,k=1\\j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)$$
$$-6n\rho\tau(2\rho^2-\rho-1)=0 \qquad (3.30)$$

Now, adding equations (3.30) and the product of equation (3.27) by 2ρ, we obtain

$$-2\rho\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\mu_j)^2+2\sum_{i=1}^{n}\sum_{\substack{j,k=1\\j\neq k}}^{3}(y_{ij}-\mu_j)(y_{ik}-\mu_k)=0 \qquad (3.31)$$

and solving it for ρ we get

$$\hat{\rho}=\frac{\sum\limits_{i=1}^{n}\sum\limits_{\substack{j,k=1\\j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)}{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{3}(y_{ij}-\overline{y}_j)^2} \qquad (3.32)$$

which is equivalent to

$$\hat{\rho} = \frac{\sum_{i=1}^{n}\left[(y_{i1}-\overline{y}_1)(y_{i2}-\overline{y}_2)+(y_{i1}-\overline{y}_1)(y_{i3}-\overline{y}_3)+(y_{i2}-\overline{y}_2)(y_{i3}-\overline{y}_3)\right]}{\sum_{i=1}^{n}\left[(y_{i1}-\overline{y}_1)^2+(y_{i2}-\overline{y}_2)^2+(y_{i3}-\overline{y}_3)^2\right]} \tag{3.33}$$

Replacing ρ in (3.27) by $\hat{\rho}$ in(3.33) and solving (**??**) for τ, we obtain the MLE $\hat{\tau}$ for τ

$$\hat{\tau} = \frac{\sum_{i=1}^{n}\left[(y_{i1}-\overline{y}_1)^2+(y_{i2}-\overline{y}_2)^2+(y_{i3}-\overline{y}_3)^2\right]}{3n} \tag{3.34}$$

and thus $\hat{\sigma}$ is

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}\left[(y_{i1}-\overline{y}_1)^2+(y_{i2}-\overline{y}_2)^2+(y_{i3}-\overline{y}_3)^2\right]}{3n}} \tag{3.35}$$

3.2.4 Properties of the MLE of ρ

Theorem 3.2.1. *It holds that $-\frac{1}{2} < \hat{\rho} < 1$ with probability one.*

Proof.

$$\begin{aligned}
\hat{\rho} \geq -\frac{1}{2} &\Leftrightarrow \frac{\sum_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)}{\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2} \geq -\frac{1}{2} \\
&\Leftrightarrow 2\sum_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k) \geq -\sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2 \\
&\Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{3}(y_{ij}-\overline{y}_j)^2 + 2\sum_{i=1}^{n}\sum_{\substack{j,k=1\\ j\neq k}}^{3}(y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k) \geq 0
\end{aligned}$$

$$\Leftrightarrow \sum_{i=1}^{n} \big[(y_{i1}-\overline{y}_1)^2+(y_{i2}-\overline{y}_2)^2+(y_{i3}-\overline{y}_3)^2+2(y_{i1}-\overline{y}_1)(y_{i2}-\overline{y}_2)$$

$$+\,2(y_{i1}-\overline{y}_1)(y_{i3}-\overline{y}_3)+2(y_{i2}-\overline{y}_2)(y_{i3}-\overline{y}_3)\big] \geq 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \Big((y_{i1}-\overline{y}_1)+(y_{i2}-\overline{y}_2)+(y_{i3}-\overline{y}_3)\Big)^2 \geq 0$$

which is true for all $y_{ij}, i = 1, \ldots, n; j = 1, 2, 3$.

Now, let us prove that $\hat{\rho} \leq 1$:

$$\hat{\rho} \leq 1 \Leftrightarrow \frac{\sum\limits_{i=1}^{n} \sum\limits_{\substack{j,k=1\\ j\neq k}}^{3} (y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k)}{\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{3} (y_{ij}-\overline{y}_j)^2} \leq 1$$

$$\Leftrightarrow \sum_{i=1}^{n} \sum_{\substack{j,k=1\\ j\neq k}}^{3} (y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k) \leq \sum_{i=1}^{n} \sum_{j=1}^{3} (y_{ij}-\overline{y}_j)^2$$

$$\Leftrightarrow \sum_{i=1}^{n} \sum_{j=1}^{3} (y_{ij}-\overline{y}_j)^2 - \sum_{i=1}^{n} \sum_{\substack{j,k=1\\ j\neq k}}^{3} (y_{ij}-\overline{y}_j)(y_{ik}-\overline{y}_k) \geq 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \big[2(y_{i1}-\overline{y}_1)^2+2(y_{i2}-\overline{y}_2)^2+2(y_{i3}-\overline{y}_3)^2-2(y_{i1}-\overline{y}_1)(y_{i2}-\overline{y}_2)$$

$$-\,2(y_{i1}-\overline{y}_1)(y_{i3}-\overline{y}_3)-2(y_{i2}-\overline{y}_2)(y_{i3}-\overline{y}_3)\big] \geq 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \Big[\Big((y_{i1}-\overline{y}_1)-(y_{i2}-\overline{y}_2)\Big)^2+\Big((y_{i1}-\overline{y}_1)-(y_{i3}-\overline{y}_3)\Big)^2$$

$$+\,\Big((y_{i2}-\overline{y}_2)-(y_{i3}-\overline{y}_3)\Big)^2\Big] \geq 0$$

which is also true for all $y_{ij}, i = 1, \ldots, n; j = 1, 2, 3$.

Note that $P\Big(\hat{\rho} = -\frac{1}{2}\ or\ 1\Big) = 0$, therefore, we can conclude that $-\frac{1}{2} < \hat{\rho} < 1$ with probability one. $\square$

3.3 Sampling Distribution of $\hat{\rho}$ of Trivariate Normal Distribution

In Section 3.2.3 we derived the maximum likelihood estimator $\hat{\rho}$ of the correlation coefficient ρ. Now we will derive its limiting distribution.

3.3.1 The Limiting Distribution of $\hat{\rho}$

Mi Mie has shown that under the covariance-matrix symmetric Trivariate Normal Distribution the MLE $\hat{\rho}_n$ of the correlation coefficient ρ is asymptotically normal, that is

$$\sqrt{n}\Big(\hat{\rho_n} - \rho\Big) \to N\Big(0, \frac{(1-\rho)^2(1+2\rho)^2}{3}\Big) \tag{3.36}$$

in distribution, but since the variance in the limiting distribution depends on the unknown parameter ρ, we want to find a function $h(\cdot)$ such that the limiting distribution of $\sqrt{n}(h(\hat{\rho_n}) - h(\rho))$ will be parameter-free, that is

$$\sqrt{n}\Big(h(\hat{\rho_n}) - h(\rho)\Big) \to N(0,1) \tag{3.37}$$

For our purpose, we will use the Variance-Stabilizing Transformation in the next section.

3.3.2 Variance-stabilizing transformation: The limiting distribution of the z- transformation of $\hat{\rho}$

Using the Variance - Stabilizing transformation explained in Section 2.2.7, we have to find the function $h(\cdot)$ such that

$$\sqrt{n}\Big(h(\hat{\theta_n}) - h(\theta)\Big) \to N\Big(0, (h'(\theta)^2\tau^2(\theta))\Big)$$

which in our desired case $[h'(\theta)]^2\tau^2(\theta) = 1$ and where $\tau^2(\rho) = \frac{(1-\rho)^2(1+2\rho)^2}{3}$. Solving for $h'(\rho)$ in the previous equation

$$h'(\rho) = \frac{\sqrt{3}}{(1-\rho)(1+2\rho)}$$

and thus

$$\begin{aligned}
h(\rho) &= \int h'(\rho)d\rho \\
&= \sqrt{3}\int \frac{1}{(1-\rho)(1+2\rho)}d\rho \\
&= \sqrt{3}\int \frac{1}{3}\Big(\frac{2}{1+2\rho} + \frac{1}{1-\rho}\Big)d\rho \\
&= \frac{1}{\sqrt{3}}\Big[\ln(1+2\rho) - \ln(1-\rho)\Big] \\
&= \frac{1}{\sqrt{3}}\ln\left(\frac{1+2\rho}{1-\rho}\right)
\end{aligned}$$

Therefore, applying the previous result, we can conclude that

$$\sqrt{\frac{n}{3}}\left(\ln\left(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\right) - \ln\left(\frac{1+2\rho}{1-\rho}\right)\right) \to N(0,1) \tag{3.38}$$

in distribution.

CHAPTER 4

NUMERICAL STUDY ON THE CASE OF BIVARIATE NORMAL DISTRIBUTION

In Chapter 4 a simulation study is conducted to compare the performance of the estimators $\hat{\rho}_n$ and $\hat{r}_n$ of the correlation coefficient in the case of equal variances for the Bivariate Normal Distribution discussed in Chapter 2.

We compare these two point estimators in Section 4.1.1 and the performance of those have been evaluated using the Bias and Mean Square Error (MSE) in Section 4.1.2. Since in Theorem 2.2.7 was discussed the limiting distribution of $\hat{\rho}$ and in Section 2.2.7 was presented a parameter-free distribution for $\hat{\rho}$, a study on the confidence intervals for ρ are made in Section 4.2 and Test of Hypothesis in Section 4.3.

4.1 Comparison between $\hat{r}_n$ and $\hat{\rho}_n$

?? In Section **??** we conduct simulations to compare (a) the Pearson Correlation Coefficient r_n defined in Equation (2.7) and (b) the MLE defined in the Equation (2.33).

Without loss of generality, we assume $\mu^T = (0, 0)$ and $\sigma^2 = 4$. We picked seven different sample sizes (n), with n ranging from 20 to 50 in increments of 5 and the 19 values ρ that ranges from -0.9 to 0.9 in 0.1 increments. The simulations were run 1000 times.

4.1.1 Comparison Point Estimators of the Correlation Coefficient ρ

Figure 4.1 shows the comparison of the true value of ρ and the two estimators considered here: (a) The MLE $\hat{\rho}_n$ and (b) the Pearson Correlation Coefficient r_n. Table 4.1 shows the results of the simulation.

In this figure we can see that the value of the estimators are close to the true value of ρ; however, the Pearson Correlation Coeffient is closer to ρ than the MLE except when $\rho = 0$ where both are the same. Note also that as the sample size increases, the difference between the estimators and ρ decreases.

Table 4.1: Comparison Point Estimators

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.9	-0.8910	-0.8929	-0.8934	-0.8943	-0.8955	-0.8957	-0.8967
	-0.8	-0.7830	-0.7886	-0.7895	-0.7919	-0.7922	-0.7929	-0.7938
	-0.7	-0.6810	-0.6845	-0.6881	-0.6902	-0.6892	-0.6907	-0.6908
	-0.6	-0.5778	-0.5853	-0.5869	-0.5877	-0.5903	-0.5919	-0.5921
	-0.5	-0.4795	-0.4842	-0.4877	-0.4898	-0.4919	-0.4913	-0.4938
	-0.4	-0.3861	-0.3866	-0.3880	-0.3916	-0.3858	-0.3909	-0.3944
	-0.3	-0.2871	-0.2874	-0.2914	-0.2917	-0.2933	-0.2944	-0.2944
	-0.2	-0.1876	-0.1927	-0.1928	-0.1938	-0.1994	-0.1952	-0.1971
	-0.1	-0.0947	-0.0977	-0.0959	-0.0957	-0.0974	-0.0975	-0.0968
	0	0.0019	0.0018	-0.0030	-0.0016	-0.0015	0.0025	-0.0009
	0.1	0.0952	0.0940	0.0957	0.0994	0.0974	0.0992	0.0983
	0.2	0.1913	0.1887	0.1945	0.1955	0.1970	0.1932	0.1964
	0.3	0.2837	0.2879	0.2918	0.2918	0.2948	0.2930	0.2948
	0.4	0.3792	0.3861	0.3894	0.3915	0.3909	0.3931	0.3942
	0.5	0.4793	0.4847	0.4878	0.4875	0.4909	0.4925	0.4935
	0.6	0.5828	0.5821	0.5870	0.5900	0.5905	0.5906	0.5917
	0.7	0.6813	0.6859	0.6866	0.6881	0.6903	0.6926	0.6929
	0.8	0.7846	0.7879	0.7896	0.7915	0.7922	0.7936	0.7941
	0.9	0.8902	0.8923	0.8935	0.8948	0.8957	0.8956	0.8964
(b)	-0.9	-0.8959	-0.8967	-0.8965	-0.8969	-0.8978	-0.8977	-0.8984
	-0.8	-0.7908	-0.7946	-0.7946	-0.7963	-0.7959	-0.7963	-0.7968
	-0.7	-0.6904	-0.6920	-0.6943	-0.6955	-0.6939	-0.6947	-0.6944
	-0.6	-0.5875	-0.5930	-0.5935	-0.5934	-0.5952	-0.5961	-0.5960
	-0.5	-0.4889	-0.4918	-0.4940	-0.4951	-0.4966	-0.4956	-0.4976
	-0.4	-0.3944	-0.3931	-0.3938	-0.3966	-0.3900	-0.3946	-0.3977
	-0.3	-0.2937	-0.2928	-0.2959	-0.2956	-0.2966	-0.2975	-0.2970
	-0.2	-0.1923	-0.1965	-0.1959	-0.1966	-0.2018	-0.1973	-0.1989
	-0.1	-0.0970	-0.0997	-0.0974	-0.0971	-0.0987	-0.0986	-0.0977
	0	0.0021	0.0019	-0.0030	-0.0016	-0.0015	0.0025	-0.0010
	0.1	0.0976	0.0960	0.0973	0.1008	0.0987	0.1003	0.0993
	0.2	0.1960	0.1924	0.1977	0.1982	0.1994	0.1953	0.1983
	0.3	0.2905	0.2933	0.2963	0.2956	0.2982	0.2961	0.2975
	0.4	0.3871	0.3927	0.3950	0.3964	0.3951	0.3968	0.3975
	0.5	0.4885	0.4923	0.4941	0.4929	0.4956	0.4968	0.4973
	0.6	0.5927	0.5901	0.5937	0.5956	0.5954	0.5948	0.5956
	0.7	0.6906	0.6932	0.6928	0.6933	0.6949	0.6967	0.6966
	0.8	0.7923	0.7940	0.7946	0.7957	0.7959	0.7970	0.7972
	0.9	0.8950	0.8961	0.8966	0.8974	0.8979	0.8976	0.8982

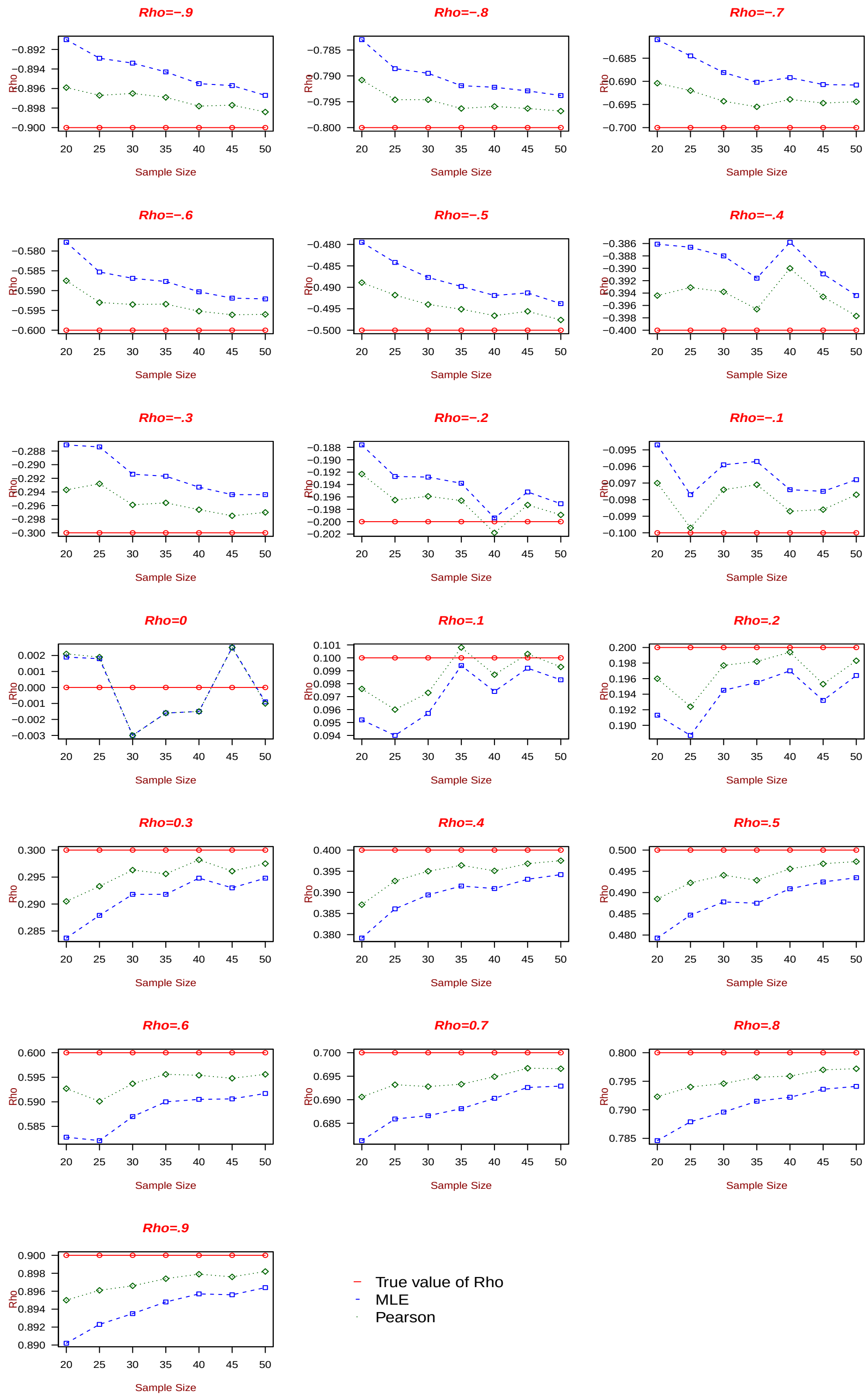

Figure 4.1: Comparison Point Estimators of Rho

4.1.2 Study on the Performance of the Estimators

In order to compare the MLE and Pearson Correlation Coefficient, we evaluate two properties of the estimators: The Bias and the Mean Square Error.

In Table 4.2 and in Figure 4.2, the bias of both estimators are shown. The Pearson Correlation Coefficient always has smaller bias than the MLE except when $\rho = 0$. In Figure 4.3 the absolute value of the Bias can be observed. From this Figure we can see that the absolute value of the bias decreases as the sample size increases.

Since the Bias is not enough to compare the performance of both estimators, the MSE of the estimators were found. Table 4.3 describes those values and are represented in Figure 4.4. It can be observed that there is not a noticeable difference on the MSE of the MLE compared to the Pearson Correlation Coefficient.

Table 4.2: Comparison Bias Estimators

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.9	-0.0090	-0.0071	-0.0066	-0.0057	-0.0045	-0.0043	-0.0033
	-0.8	-0.0170	-0.0114	-0.0105	-0.0081	-0.0078	-0.0071	-0.0062
	-0.7	-0.0190	-0.0155	-0.0119	-0.0098	-0.0108	-0.0093	-0.0092
	-0.6	-0.0222	-0.0147	-0.0131	-0.0123	-0.0097	-0.0081	-0.0079
	-0.5	-0.0205	-0.0158	-0.0123	-0.0102	-0.0081	-0.0087	-0.0062
	-0.4	-0.0139	-0.0134	-0.0120	-0.0084	-0.0142	-0.0091	-0.0056
	-0.3	-0.0129	-0.0126	-0.0086	-0.0083	-0.0067	-0.0056	-0.0056
	-0.2	-0.0124	-0.0073	-0.0072	-0.0062	-0.0006	-0.0048	-0.0029
	-0.1	-0.0053	-0.0023	-0.0041	-0.0043	-0.0026	-0.0025	-0.0032
	0	-0.0019	-0.0018	0.0030	0.0016	0.0015	-0.0025	0.0009
	0.1	0.0048	0.0060	0.0043	0.0006	0.0026	0.0008	0.0017
	0.2	0.0087	0.0113	0.0055	0.0045	0.0030	0.0068	0.0036
	0.3	0.0163	0.0121	0.0082	0.0082	0.0052	0.0070	0.0052
	0.4	0.0208	0.0139	0.0106	0.0085	0.0091	0.0069	0.0058
	0.5	0.0207	0.0153	0.0122	0.0125	0.0091	0.0075	0.0065
	0.6	0.0172	0.0179	0.0130	0.0100	0.0095	0.0094	0.0083
	0.7	0.0187	0.0141	0.0134	0.0119	0.0097	0.0074	0.0071
	0.8	0.0154	0.0121	0.0104	0.0085	0.0078	0.0064	0.0059
	0.9	0.0098	0.0077	0.0065	0.0052	0.0043	0.0044	0.0036
(b)	-0.9	-0.0041	-0.0033	-0.0035	-0.0031	-0.0022	-0.0023	-0.0016
	-0.8	-0.0092	-0.0054	-0.0054	-0.0037	-0.0041	-0.0037	-0.0032
	-0.7	-0.0096	-0.0080	-0.0057	-0.0045	-0.0061	-0.0053	-0.0056
	-0.6	-0.0125	-0.0070	-0.0065	-0.0066	-0.0048	-0.0039	-0.0040
	-0.5	-0.0111	-0.0082	-0.0060	-0.0049	-0.0034	-0.0044	-0.0024
	-0.4	-0.0056	-0.0069	-0.0062	-0.0034	-0.0100	-0.0054	-0.0023
	-0.3	-0.0063	-0.0072	-0.0041	-0.0044	-0.0034	-0.0025	-0.0030
	-0.2	-0.0077	-0.0035	-0.0041	-0.0034	0.0018	-0.0027	-0.0011
	-0.1	-0.0030	-0.0003	-0.0026	-0.0029	-0.0013	-0.0014	-0.0023
	0	-0.0021	-0.0019	0.0030	0.0016	0.0015	-0.0025	0.0010
	0.1	0.0024	0.0040	0.0027	-0.0008	0.0013	-0.0003	0.0007
	0.2	0.0040	0.0076	0.0023	0.0018	0.0006	0.0047	0.0017
	0.3	0.0095	0.0067	0.0037	0.0044	0.0018	0.0039	0.0025
	0.4	0.0129	0.0073	0.0050	0.0036	0.0049	0.0032	0.0025
	0.5	0.0115	0.0077	0.0059	0.0071	0.0044	0.0032	0.0027
	0.6	0.0073	0.0099	0.0063	0.0044	0.0046	0.0052	0.0044
	0.7	0.0094	0.0068	0.0072	0.0067	0.0051	0.0033	0.0034
	0.8	0.0077	0.0060	0.0054	0.0043	0.0041	0.0030	0.0028
	0.9	0.0050	0.0039	0.0034	0.0026	0.0021	0.0024	0.0018

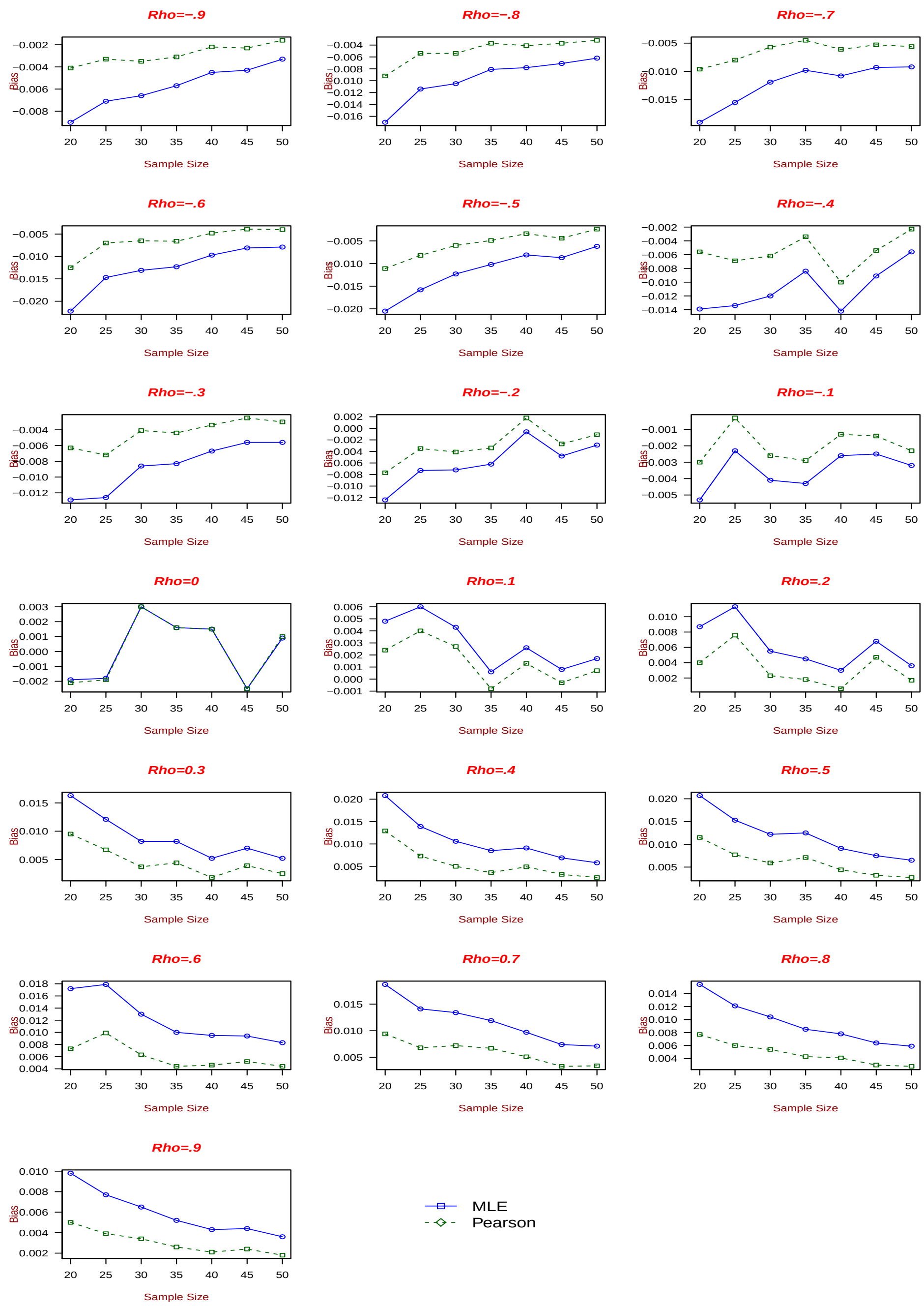

Figure 4.2: Comparison Bias of Point Estimators

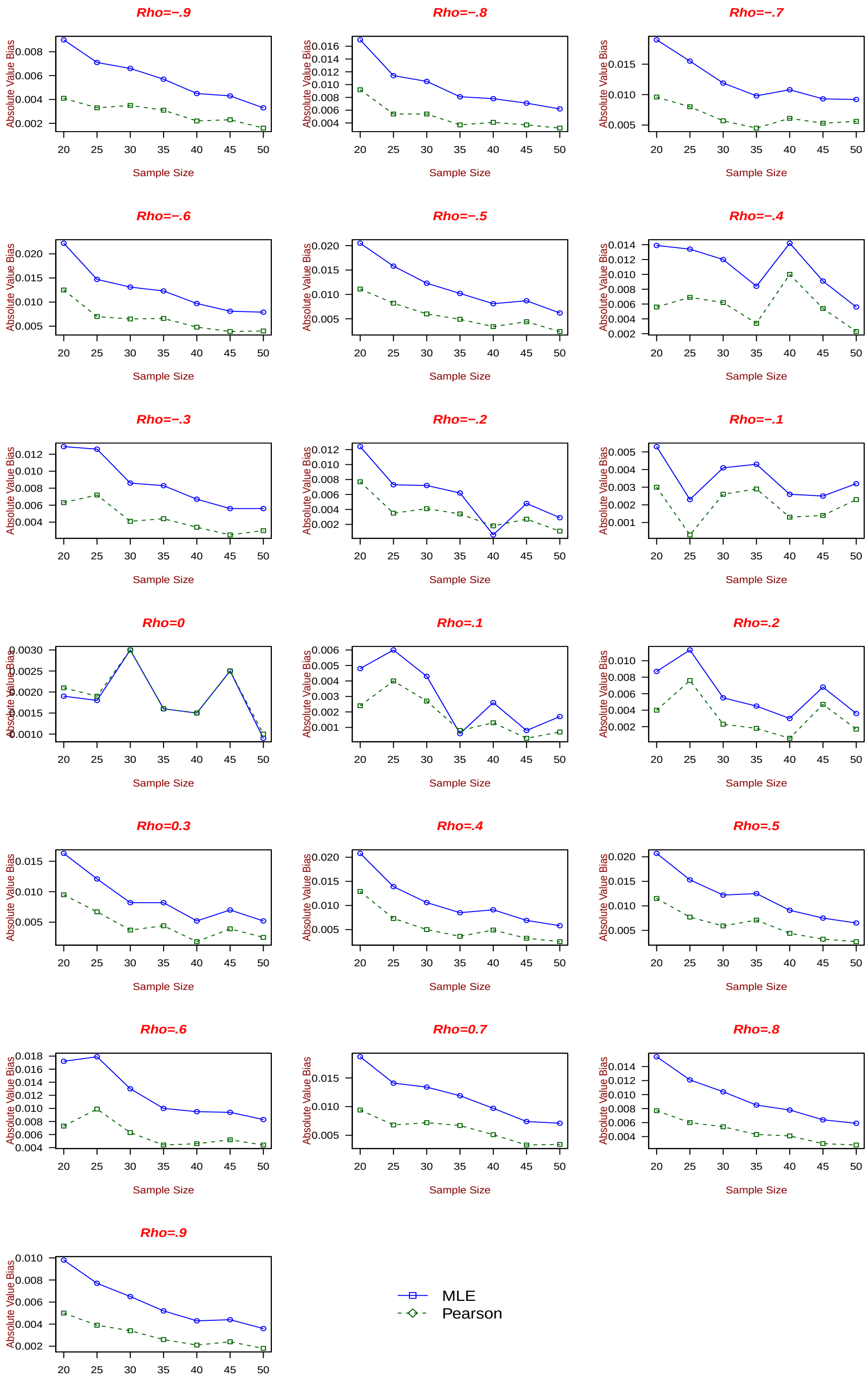

Figure 4.3: Comparison Absolute Value of Bias

Table 4.3: Comparison MSE Estimators

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.9	0.0026	0.0019	0.0016	0.0013	0.0011	0.0010	0.0008
	-0.8	0.0086	0.0064	0.0052	0.0043	0.0038	0.0032	0.0029
	-0.7	0.0162	0.0122	0.0097	0.0081	0.0073	0.0066	0.0059
	-0.6	0.0240	0.0186	0.0159	0.0129	0.0108	0.0098	0.0086
	-0.5	0.0313	0.0246	0.0200	0.0162	0.0146	0.0128	0.0119
	-0.4	0.0366	0.0308	0.0246	0.0205	0.0189	0.0160	0.0143
	-0.3	0.0429	0.0346	0.0279	0.0240	0.0209	0.0190	0.0170
	-0.2	0.0463	0.0367	0.0309	0.0267	0.0231	0.0211	0.0186
	-0.1	0.0485	0.0393	0.0331	0.0278	0.0243	0.0221	0.0193
	0	0.0493	0.0389	0.0333	0.0286	0.0252	0.0221	0.0202
	0.1	0.0490	0.0391	0.0326	0.0284	0.0243	0.0218	0.0201
	0.2	0.0466	0.0372	0.0313	0.0265	0.0231	0.0206	0.0187
	0.3	0.0430	0.0331	0.0277	0.0240	0.0212	0.0186	0.0166
	0.4	0.0385	0.0303	0.0245	0.0209	0.0183	0.0165	0.0147
	0.5	0.0312	0.0239	0.0203	0.0174	0.0151	0.0130	0.0118
	0.6	0.0235	0.0185	0.0152	0.0127	0.0109	0.0097	0.0088
	0.7	0.0157	0.0119	0.0100	0.0084	0.0073	0.0063	0.0058
	0.8	0.0085	0.0068	0.0053	0.0044	0.0037	0.0032	0.0029
	0.9	0.0026	0.0020	0.0016	0.0013	0.0011	0.0010	0.0008
(b)	-0.9	0.0024	0.0018	0.0015	0.0013	0.0011	0.0009	0.0008
	-0.8	0.0082	0.0062	0.0051	0.0042	0.0038	0.0032	0.0029
	-0.7	0.0160	0.0119	0.0096	0.0080	0.0071	0.0065	0.0058
	-0.6	0.0240	0.0185	0.0159	0.0129	0.0107	0.0098	0.0086
	-0.5	0.0319	0.0249	0.0202	0.0164	0.0147	0.0129	0.0119
	-0.4	0.0377	0.0314	0.0251	0.0209	0.0191	0.0162	0.0145
	-0.3	0.0446	0.0357	0.0287	0.0245	0.0213	0.0193	0.0172
	-0.2	0.0484	0.0381	0.0318	0.0274	0.0237	0.0215	0.0189
	-0.1	0.0511	0.0410	0.0342	0.0286	0.0249	0.0225	0.0197
	0	0.0519	0.0405	0.0345	0.0294	0.0259	0.0226	0.0206
	0.1	0.0515	0.0407	0.0338	0.0293	0.0249	0.0222	0.0205
	0.2	0.0488	0.0385	0.0323	0.0272	0.0237	0.0210	0.0191
	0.3	0.0448	0.0341	0.0284	0.0245	0.0216	0.0190	0.0168
	0.4	0.0395	0.0310	0.0250	0.0213	0.0186	0.0167	0.0149
	0.5	0.0317	0.0241	0.0205	0.0175	0.0152	0.0131	0.0118
	0.6	0.0235	0.0184	0.0151	0.0127	0.0109	0.0097	0.0088
	0.7	0.0154	0.0117	0.0098	0.0082	0.0073	0.0062	0.0057
	0.8	0.0081	0.0065	0.0052	0.0042	0.0036	0.0032	0.0028
	0.9	0.0024	0.0018	0.0015	0.0012	0.0011	0.0009	0.0008

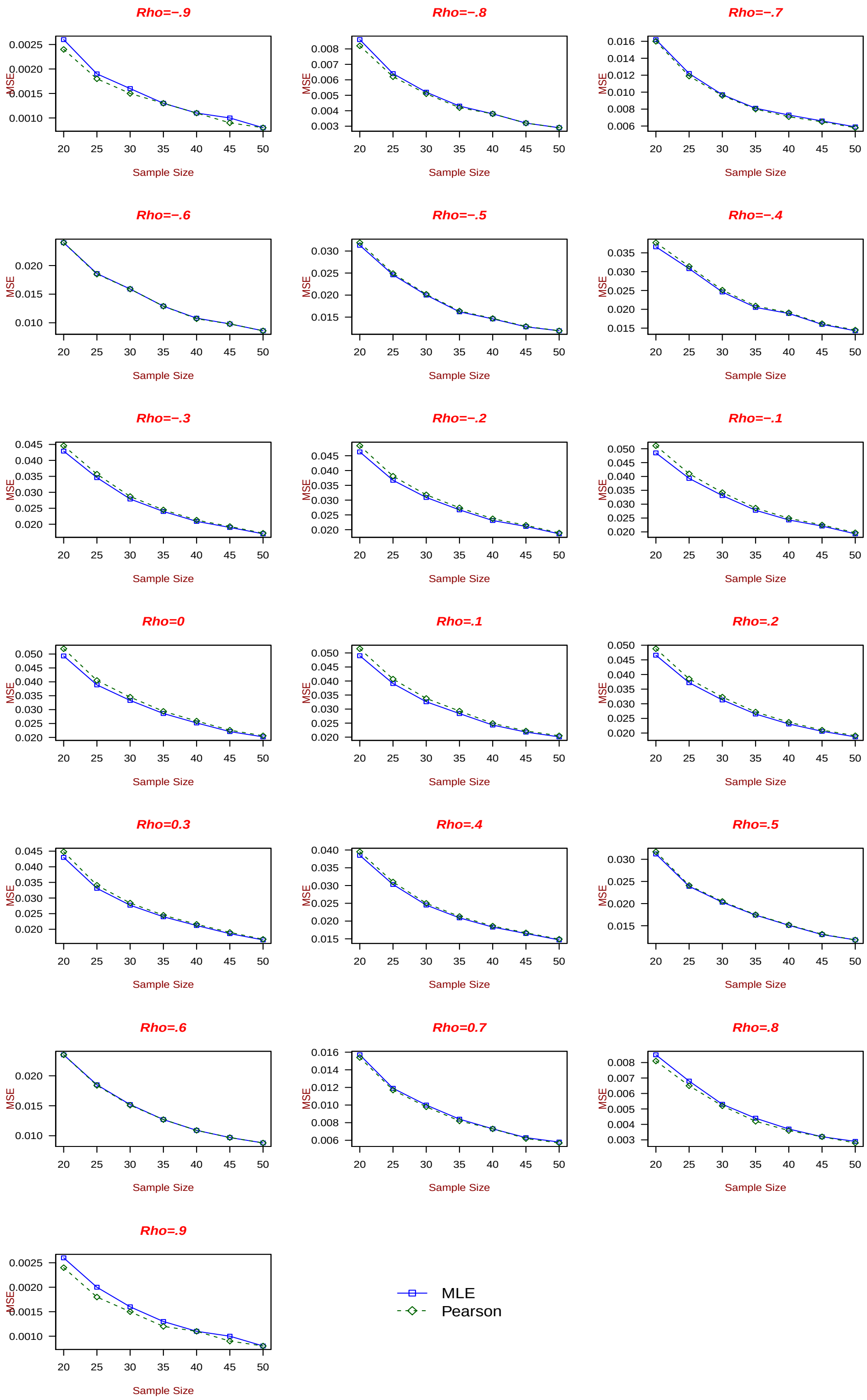

Figure 4.4: Comparison MSE

4.2 Confidence Interval for ρ

As noted in Theorem 2.2.7, $\hat{\rho}_n \to N\Big(\rho, \frac{(1-\rho^2)^2}{n}\Big)$ in distribution; however, since the variance of it depends on ρ, a variance stabilizing transformation was used in Section 2.2.7 and it was found that $\frac{1}{2}\ln\Big(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\Big) \sim N\Big(\frac{1}{2}\ln\Big(\frac{1+\rho}{1-\rho}\Big), \frac{1}{n}\Big)$.

Using the previous two distributions, a $(1-\alpha)100\%$ Confidence Interval for ρ can be found as follow:

$$\Big(\hat{\rho} - z_{\alpha/2}\frac{1-\rho^2}{\sqrt{n}}, \hat{\rho} + z_{\alpha/2}\frac{1-\rho^2}{\sqrt{n}}\Big) \tag{4.1}$$

or

$$\Big(\frac{e^{2l}-1}{e^{2l}+1}, \frac{e^{2u}-1}{e^{2u}+1}\Big) \tag{4.2}$$

respectively, where $l = \frac{1}{2}\ln\Big(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\Big) - z_{\alpha/2}\frac{1}{\sqrt{n}}$ and $u = \frac{1}{2}\ln\Big(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\Big) + z_{\alpha/2}\frac{1}{\sqrt{n}}$; however the desired confidence level is not attained using these intervals. Therefore, to achieve the $(1-\alpha)100\%$ level of confidence, a study on the denominators of the sampling errors is performed.

4.2.1 Study on the Denominator for the Confidence Intervals

Our overaching goal is to find a 95% confidence interval for ρ. For this purpose, the coverage probability of the intervals

$$\Big(\hat{\rho} - z_{\alpha/2}\frac{1-\hat{\rho}^2}{\sqrt{n-k}}, \hat{\rho} + z_{\alpha/2}\frac{1-\hat{\rho}^2}{\sqrt{n-k}}\Big) \tag{4.3}$$

for $k = 0, 1, 2, 3, 4, 5$ is calculated and recorded in Tables 4.4-4.5 and 4.6-4.7 where the MLE and Pearson Correlation Coefficient are used, respectively.

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
-0.9	20	0.9075	0.913	0.9184	0.923	0.9267	0.9322
	25	0.9204	0.9241	0.9268	0.9307	0.9346	0.938
	30	0.9253	0.9277	0.9311	0.9347	0.9379	0.9413
	35	0.9243	0.9273	0.9299	0.9322	0.935	0.9389
	40	0.9286	0.9316	0.9342	0.9358	0.9398	0.9431
	45	0.9332	0.9342	0.9371	0.9387	0.9414	0.9439
	50	0.9361	0.9377	0.9401	0.9431	0.9443	0.946
-0.8	20	0.9166	0.9223	0.9277	0.9325	0.9377	0.9423
	25	0.9209	0.9248	0.9293	0.9327	0.9368	0.9398
	30	0.929	0.9318	0.9346	0.9378	0.9405	0.9434
	35	0.9328	0.9358	0.939	0.9419	0.944	0.9469
	40	0.9301	0.9319	0.9348	0.9377	0.9399	0.943
	45	0.9335	0.9367	0.9392	0.9409	0.9426	0.9442
	50	0.9345	0.9363	0.9389	0.9409	0.9431	0.9454
-0.7	20	0.9099	0.9144	0.9207	0.9272	0.933	0.9384
	25	0.9222	0.925	0.929	0.9345	0.9382	0.9421
	30	0.923	0.9269	0.9302	0.934	0.9375	0.9412
	35	0.9322	0.9343	0.9375	0.9404	0.9434	0.946
	40	0.9343	0.9367	0.9393	0.9421	0.9452	0.9471
	45	0.9323	0.9346	0.9371	0.939	0.941	0.944
	50	0.9343	0.9364	0.9392	0.9411	0.9427	0.945
-0.6	20	0.9115	0.9167	0.9228	0.929	0.9346	0.9405
	25	0.918	0.922	0.9262	0.9326	0.9374	0.9398
	30	0.9157	0.9199	0.9241	0.928	0.9317	0.9348
	35	0.9259	0.9301	0.9334	0.9363	0.9393	0.9441
	40	0.9352	0.9379	0.9398	0.9422	0.9456	0.9476
	45	0.9298	0.9335	0.936	0.9382	0.9401	0.9428
	50	0.9347	0.938	0.9394	0.9419	0.9441	0.9462
-0.5	20	0.9141	0.921	0.9255	0.9305	0.9363	0.9425
	25	0.9158	0.9204	0.9251	0.9296	0.9331	0.9379
	30	0.9217	0.9254	0.9297	0.9336	0.9377	0.9415
	35	0.9305	0.9336	0.937	0.94	0.9429	0.9459
	40	0.9303	0.9334	0.9353	0.9389	0.9414	0.9436
	45	0.9334	0.9363	0.9387	0.9416	0.9441	0.9474
	50	0.9319	0.934	0.9364	0.9385	0.9408	0.9433
-0.4	20	0.9134	0.9201	0.925	0.9321	0.938	0.9424
	25	0.9136	0.9183	0.9228	0.9282	0.9327	0.9375
	30	0.9251	0.9289	0.9331	0.9374	0.9422	0.9463
	35	0.9293	0.9327	0.9356	0.9399	0.9436	0.9477
	40	0.9265	0.9295	0.9333	0.9365	0.9397	0.9429
	45	0.9359	0.9384	0.9407	0.9437	0.9474	0.9486
	50	0.9349	0.937	0.9386	0.9412	0.9436	0.946
-0.3	20	0.9105	0.9186	0.9242	0.9298	0.935	0.9405
	25	0.9184	0.9244	0.9291	0.9349	0.939	0.9437
	30	0.9222	0.9262	0.9304	0.9339	0.939	0.9429
	35	0.9282	0.9311	0.9344	0.9376	0.9419	0.945
	40	0.9281	0.9309	0.9337	0.9372	0.9399	0.9427
	45	0.9319	0.9351	0.9374	0.9403	0.9434	0.9464
	50	0.9334	0.9363	0.9382	0.94	0.9425	0.9446

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
-0.2	20	0.9093	0.9147	0.9208	0.9276	0.935	0.9414
	25	0.9199	0.9257	0.9309	0.9354	0.9411	0.9461
	30	0.9242	0.9296	0.9354	0.9387	0.9433	0.947
	35	0.9293	0.9329	0.9357	0.9382	0.9423	0.9455
	40	0.9272	0.9301	0.9325	0.9359	0.9384	0.9417
	45	0.9282	0.9303	0.9334	0.9368	0.9386	0.9422
	50	0.9336	0.9356	0.9372	0.9399	0.9428	0.9459
-0.1	20	0.9107	0.9172	0.9236	0.9304	0.9361	0.9422
	25	0.9165	0.9229	0.9284	0.9334	0.9371	0.9424
	30	0.9214	0.9256	0.9291	0.9343	0.9382	0.9425
	35	0.9289	0.9327	0.936	0.9389	0.9428	0.9474
	40	0.9312	0.9345	0.937	0.9402	0.9432	0.9458
	45	0.9294	0.9328	0.9358	0.9379	0.9403	0.9428
	50	0.9324	0.9352	0.9377	0.9404	0.9431	0.9453
0	20	0.909	0.9163	0.9233	0.9309	0.9376	0.9434
	25	0.9217	0.926	0.9318	0.9356	0.939	0.9436
	30	0.9231	0.9274	0.9312	0.9348	0.9394	0.9438
	35	0.9264	0.9302	0.9339	0.9372	0.9412	0.9444
	40	0.9274	0.9301	0.9329	0.9356	0.9393	0.942
	45	0.9339	0.9362	0.9395	0.9422	0.9446	0.9473
	50	0.932	0.9345	0.9368	0.939	0.9417	0.9441
0.1	20	0.9069	0.9132	0.9191	0.9264	0.9342	0.9414
	25	0.9149	0.9204	0.9258	0.9326	0.9365	0.9418
	30	0.9237	0.9287	0.9336	0.9368	0.9408	0.9451
	35	0.9229	0.9259	0.9294	0.9324	0.9363	0.9402
	40	0.9311	0.9348	0.9373	0.9388	0.9423	0.9446
	45	0.9302	0.9333	0.9353	0.9386	0.9421	0.9458
	50	0.9333	0.9353	0.9375	0.9387	0.9412	0.9439
0.2	20	0.9116	0.9183	0.9249	0.9305	0.9361	0.9422
	25	0.9192	0.9243	0.9293	0.9351	0.9395	0.9445
	30	0.9199	0.9252	0.9281	0.9322	0.937	0.9406
	35	0.9255	0.9282	0.9323	0.9355	0.9396	0.9428
	40	0.9307	0.9339	0.9366	0.9396	0.9425	0.9453
	45	0.9322	0.9349	0.9375	0.9414	0.9443	0.9466
	50	0.9311	0.9341	0.9365	0.9385	0.9409	0.9436
0.3	20	0.9099	0.9158	0.9221	0.93	0.9358	0.9425
	25	0.9229	0.9264	0.9311	0.9353	0.9411	0.9447
	30	0.9235	0.9265	0.9303	0.9343	0.9387	0.9431
	35	0.9304	0.9341	0.9388	0.9417	0.9444	0.9476
	40	0.9292	0.9319	0.9344	0.9379	0.9408	0.9442
	45	0.93	0.933	0.9356	0.9385	0.941	0.9432
	50	0.9367	0.9392	0.9423	0.9445	0.9457	0.9478
0.4	20	0.9085	0.9147	0.9206	0.927	0.9344	0.9412
	25	0.9131	0.9179	0.9233	0.9286	0.9343	0.9393
	30	0.9236	0.9282	0.9328	0.9359	0.94	0.9441
	35	0.9297	0.9325	0.936	0.9388	0.9424	0.9455
	40	0.9279	0.9313	0.9343	0.9365	0.9389	0.9422
	45	0.9291	0.9317	0.9342	0.9365	0.9385	0.9411
	50	0.9313	0.9335	0.9359	0.9382	0.9412	0.9436

Table 4.4: Comparison Denominadors MLE using Confidence Interval in Expression (4.3)

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
0.5	20	0.9115	0.9174	0.9225	0.9298	0.9355	0.9413
	25	0.9218	0.928	0.9327	0.9375	0.9414	0.9448
	30	0.9243	0.9277	0.9316	0.9346	0.9379	0.9425
	35	0.927	0.9304	0.9337	0.9372	0.941	0.9449
	40	0.9291	0.9324	0.935	0.9376	0.9401	0.9427
	45	0.9316	0.9335	0.9349	0.9374	0.9407	0.9435
	50	0.9333	0.9358	0.9383	0.9403	0.9422	0.9447
0.6	20	0.9098	0.9165	0.9218	0.9265	0.9335	0.9389
	25	0.9198	0.9253	0.9304	0.9349	0.9393	0.943
	30	0.9249	0.9279	0.9316	0.9361	0.9396	0.9429
	35	0.9271	0.9303	0.9333	0.9377	0.9409	0.9439
	40	0.9339	0.9366	0.9392	0.9425	0.9447	0.9462
	45	0.9334	0.9361	0.9379	0.9406	0.9434	0.9455
	50	0.9374	0.9391	0.9416	0.9444	0.946	0.9475
0.7	20	0.9155	0.9214	0.9277	0.9327	0.9384	0.9437
	25	0.921	0.9254	0.9287	0.9329	0.9365	0.9402
	30	0.9265	0.9316	0.9358	0.9391	0.9424	0.946
	35	0.9303	0.9329	0.9359	0.9374	0.9402	0.9431
	40	0.9289	0.9305	0.9341	0.9373	0.9406	0.9428
	45	0.9339	0.9357	0.9378	0.9405	0.9425	0.9448
	50	0.9318	0.9342	0.9363	0.9383	0.9411	0.9434

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
0.8	20	0.9094	0.9153	0.9211	0.9263	0.9321	0.9376
	25	0.9131	0.9175	0.9227	0.927	0.9318	0.9357
	30	0.9254	0.928	0.932	0.9345	0.9383	0.9409
	35	0.9276	0.9311	0.9335	0.9355	0.9382	0.9413
	40	0.9317	0.9341	0.9364	0.9385	0.9412	0.9435
	45	0.9376	0.9399	0.9414	0.9438	0.9459	0.9476
	50	0.9366	0.9384	0.9403	0.9422	0.9445	0.9471
0.9	20	0.911	0.915	0.9213	0.9259	0.9303	0.9357
	25	0.9199	0.9232	0.9265	0.9305	0.9337	0.9373
	30	0.9224	0.9257	0.9294	0.9322	0.9358	0.9394
	35	0.926	0.9288	0.9314	0.9346	0.9379	0.9402
	40	0.9283	0.9308	0.9338	0.9366	0.9389	0.9417
	45	0.9375	0.9391	0.9411	0.9432	0.9449	0.9467
	50	0.9323	0.9348	0.937	0.9387	0.9414	0.9439

Table 4.5: Cont. Comparison Denominators MLE using Confidence Interval in Expression. (4.3)

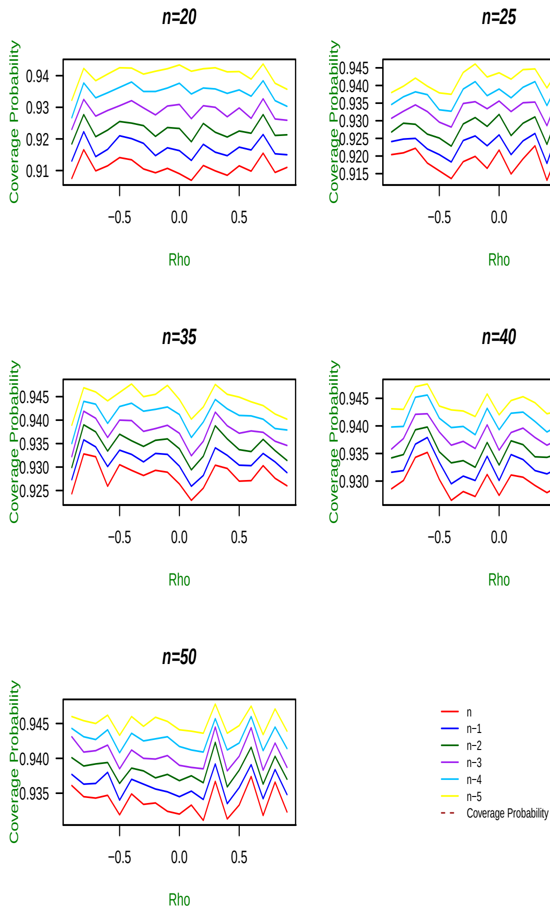

Figure 4.5: Comparison Denominators for MLE using Confidence Interval in Expression. (4.3)

It can be observed from Figure 4.5 that the 95% level for the confidence interval for ρ is not attained for the values of k and all sample sizes.

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
	20	0.8832	0.8883	0.8938	0.8998	0.9066	0.9131
	25	0.9039	0.9075	0.9112	0.9149	0.9191	0.9227
	30	0.9084	0.9112	0.9141	0.9175	0.9222	0.9266
-0.9	35	0.9117	0.9146	0.9177	0.9204	0.9234	0.9273
	40	0.9169	0.9191	0.9219	0.9243	0.9267	0.9293
	45	0.9245	0.9269	0.9289	0.9310	0.9331	0.9347
	50	0.9270	0.9291	0.9309	0.9338	0.9351	0.9373
	20	0.8956	0.9021	0.9078	0.9141	0.9200	0.9256
	25	0.9063	0.9096	0.9141	0.9192	0.9246	0.9283
	30	0.9167	0.9199	0.9233	0.9265	0.9296	0.9331
-0.8	35	0.9214	0.9244	0.9269	0.9301	0.9328	0.9366
	40	0.9201	0.9220	0.9259	0.9292	0.9318	0.9341
	45	0.9243	0.9269	0.9291	0.9309	0.9330	0.9351
	50	0.9268	0.9286	0.9312	0.9334	0.9355	0.9379
	20	0.8929	0.8976	0.9048	0.9107	0.9166	0.9223
	25	0.9078	0.9126	0.9185	0.9234	0.9272	0.9309
	30	0.9144	0.9176	0.9217	0.9254	0.9283	0.9324
-0.7	35	0.9228	0.9265	0.9290	0.9316	0.9346	0.9382
	40	0.9272	0.9295	0.9322	0.9344	0.9370	0.9399
	45	0.9259	0.9285	0.9305	0.9327	0.9358	0.9380
	50	0.9290	0.9311	0.9340	0.9362	0.9383	0.9397
	20	0.8977	0.9042	0.9115	0.9177	0.9234	0.9287
	25	0.9072	0.9109	0.9163	0.9224	0.9268	0.9319
	30	0.9054	0.9099	0.9145	0.9181	0.9221	0.9255
-0.6	35	0.9191	0.9220	0.9251	0.9282	0.9304	0.9351
	40	0.9286	0.9307	0.9339	0.9361	0.9397	0.9420
	45	0.9254	0.9283	0.9307	0.9339	0.9355	0.9383
	50	0.9284	0.9307	0.9322	0.9347	0.9369	0.9393
	20	0.9025	0.9086	0.9138	0.9196	0.9250	0.9310
	25	0.9058	0.9111	0.9148	0.9201	0.9243	0.9287
	30	0.9124	0.9172	0.9220	0.9255	0.9298	0.9323
-0.5	35	0.9245	0.9281	0.9317	0.9346	0.9368	0.9400
	40	0.9241	0.9274	0.9308	0.9338	0.9365	0.9389
	45	0.9283	0.9311	0.9340	0.9365	0.9394	0.9422
	50	0.9276	0.9299	0.9319	0.9336	0.9368	0.9400
	20	0.9021	0.9080	0.9141	0.9203	0.9268	0.9339
	25	0.9044	0.9102	0.9153	0.9197	0.9256	0.9305
	30	0.9161	0.9199	0.9240	0.9294	0.9334	0.9374
-0.4	35	0.9247	0.9281	0.9309	0.9354	0.9387	0.9417
	40	0.9213	0.9241	0.9282	0.9316	0.9350	0.9380
	45	0.9320	0.9344	0.9365	0.9396	0.9435	0.9457
	50	0.9309	0.9331	0.9350	0.9374	0.9401	0.9429
	20	0.8995	0.9077	0.9143	0.9207	0.9268	0.9321
	25	0.9099	0.9155	0.9211	0.9267	0.9313	0.9362
	30	0.9177	0.9224	0.9253	0.9289	0.9334	0.9372
-0.3	35	0.9229	0.9262	0.9289	0.9327	0.9364	0.9395
	40	0.9240	0.9270	0.9294	0.9324	0.9351	0.9384
	45	0.9279	0.9308	0.9333	0.9358	0.9383	0.9419
	50	0.9306	0.9337	0.9352	0.9373	0.9399	0.9427

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
	20	0.9002	0.9054	0.9120	0.9193	0.9269	0.9332
	25	0.9130	0.9187	0.9237	0.9293	0.9357	0.9399
	30	0.9173	0.9225	0.9287	0.9328	0.9379	0.9419
-0.2	35	0.9232	0.9281	0.9312	0.9347	0.9378	0.9416
	40	0.9220	0.9256	0.9289	0.9318	0.9351	0.9381
	45	0.9240	0.9266	0.9293	0.9325	0.9354	0.9395
	50	0.9298	0.9328	0.9342	0.9366	0.9399	0.9430
	20	0.9007	0.9079	0.9149	0.9223	0.9282	0.9355
	25	0.9089	0.9150	0.9216	0.9264	0.9319	0.9366
	30	0.9169	0.9207	0.9239	0.9272	0.9328	0.9373
-0.1	35	0.9242	0.9281	0.9323	0.9351	0.9387	0.9420
	40	0.9270	0.9301	0.9329	0.9367	0.9392	0.9423
	45	0.9264	0.9291	0.9319	0.9352	0.9378	0.9404
	50	0.9296	0.9323	0.9344	0.9378	0.9404	0.9433
	20	0.8993	0.9073	0.9148	0.9225	0.9300	0.9369
	25	0.9157	0.9206	0.9257	0.9306	0.9345	0.9391
	30	0.9175	0.9218	0.9256	0.9294	0.9344	0.9388
0	35	0.9223	0.9263	0.9299	0.9330	0.9371	0.9406
	40	0.9240	0.9271	0.9299	0.9323	0.9356	0.9390
	45	0.9289	0.9321	0.9361	0.9388	0.9418	0.9448
	50	0.9293	0.9320	0.9344	0.9373	0.9400	0.9424
	20	0.8983	0.9050	0.9104	0.9173	0.9245	0.9322
	25	0.9085	0.9129	0.9197	0.9259	0.9307	0.9352
	30	0.9189	0.9234	0.9280	0.9317	0.9362	0.9398
0.1	35	0.9183	0.9219	0.9251	0.9289	0.9328	0.9364
	40	0.9279	0.9314	0.9348	0.9372	0.9397	0.9415
	45	0.9273	0.9300	0.9330	0.9360	0.9390	0.9429
	50	0.9294	0.9317	0.9344	0.9364	0.9385	0.9412
	20	0.9005	0.9082	0.9148	0.9209	0.9279	0.9341
	25	0.9125	0.9173	0.9222	0.9282	0.9340	0.9389
	30	0.9152	0.9202	0.9233	0.9280	0.9330	0.9362
0.2	35	0.9216	0.9247	0.9285	0.9319	0.9347	0.9383
	40	0.9272	0.9307	0.9333	0.9357	0.9390	0.9420
	45	0.9282	0.9318	0.9341	0.9373	0.9400	0.9436
	50	0.9282	0.9309	0.9337	0.9365	0.9385	0.9410
	20	0.8993	0.9044	0.9122	0.9192	0.9259	0.9327
	25	0.9145	0.9183	0.9235	0.9291	0.9350	0.9387
	30	0.9165	0.9204	0.9243	0.9287	0.9321	0.9368
0.3	35	0.9255	0.9278	0.9323	0.9353	0.9388	0.9432
	40	0.9249	0.9276	0.9305	0.9343	0.9374	0.9406
	45	0.9257	0.9284	0.9308	0.9336	0.9365	0.9397
	50	0.9324	0.9354	0.9393	0.9413	0.9430	0.9449
	20	0.8962	0.9020	0.9093	0.9168	0.9238	0.9319
	25	0.9028	0.9085	0.9137	0.9193	0.9258	0.9312
	30	0.9171	0.9225	0.9269	0.9303	0.9338	0.9370
0.4	35	0.9221	0.9255	0.9291	0.9325	0.9354	0.9390
	40	0.9227	0.9256	0.9286	0.9312	0.9338	0.9364
	45	0.9248	0.9272	0.9298	0.9328	0.9349	0.9373
	50	0.9281	0.9301	0.9324	0.9346	0.9372	0.9404

Table 4.6: Comparison Denominadors Pearson using Confidence Interval in Expression. (4.3)

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
0.5	20	0.8987	0.9048	0.9102	0.9179	0.9243	0.9301
	25	0.9127	0.9179	0.9231	0.9284	0.9328	0.9375
	30	0.916	0.9201	0.9248	0.9274	0.931	0.9352
	35	0.9186	0.923	0.9266	0.9295	0.9337	0.9376
	40	0.922	0.9263	0.9285	0.9308	0.9332	0.9367
	45	0.9265	0.9285	0.9312	0.9331	0.9361	0.9389
	50	0.9285	0.9314	0.9337	0.9358	0.9374	0.9399
0.6	20	0.8931	0.8987	0.9062	0.9119	0.9197	0.9258
	25	0.9063	0.9118	0.9184	0.9239	0.9296	0.9349
	30	0.9141	0.9178	0.9218	0.9266	0.9307	0.9355
	35	0.9199	0.9228	0.926	0.93	0.9332	0.9368
	40	0.9262	0.9285	0.9318	0.9351	0.938	0.9413
	45	0.9291	0.931	0.9335	0.9354	0.9384	0.9407
	50	0.9309	0.9338	0.9365	0.9392	0.9412	0.9425
0.7	20	0.8997	0.9048	0.912	0.9174	0.9227	0.9296
	25	0.9096	0.914	0.9168	0.9208	0.9259	0.9299
	30	0.9164	0.9206	0.9243	0.9284	0.9319	0.9358
	35	0.921	0.9239	0.9262	0.9296	0.9327	0.9352
	40	0.9202	0.923	0.9256	0.9282	0.9319	0.9337
	45	0.9252	0.928	0.93	0.933	0.936	0.9387
	50	0.9244	0.9269	0.9296	0.9322	0.935	0.9366

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
0.8	20	0.8893	0.8964	0.9027	0.9093	0.9145	0.9212
	25	0.8977	0.9022	0.908	0.9129	0.9187	0.9238
	30	0.9143	0.9166	0.9205	0.9243	0.9276	0.931
	35	0.9185	0.9212	0.9242	0.9265	0.9292	0.9324
	40	0.9233	0.9251	0.9283	0.9302	0.9324	0.9351
	45	0.9289	0.932	0.9345	0.9363	0.9379	0.9396
	50	0.9299	0.9318	0.9332	0.9352	0.937	0.9394
0.9	20	0.8881	0.8927	0.8978	0.9031	0.9088	0.9163
	25	0.901	0.905	0.9095	0.914	0.9181	0.9215
	30	0.9091	0.9119	0.9152	0.9179	0.9217	0.9262
	35	0.9128	0.9167	0.9201	0.9222	0.9251	0.9282
	40	0.9177	0.9202	0.923	0.9257	0.9286	0.9318
	45	0.9264	0.9294	0.9315	0.9344	0.9364	0.9393
	50	0.925	0.9277	0.9297	0.9322	0.9346	0.9366

Table 4.7: Cont. Comparison Denominadors Pearson using Confidence Interval in Expression. (4.3)

Similarly, Figure 4.6 shows us that the 95% level for the confidence interval for ρ using the Pearson Estimator is not attained for neither of the values of k in all sample sizes considered.

For the previous observations, since the desired level is not achieved, this distribution for $\hat{\rho}$ in Expression (4.3) will not be considered further.

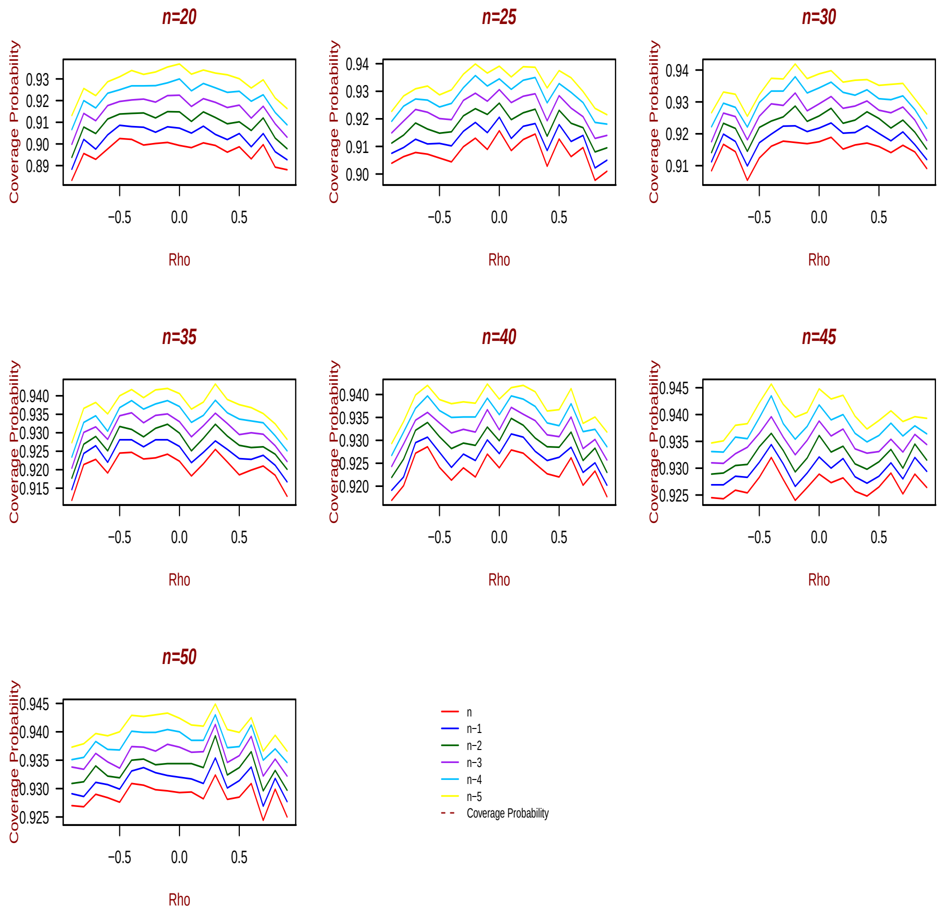

Figure 4.6: Comparison Denominators for Pearson Estimator using Confidence Interval in Espression. (4.3)

Likewise, a coverage probability is measured using the 95% confidence interval given in the expression (4.2):

$$\left(\frac{e^{2l}-1}{e^{2l}+1}, \frac{e^{2u}-1}{e^{2u}+1}\right) \tag{4.4}$$

where $l = \frac{1}{2}\ln\left(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\right) - 1.96\frac{1}{\sqrt{n-k}}$ and $u = \frac{1}{2}\ln\left(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\right) + 1.96\frac{1}{\sqrt{n-k}}$ for $k = 0, \ldots, 5$ using both estimators.

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
-0.9	20	0.9379	0.9448	0.9517	0.9580	0.9642	0.9688
	25	0.9392	0.9443	0.9500	0.9556	0.9601	0.9659
	30	0.9406	0.9445	0.9488	0.9529	0.9567	0.9621
	35	0.9406	0.9438	0.9481	0.9521	0.9551	0.9586
	40	0.9424	0.9453	0.9478	0.9505	0.9540	0.9568
	45	0.9431	0.9448	0.9483	0.9511	0.9538	0.9562
	50	0.9448	0.9478	0.9506	0.9523	0.9546	0.9570
-0.8	20	0.9424	0.9469	0.9537	0.9606	0.9662	0.9719
	25	0.9406	0.9458	0.9507	0.9552	0.9602	0.9644
	30	0.9423	0.9455	0.9505	0.9549	0.9589	0.9624
	35	0.9448	0.9481	0.9512	0.9542	0.9586	0.9614
	40	0.9430	0.9461	0.9489	0.9533	0.9545	0.9572
	45	0.9436	0.9466	0.9491	0.9516	0.9539	0.9567
	50	0.9446	0.9463	0.9485	0.9511	0.9537	0.9563
-0.7	20	0.9331	0.9400	0.9470	0.9518	0.9593	0.9648
	25	0.9417	0.9462	0.9504	0.9568	0.9615	0.9653
	30	0.9440	0.9479	0.9514	0.9555	0.9593	0.9631
	35	0.9463	0.9500	0.9526	0.9565	0.9598	0.9633
	40	0.9451	0.9476	0.9502	0.9531	0.9563	0.9597
	45	0.9391	0.9414	0.9440	0.9482	0.9502	0.9535
	50	0.9419	0.9449	0.9470	0.9505	0.9529	0.9556
-0.6	20	0.9393	0.9446	0.9501	0.9545	0.9601	0.9664
	25	0.9380	0.9434	0.9488	0.9547	0.9596	0.9652
	30	0.9332	0.9387	0.9428	0.9478	0.9522	0.9557
	35	0.9406	0.9437	0.9472	0.9507	0.9540	0.9580
	40	0.9459	0.9482	0.9510	0.9540	0.9567	0.9602
	45	0.9409	0.9436	0.9458	0.9482	0.9516	0.9537
	50	0.9469	0.9494	0.9519	0.9540	0.9563	0.9578
-0.5	20	0.9390	0.9450	0.9502	0.9564	0.9622	0.9678
	25	0.9379	0.9436	0.9492	0.9539	0.9575	0.9627
	30	0.9410	0.9453	0.9497	0.9527	0.9565	0.9596
	35	0.9462	0.9496	0.9533	0.9568	0.9605	0.9630
	40	0.9435	0.9471	0.9507	0.9537	0.9573	0.9600
	45	0.9487	0.9515	0.9546	0.9564	0.9588	0.9605
	50	0.9400	0.9425	0.9445	0.9466	0.9487	0.9514
-0.4	20	0.9397	0.9463	0.9519	0.9582	0.9629	0.9691
	25	0.9343	0.9407	0.9453	0.9505	0.9559	0.9603
	30	0.9436	0.9472	0.9514	0.9546	0.9586	0.9628
	35	0.9455	0.9496	0.9519	0.9546	0.9573	0.9614
	40	0.9396	0.9422	0.9440	0.9477	0.9502	0.9545
	45	0.9477	0.9504	0.9523	0.9544	0.9574	0.9595
	50	0.9456	0.9476	0.9501	0.9526	0.9554	0.9581
-0.3	20	0.9391	0.9463	0.9513	0.9560	0.9611	0.9659
	25	0.9404	0.9458	0.9514	0.9551	0.9598	0.9639
	30	0.9418	0.9469	0.9503	0.9538	0.9581	0.9627
	35	0.9447	0.9484	0.9513	0.9542	0.9569	0.9604
	40	0.9431	0.9469	0.9498	0.9522	0.9550	0.9587
	45	0.9446	0.9470	0.9488	0.9515	0.9542	0.9569
	50	0.9456	0.9476	0.9505	0.9533	0.9551	0.9568

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
-0.2	20	0.9372	0.9439	0.9498	0.9563	0.9623	0.9678
	25	0.9436	0.9480	0.9529	0.9578	0.9611	0.9662
	30	0.9439	0.9495	0.9520	0.9554	0.9591	0.9625
	35	0.9456	0.9485	0.9521	0.9551	0.9589	0.9613
	40	0.9421	0.9459	0.9494	0.9531	0.9545	0.9567
	45	0.9403	0.9420	0.9451	0.9488	0.9509	0.9539
	50	0.9449	0.9469	0.9484	0.9507	0.9534	0.9561
-0.1	20	0.9382	0.9446	0.9499	0.9567	0.9624	0.9676
	25	0.9412	0.9456	0.9507	0.9555	0.9610	0.9659
	30	0.9423	0.9466	0.9496	0.9541	0.9570	0.9618
	35	0.9448	0.9481	0.9520	0.9546	0.9574	0.9604
	40	0.9462	0.9487	0.9511	0.9545	0.9573	0.9598
	45	0.9427	0.9455	0.9482	0.9505	0.9541	0.9568
	50	0.9444	0.9465	0.9485	0.9503	0.9521	0.9549
0	20	0.9400	0.9461	0.9514	0.9565	0.9616	0.9680
	25	0.9419	0.9462	0.9517	0.9552	0.9602	0.9650
	30	0.9420	0.9455	0.9479	0.9518	0.9559	0.9592
	35	0.9432	0.9468	0.9499	0.9540	0.9573	0.9608
	40	0.9413	0.9444	0.9479	0.9504	0.9534	0.9568
	45	0.9463	0.9493	0.9519	0.9547	0.9574	0.9596
	50	0.9439	0.9469	0.9485	0.9506	0.9521	0.9547
0.1	20	0.9366	0.9422	0.9496	0.9557	0.9627	0.9686
	25	0.9399	0.9442	0.9496	0.9551	0.9606	0.9657
	30	0.9427	0.9468	0.9512	0.9544	0.9586	0.9626
	35	0.9397	0.9424	0.9463	0.9495	0.9537	0.9567
	40	0.9444	0.9478	0.9494	0.9525	0.9551	0.9579
	45	0.9440	0.9478	0.9505	0.9534	0.9575	0.9593
	50	0.9436	0.9455	0.9482	0.9507	0.9526	0.9552
0.2	20	0.9388	0.9445	0.9521	0.9560	0.9618	0.9676
	25	0.9409	0.9467	0.9531	0.9573	0.9616	0.9648
	30	0.9399	0.9433	0.9465	0.9500	0.9539	0.9583
	35	0.9420	0.9457	0.9497	0.9525	0.9557	0.9585
	40	0.9442	0.9479	0.9504	0.9536	0.9568	0.9588
	45	0.9451	0.9480	0.9500	0.9535	0.9557	0.9579
	50	0.9427	0.9451	0.9476	0.9509	0.9524	0.9554
0.3	20	0.9352	0.9408	0.9478	0.9557	0.9625	0.9687
	25	0.9439	0.9497	0.9537	0.9582	0.9627	0.9663
	30	0.9420	0.9459	0.9507	0.9547	0.9584	0.9617
	35	0.9458	0.9488	0.9519	0.9547	0.9578	0.9615
	40	0.9437	0.9465	0.9498	0.9520	0.9536	0.9574
	45	0.9444	0.9483	0.9501	0.9524	0.9547	0.9570
	50	0.9475	0.9497	0.9513	0.9541	0.9562	0.9593
0.4	20	0.9356	0.9425	0.9490	0.9544	0.9610	0.9660
	25	0.9397	0.9439	0.9492	0.9532	0.9569	0.9606
	30	0.9414	0.9458	0.9499	0.9531	0.9577	0.9627
	35	0.9447	0.9472	0.9507	0.9543	0.9568	0.9591
	40	0.9445	0.9477	0.9513	0.9540	0.9567	0.9591
	45	0.9412	0.9449	0.9471	0.9498	0.9523	0.9550
	50	0.9425	0.9455	0.9483	0.9506	0.9528	0.9544

Table 4.8: Comparison Denominadors MLE using Fisher Transformation

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
	20	0.9376	0.9428	0.9486	0.9556	0.9612	0.9678
	25	0.9417	0.9477	0.9519	0.9570	0.9617	0.9664
	30	0.9422	0.9470	0.9514	0.9551	0.9589	0.9617
0.5	35	0.9418	0.9449	0.9492	0.9538	0.9562	0.9593
	40	0.9409	0.9439	0.9479	0.9514	0.9554	0.9577
	45	0.9443	0.9477	0.9502	0.9529	0.9559	0.9578
	50	0.9451	0.9478	0.9503	0.9524	0.9550	0.9563
	20	0.9366	0.9438	0.9495	0.9551	0.9616	0.9678
	25	0.9396	0.9449	0.9500	0.9540	0.9585	0.9632
	30	0.9418	0.9455	0.9493	0.9542	0.9586	0.9626
0.6	35	0.9443	0.9480	0.9519	0.9556	0.9582	0.9617
	40	0.9473	0.9498	0.9520	0.9550	0.9581	0.9607
	45	0.9446	0.9483	0.9512	0.9537	0.9564	0.9590
	50	0.9456	0.9484	0.9500	0.9516	0.9543	0.9562
	20	0.9409	0.9482	0.9536	0.9587	0.9632	0.9696
	25	0.9415	0.9458	0.9517	0.9567	0.9608	0.9649
	30	0.9441	0.9490	0.9538	0.9568	0.9602	0.9636
0.7	35	0.9458	0.9491	0.9520	0.9552	0.9584	0.9625
	40	0.9420	0.9455	0.9481	0.9507	0.9536	0.9565
	45	0.9458	0.9493	0.9512	0.9539	0.9559	0.9594
	50	0.9445	0.9464	0.9479	0.9507	0.9529	0.9558

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
	20	0.9396	0.9477	0.9530	0.9583	0.9647	0.9700
	25	0.9346	0.9401	0.9443	0.9499	0.9546	0.9601
	30	0.9420	0.9475	0.9509	0.9549	0.9601	0.9635
0.8	35	0.9425	0.9467	0.9494	0.9535	0.9571	0.9594
	40	0.9451	0.9479	0.9513	0.9541	0.9571	0.9590
	45	0.9444	0.9476	0.9499	0.9531	0.9550	0.9573
	50	0.9470	0.9488	0.9508	0.9534	0.9558	0.9576
	20	0.9336	0.9393	0.9474	0.9547	0.9612	0.9668
	25	0.9384	0.9430	0.9478	0.9530	0.9574	0.9631
	30	0.9415	0.9452	0.9493	0.9540	0.9585	0.9624
0.9	35	0.9419	0.9449	0.9479	0.9514	0.9563	0.9601
	40	0.9433	0.9463	0.9485	0.9511	0.9539	0.9566
	45	0.9471	0.9495	0.9516	0.9537	0.9562	0.9591
	50	0.9437	0.9470	0.9489	0.9509	0.9531	0.9543

Table 4.9: Cont. Comparison Denominadors MLE using Fisher Transformation

Tables 4.8 and 4.9 include the coverages of the intervals using the MLE.

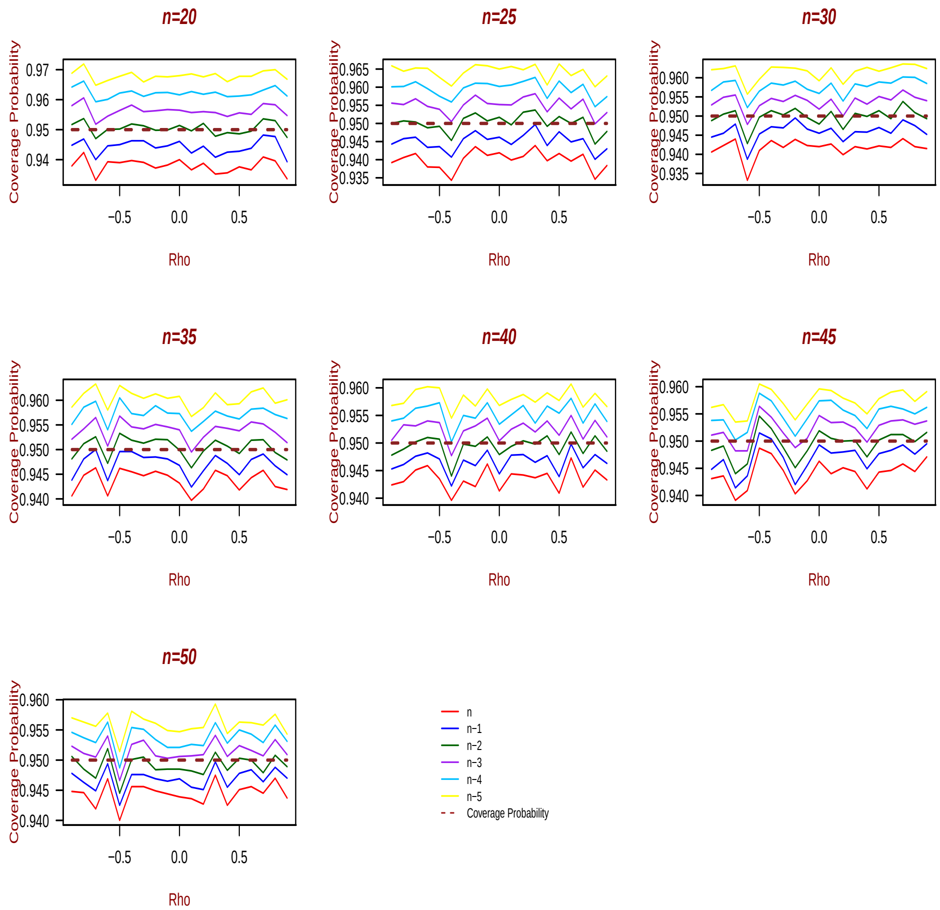

Figure 4.7: Comparison Denominators for MLE using Fisher Transformation

In Figure 4.7 is easily observable that for $k = 2$, the 95% level is achieved. Therefore, it can be conclude that

$$\frac{1}{2}\ln\left(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\right) \sim N\left(\frac{1}{2}\ln\left(\frac{1+\rho}{1-\rho}\right), \frac{1}{n-2}\right) \tag{4.5}$$

is more appropiate.

Table 4.10: Comparison Denominadors Pearson using Fisher Dist.

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
-0.9	20	0.9301	0.9361	0.9437	0.9522	0.9588	0.9638
	25	0.9343	0.9405	0.9447	0.9508	0.9554	0.9603
	30	0.9353	0.9387	0.9442	0.9489	0.9543	0.9587
	35	0.9374	0.9421	0.9455	0.9491	0.9521	0.9564
	40	0.9396	0.9427	0.9465	0.9488	0.9517	0.9551
	45	0.9417	0.9443	0.9465	0.9497	0.9522	0.9545
	50	0.9435	0.9458	0.9482	0.9501	0.9521	0.9551
-0.8	20	0.9362	0.9420	0.9486	0.9556	0.9614	0.9676
	25	0.9359	0.9404	0.9465	0.9510	0.9561	0.9605
	30	0.9390	0.9434	0.9481	0.9521	0.9567	0.9610
	35	0.9435	0.9463	0.9490	0.9521	0.9553	0.9583
	40	0.9396	0.9425	0.9466	0.9493	0.9525	0.9551
	45	0.9415	0.9445	0.9478	0.9504	0.9528	0.9548
	50	0.9427	0.9447	0.9464	0.9495	0.9511	0.9527
-0.7	20	0.9249	0.9318	0.9384	0.9462	0.9532	0.9600
	25	0.9356	0.9413	0.9460	0.9519	0.9567	0.9612
	30	0.9382	0.9424	0.9476	0.9514	0.9559	0.9604
	35	0.9420	0.9456	0.9492	0.9529	0.9567	0.9603
	40	0.9427	0.9452	0.9483	0.9511	0.9537	0.9573
	45	0.9370	0.9396	0.9421	0.9458	0.9491	0.9518
	50	0.9400	0.9422	0.9449	0.9476	0.9500	0.9526
-0.6	20	0.9319	0.9376	0.9427	0.9494	0.9555	0.9616
	25	0.9340	0.9392	0.9440	0.9510	0.9552	0.9610
	30	0.9287	0.9329	0.9385	0.9431	0.9478	0.9518
	35	0.9369	0.9395	0.9435	0.9470	0.9508	0.9551
	40	0.9441	0.9470	0.9491	0.9520	0.9543	0.9580
	45	0.9377	0.9400	0.9430	0.9461	0.9489	0.9515
	50	0.9441	0.9464	0.9490	0.9509	0.9527	0.9554
-0.5	20	0.9306	0.9380	0.9450	0.9511	0.9577	0.9628
	25	0.9334	0.9383	0.9433	0.9488	0.9536	0.9585
	30	0.9362	0.9413	0.9451	0.9487	0.9522	0.9564
	35	0.9431	0.9459	0.9489	0.9534	0.9570	0.9599
	40	0.9412	0.9444	0.9477	0.9503	0.9541	0.9573
	45	0.9452	0.9490	0.9520	0.9544	0.9563	0.9595
	50	0.9381	0.9407	0.9436	0.9456	0.9478	0.9503
-0.4	20	0.9322	0.9395	0.9459	0.9525	0.9580	0.9638
	25	0.9300	0.9358	0.9412	0.9459	0.9519	0.9569
	30	0.9383	0.9436	0.9483	0.9513	0.9551	0.9593
	35	0.9408	0.9451	0.9478	0.9517	0.9540	0.9582
	40	0.9362	0.9393	0.9423	0.9456	0.9474	0.9509
	45	0.9450	0.9474	0.9493	0.9519	0.9550	0.9574
	50	0.9421	0.9443	0.9476	0.9502	0.9530	0.9556
-0.3	20	0.9317	0.9384	0.9455	0.9506	0.9564	0.9612
	25	0.9340	0.9407	0.9452	0.9499	0.9551	0.9599
	30	0.9366	0.9417	0.9453	0.9493	0.9534	0.9589
	35	0.9416	0.9450	0.9480	0.9514	0.9542	0.9579
	40	0.9392	0.9434	0.9464	0.9492	0.9517	0.9550
	45	0.9419	0.9444	0.9466	0.9494	0.9516	0.9541
	50	0.9434	0.9456	0.9482	0.9511	0.9537	0.9553

RHO	SAMPLE SIZE	DENOMINADOR n	n-1	n-2	n-3	n-4	n-5
-0.2	20	0.9315	0.9370	0.9442	0.9502	0.9571	0.9630
	25	0.9390	0.9431	0.9475	0.9530	0.9563	0.9626
	30	0.9395	0.9450	0.9487	0.9527	0.9559	0.9589
	35	0.9416	0.9458	0.9491	0.9516	0.9553	0.9578
	40	0.9385	0.9418	0.9458	0.9497	0.9513	0.9536
	45	0.9384	0.9398	0.9418	0.9456	0.9486	0.9514
	50	0.9431	0.9452	0.9464	0.9483	0.9515	0.9542
-0.1	20	0.9308	0.9371	0.9438	0.9495	0.9570	0.9628
	25	0.9346	0.9402	0.9452	0.9507	0.9553	0.9619
	30	0.9364	0.9416	0.9450	0.9500	0.9540	0.9587
	35	0.9420	0.9448	0.9477	0.9513	0.9542	0.9582
	40	0.9415	0.9450	0.9485	0.9515	0.9546	0.9573
	45	0.9400	0.9431	0.9457	0.9479	0.9512	0.9546
	50	0.9415	0.9439	0.9464	0.9487	0.9504	0.9524
0	20	0.9326	0.9393	0.9458	0.9521	0.9574	0.9632
	25	0.9369	0.9412	0.9470	0.9511	0.9570	0.9616
	30	0.9367	0.9412	0.9453	0.9484	0.9518	0.9555
	35	0.9393	0.9437	0.9474	0.9504	0.9545	0.9582
	40	0.9385	0.9414	0.9449	0.9474	0.9504	0.9533
	45	0.9441	0.9469	0.9488	0.9519	0.9550	0.9573
	50	0.9421	0.9439	0.9466	0.9492	0.9507	0.9526
0.1	20	0.9287	0.9342	0.9415	0.9492	0.9568	0.9625
	25	0.9335	0.9378	0.9435	0.9490	0.9549	0.9606
	30	0.9381	0.9425	0.9470	0.9501	0.9542	0.9584
	35	0.9353	0.9376	0.9423	0.9464	0.9509	0.9534
	40	0.9415	0.9444	0.9471	0.9497	0.9523	0.9550
	45	0.9405	0.9446	0.9476	0.9505	0.9535	0.9566
	50	0.9414	0.9437	0.9453	0.9475	0.9505	0.9530
0.2	20	0.9308	0.9373	0.9446	0.9495	0.9569	0.9631
	25	0.9363	0.9417	0.9469	0.9518	0.9575	0.9614
	30	0.9357	0.9387	0.9423	0.9464	0.9503	0.9549
	35	0.9382	0.9417	0.9459	0.9500	0.9534	0.9566
	40	0.9418	0.9452	0.9476	0.9513	0.9548	0.9572
	45	0.9423	0.9460	0.9477	0.9509	0.9536	0.9558
	50	0.9405	0.9427	0.9454	0.9474	0.9498	0.9524
0.3	20	0.9287	0.9335	0.9404	0.9487	0.9565	0.9633
	25	0.9393	0.9450	0.9483	0.9540	0.9593	0.9625
	30	0.9373	0.9426	0.9464	0.9510	0.9559	0.9585
	35	0.9422	0.9457	0.9490	0.9522	0.9557	0.9589
	40	0.9408	0.9439	0.9462	0.9490	0.9507	0.9554
	45	0.9397	0.9437	0.9465	0.9491	0.9516	0.9540
	50	0.9441	0.9469	0.9495	0.9516	0.9538	0.9567
0.4	20	0.9298	0.9360	0.9431	0.9500	0.9557	0.9617
	25	0.9332	0.9392	0.9445	0.9483	0.9523	0.9558
	30	0.9364	0.9403	0.9449	0.9491	0.9528	0.9589
	35	0.9411	0.9446	0.9478	0.9510	0.9547	0.9572
	40	0.9401	0.9449	0.9481	0.9510	0.9536	0.9566
	45	0.9380	0.9415	0.9443	0.9473	0.9499	0.9527
	50	0.9402	0.9430	0.9458	0.9488	0.9507	0.9526

Table 4.11: Cont. Comparison Denominadors Pearson using Fisher Dist.

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
0.5	20	0.9312	0.9369	0.9433	0.9498	0.9561	0.9626
	25	0.9369	0.9425	0.9471	0.9528	0.9575	0.9621
	30	0.9379	0.9429	0.9477	0.9517	0.9549	0.9586
	35	0.9385	0.9422	0.9458	0.9498	0.9534	0.9567
	40	0.9373	0.9407	0.9447	0.9474	0.9514	0.955
	45	0.941	0.9437	0.9469	0.9499	0.9531	0.9555
	50	0.9433	0.9453	0.948	0.9507	0.9525	0.9545
0.6	20	0.9295	0.9363	0.9438	0.9503	0.9559	0.9627
	25	0.9358	0.9415	0.9464	0.9505	0.9550	0.9597
	30	0.9390	0.9421	0.9461	0.9511	0.9557	0.9596
	35	0.9413	0.9456	0.9503	0.9531	0.9559	0.9594
	40	0.9450	0.9474	0.9492	0.9521	0.9549	0.9578
	45	0.9430	0.9459	0.9484	0.9509	0.9535	0.9565
	50	0.9434	0.946	0.9487	0.9501	0.9525	0.9545
0.7	20	0.9362	0.943	0.9492	0.9556	0.9605	0.9659
	25	0.9378	0.9435	0.9483	0.9526	0.9567	0.9606
	30	0.9403	0.9457	0.949	0.9522	0.9573	0.9615
	35	0.9421	0.9454	0.9496	0.9531	0.9568	0.9601
	40	0.9384	0.942	0.9454	0.9483	0.9519	0.9543
	45	0.9440	0.9462	0.9489	0.9504	0.9532	0.956
	50	0.9411	0.9427	0.9453	0.9483	0.9507	0.9539

		DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
0.8	20	0.9325	0.9381	0.9448	0.9514	0.958	0.9639
	25	0.9298	0.935	0.9407	0.9455	0.9504	0.9543
	30	0.9380	0.9427	0.9470	0.9513	0.9555	0.9595
	35	0.9373	0.9415	0.9449	0.9498	0.9543	0.9562
	40	0.941	0.9448	0.9478	0.9506	0.9534	0.9567
	45	0.9433	0.9456	0.9485	0.95	0.952	0.9554
	50	0.9436	0.9467	0.9488	0.9515	0.9547	0.9577
0.9	20	0.9295	0.9362	0.9436	0.9513	0.9564	0.9619
	25	0.9342	0.9399	0.9442	0.9486	0.9542	0.9604
	30	0.9353	0.9399	0.9443	0.9490	0.9545	0.9584
	35	0.9395	0.9429	0.9465	0.9497	0.954	0.9574
	40	0.9403	0.9439	0.9465	0.9496	0.9524	0.9551
	45	0.9464	0.9488	0.9508	0.9528	0.9554	0.9580
	50	0.9416	0.9441	0.9466	0.9484	0.9505	0.9529

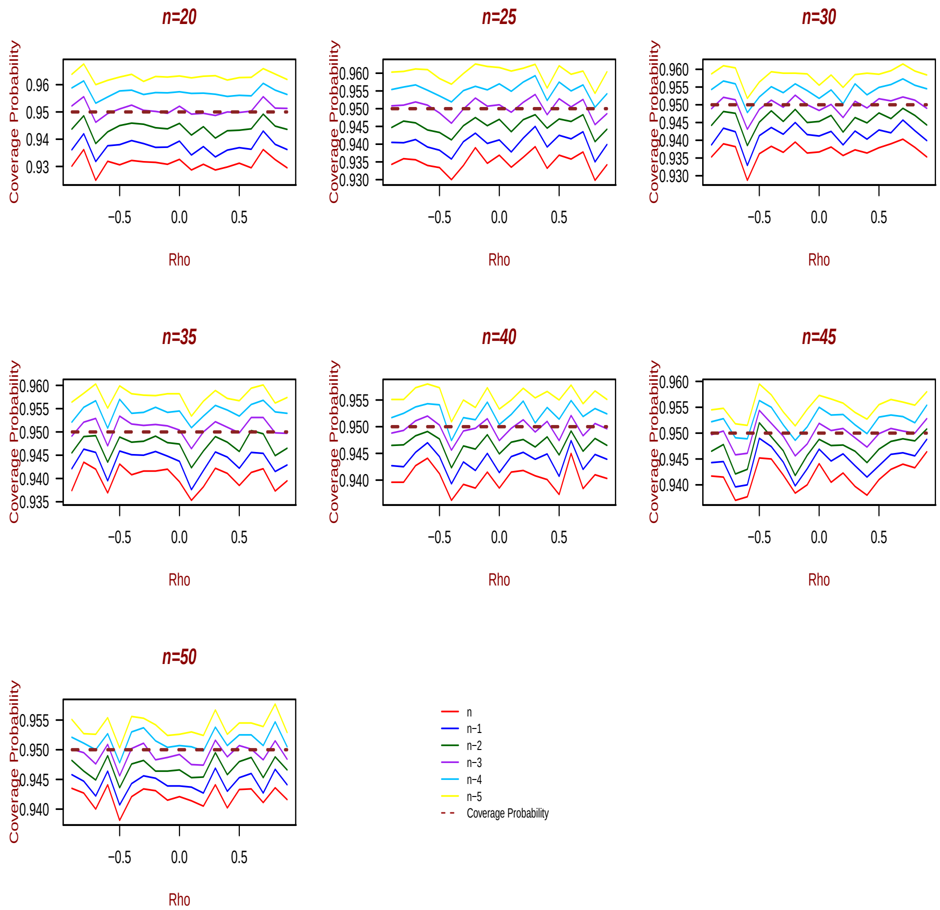

Figure 4.8: Comparison Denominators for Pearson Estimator using Fisher Transformation

The coverage probability for the confidences intervals in expression (4.4) using the Pearson Correlation Coefficient is represented Figure 4.8 whose values are presented in Tables 4.10 and 4.11. It can be seen that $n-3$ is the best denominator when the Pearson Correlation Coefficient $\hat{r}$ is used as estimator. Therefore,

$$\frac{1}{2}\ln\Bigl(\frac{1+\hat{r}_n}{1-\hat{r}_n}\Bigr) \sim N\left(\frac{1}{2}\ln\Bigl(\frac{1+\rho}{1-\rho}\Bigr), \frac{1}{n-3}\right) \tag{4.6}$$

.

4.2.2 Confidence Intervals

In previous section it was shown that the for the MLE $\hat{\rho}_n$

$$\frac{1}{2}\ln\Big(\frac{1+\hat{\rho}_n}{1-\hat{\rho}_n}\Big) \sim N\Big(\frac{1}{2}\ln\Big(\frac{1+\rho}{1-\rho}\Big), \frac{1}{n-2}\Big)$$

and for the Pearson correlation coefficient $\hat{r}_n$

$$\frac{1}{2}\ln\Big(\frac{1+\hat{r}_n}{1-\hat{r}_n}\Big) \sim N\left(\frac{1}{2}\ln\Big(\frac{1+\rho}{1-\rho}\Big), \frac{1}{n-3}\right).$$

and thus, a confidence interval for ρ can be found.

In this section a comparison between those intervals will be performed. For our purpose, the coverage probability and the average width of the intervals were recorded and showed in the next figures for (a) the MLE and (b) the Pearson Correlation Coefficient.

Table 4.12: Comparison Coverage Probability

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.9	0.9517	0.9500	0.9488	0.9481	0.9478	0.9483	0.9506
	-0.8	0.9537	0.9507	0.9505	0.9512	0.9489	0.9491	0.9485
	-0.7	0.9470	0.9504	0.9514	0.9526	0.9502	0.9440	0.9470
	-0.6	0.9501	0.9488	0.9428	0.9472	0.9510	0.9458	0.9519
	-0.5	0.9502	0.9492	0.9497	0.9533	0.9507	0.9546	0.9445
	-0.4	0.9519	0.9453	0.9514	0.9519	0.9440	0.9523	0.9501
	-0.3	0.9513	0.9514	0.9503	0.9513	0.9498	0.9488	0.9505
	-0.2	0.9498	0.9529	0.9520	0.9521	0.9494	0.9451	0.9484
	-0.1	0.9499	0.9507	0.9496	0.9520	0.9511	0.9482	0.9485
	0	0.9514	0.9517	0.9479	0.9499	0.9479	0.9519	0.9485
	0.1	0.9496	0.9496	0.9512	0.9463	0.9494	0.9505	0.9482
	0.2	0.9521	0.9531	0.9465	0.9497	0.9504	0.9500	0.9476
	0.3	0.9478	0.9537	0.9507	0.9519	0.9498	0.9501	0.9513
	0.4	0.9490	0.9492	0.9499	0.9507	0.9513	0.9471	0.9483
	0.5	0.9486	0.9519	0.9514	0.9492	0.9479	0.9502	0.9503
	0.6	0.9495	0.9500	0.9493	0.9519	0.9520	0.9512	0.9500
	0.7	0.9536	0.9517	0.9538	0.9520	0.9481	0.9512	0.9479
	0.8	0.9530	0.9443	0.9509	0.9494	0.9513	0.9499	0.9508
	0.9	0.9474	0.9478	0.9493	0.9479	0.9485	0.9516	0.9489
(b)	-0.9	0.9522	0.9508	0.9489	0.9491	0.9488	0.9497	0.9501
	-0.8	0.9556	0.9510	0.9521	0.9521	0.9493	0.9504	0.9495
	-0.7	0.9462	0.9519	0.9514	0.9529	0.9511	0.9458	0.9476
	-0.6	0.9494	0.9510	0.9431	0.9470	0.9520	0.9461	0.9509
	-0.5	0.9511	0.9488	0.9487	0.9534	0.9503	0.9544	0.9456
	-0.4	0.9525	0.9459	0.9513	0.9517	0.9456	0.9519	0.9502
	-0.3	0.9506	0.9499	0.9493	0.9514	0.9492	0.9494	0.9511
	-0.2	0.9502	0.9530	0.9527	0.9516	0.9497	0.9456	0.9483
	-0.1	0.9495	0.9507	0.9500	0.9513	0.9515	0.9479	0.9487
	0	0.9521	0.9511	0.9484	0.9504	0.9474	0.9519	0.9492
	0.1	0.9492	0.9490	0.9501	0.9464	0.9497	0.9505	0.9475
	0.2	0.9495	0.9518	0.9464	0.9500	0.9513	0.9509	0.9474
	0.3	0.9487	0.9540	0.9510	0.9522	0.9490	0.9491	0.9516
	0.4	0.9500	0.9483	0.9491	0.9510	0.9510	0.9473	0.9488
	0.5	0.9498	0.9528	0.9517	0.9498	0.9474	0.9499	0.9507
	0.6	0.9503	0.9505	0.9511	0.9531	0.9521	0.9509	0.9501
	0.7	0.9556	0.9526	0.9522	0.9531	0.9483	0.9504	0.9483
	0.8	0.9514	0.9455	0.9513	0.9498	0.9506	0.9500	0.9515
	0.9	0.9513	0.9486	0.9490	0.9497	0.9496	0.9528	0.9484

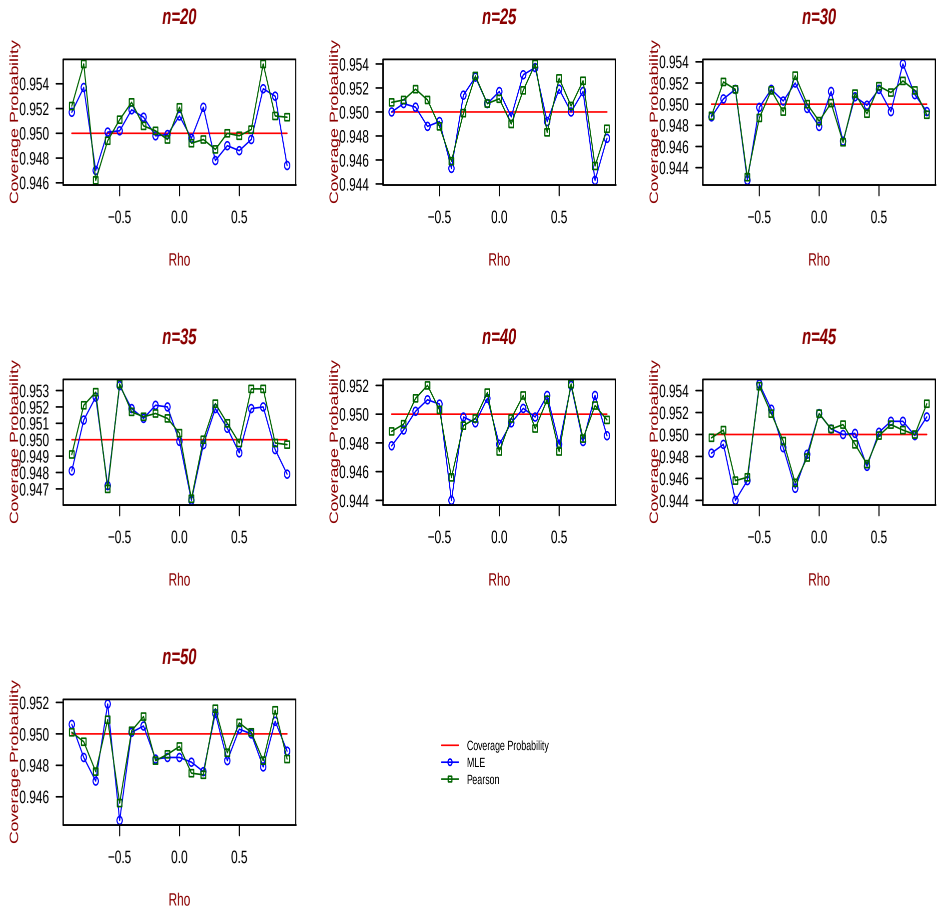

Figure 4.9: Comparison Coverage Probability Confidence Interval

It can be observed form Figure 4.9 that, in both cases (a) and (b), the coverage probabilities of the 95% confidence intervals of ρ are close to 0.95 and with a small difference between them.

Table 4.13: Comparison Average Width CI

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.9	0.2049	0.1760	0.1574	0.1428	0.1310	0.1224	0.1145
	-0.8	0.3663	0.3162	0.2847	0.2591	0.2408	0.2253	0.2121
	-0.7	0.4889	0.4313	0.3884	0.3565	0.3331	0.3120	0.2953
	-0.6	0.5900	0.5209	0.4732	0.4375	0.4075	0.3826	0.3627
	-0.5	0.6671	0.5951	0.5422	0.5018	0.4681	0.4420	0.4179
	-0.4	0.7256	0.6511	0.5967	0.5524	0.5199	0.4892	0.4632
	-0.3	0.7713	0.6951	0.6371	0.5919	0.5548	0.5236	0.4977
	-0.2	0.8040	0.7248	0.6658	0.6189	0.5798	0.5488	0.5217
	-0.1	0.8219	0.7418	0.6821	0.6352	0.5963	0.5637	0.5365
	0	0.8279	0.7486	0.6878	0.6401	0.6011	0.5688	0.5409
	0.1	0.8214	0.7426	0.6825	0.6343	0.5963	0.5637	0.5360
	0.2	0.8028	0.7256	0.6651	0.6185	0.5804	0.5495	0.5217
	0.3	0.7728	0.6959	0.6370	0.5919	0.5541	0.5243	0.4978
	0.4	0.7284	0.6517	0.5961	0.5523	0.5179	0.4879	0.4630
	0.5	0.6673	0.5953	0.5418	0.5024	0.4685	0.4412	0.4181
	0.6	0.5857	0.5238	0.4737	0.4359	0.4071	0.3835	0.3628
	0.7	0.4893	0.4301	0.3897	0.3582	0.3320	0.3107	0.2938
	0.8	0.3642	0.3167	0.2845	0.2596	0.2408	0.2246	0.2119
	0.9	0.2063	0.1769	0.1572	0.1422	0.1308	0.1225	0.1147
(b)	-0.9	0.2033	0.1747	0.1562	0.1419	0.1302	0.1218	0.1140
	-0.8	0.3667	0.3161	0.2843	0.2587	0.2403	0.2250	0.2118
	-0.7	0.4922	0.4332	0.3896	0.3572	0.3337	0.3125	0.2958
	-0.6	0.5971	0.5255	0.4763	0.4400	0.4093	0.3842	0.3640
	-0.5	0.6773	0.6022	0.5474	0.5058	0.4714	0.4447	0.4203
	-0.4	0.7388	0.6606	0.6037	0.5579	0.5245	0.4931	0.4664
	-0.3	0.7872	0.7064	0.6457	0.5988	0.5605	0.5283	0.5018
	-0.2	0.8218	0.7377	0.6757	0.6267	0.5863	0.5543	0.5265
	-0.1	0.8406	0.7556	0.6928	0.6437	0.6034	0.5696	0.5416
	0	0.8470	0.7627	0.6987	0.6489	0.6083	0.5748	0.5460
	0.1	0.8402	0.7563	0.6930	0.6429	0.6034	0.5696	0.5410
	0.2	0.8204	0.7385	0.6749	0.6265	0.5869	0.5550	0.5264
	0.3	0.7885	0.7073	0.6457	0.5988	0.5597	0.5290	0.5019
	0.4	0.7419	0.6611	0.6031	0.5579	0.5225	0.4917	0.4664
	0.5	0.6777	0.6024	0.5471	0.5065	0.4717	0.4439	0.4204
	0.6	0.5922	0.5282	0.4768	0.4383	0.4091	0.3851	0.3641
	0.7	0.4926	0.4322	0.3909	0.3591	0.3327	0.3111	0.2942
	0.8	0.3645	0.3165	0.2843	0.2592	0.2405	0.2243	0.2115
	0.9	0.2049	0.1756	0.1560	0.1413	0.1301	0.1219	0.1141

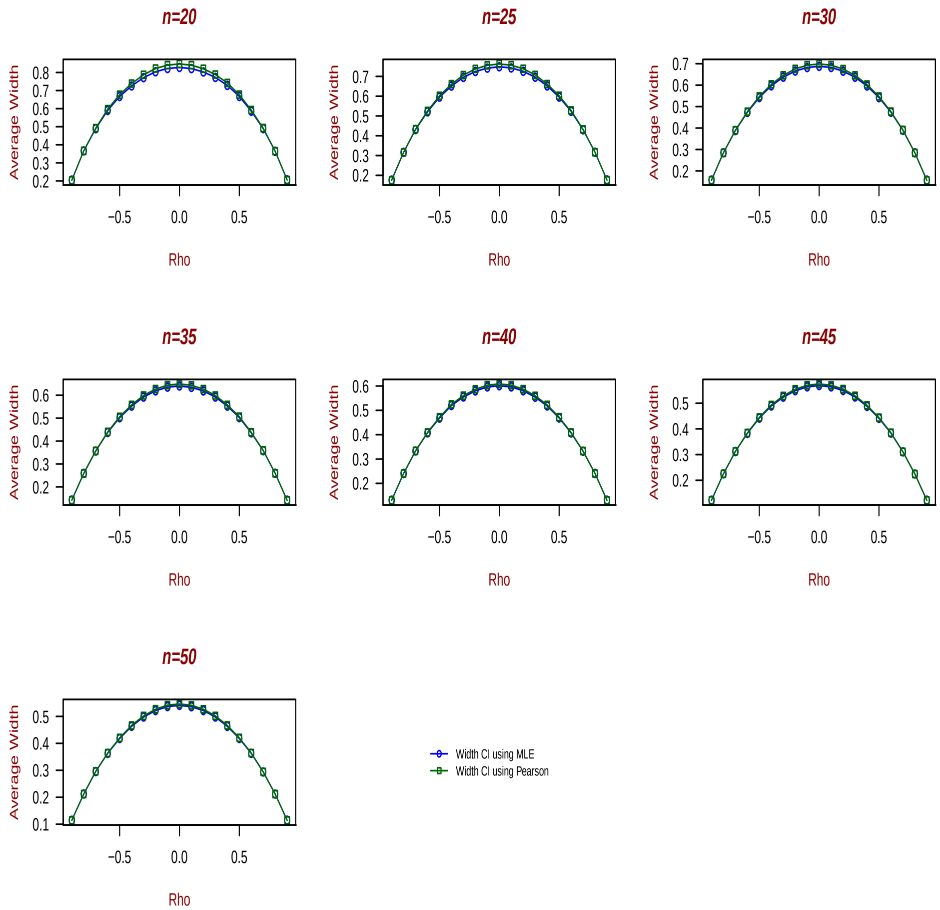

Figure 4.10: Comparison Average Width of the 95% Confidence Interval

As presented in Figure 4.10, the average width of confidence intervals of ρ is smaller when the MLE is used except for the extreme values of ρ.

In conclusion, it is better to use the MLE than the Pearson Correlation Coefficient if we want to find confidence intervals to estimate the true value of ρ.

4.3 Testing hypothesis

A continuation we compare the probability of type I error of testing

$$H_0 : \rho = \rho_0 \quad vs. \quad Ha : \rho \neq \rho_0 \tag{4.7}$$

and power of testing

$$H_0 : \rho = \rho_0 \quad vs. \quad Ha : \rho \neq \rho_0 \tag{4.8}$$

where ρ_0 is the true value of ρ using both estimators mentioned before: (a) the MLE and (b) the Pearson Correlation Coefficient.

4.3.1 Probability of a Type I error

In Table 4.14 the probability of type I error is presented when testing the hypothesis defined in (5.5). The values of ρ were chosen to be from -0.9 to 0.9 in 0.1 increments and it was used $\alpha = 0.05$ as the level of significance. In Figure 4.11 is evident that the type I errors for both (a) and (b) are close to 0.05

Table 4.14: Comparison Type I Error

	RHO	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.9	0.0483	0.0500	0.0512	0.0519	0.0522	0.0517	0.0494
	-0.8	0.0463	0.0493	0.0495	0.0488	0.0511	0.0509	0.0515
	-0.7	0.0530	0.0496	0.0486	0.0474	0.0498	0.0560	0.0530
	-0.6	0.0499	0.0512	0.0572	0.0528	0.0490	0.0542	0.0481
	-0.5	0.0498	0.0508	0.0503	0.0467	0.0493	0.0454	0.0555
	-0.4	0.0481	0.0547	0.0486	0.0481	0.0560	0.0477	0.0499
	-0.3	0.0487	0.0486	0.0497	0.0487	0.0502	0.0512	0.0495
	-0.2	0.0502	0.0471	0.0480	0.0479	0.0506	0.0549	0.0516
	-0.1	0.0501	0.0493	0.0504	0.0480	0.0489	0.0518	0.0515
	0	0.0486	0.0483	0.0521	0.0501	0.0521	0.0481	0.0515
	0.1	0.0504	0.0504	0.0488	0.0537	0.0506	0.0495	0.0518
	0.2	0.0479	0.0469	0.0535	0.0503	0.0496	0.0500	0.0524
	0.3	0.0522	0.0463	0.0493	0.0481	0.0502	0.0499	0.0487
	0.4	0.0510	0.0508	0.0501	0.0493	0.0487	0.0529	0.0517
	0.5	0.0514	0.0481	0.0486	0.0508	0.0521	0.0498	0.0497
	0.6	0.0505	0.0500	0.0507	0.0481	0.0480	0.0488	0.0500
	0.7	0.0464	0.0483	0.0462	0.0480	0.0519	0.0488	0.0521
	0.8	0.0470	0.0557	0.0491	0.0506	0.0487	0.0501	0.0492
	0.9	0.0526	0.0522	0.0507	0.0521	0.0515	0.0484	0.0511
(b)	-0.9	0.0478	0.0492	0.0511	0.0509	0.0512	0.0503	0.0499
	-0.8	0.0444	0.0490	0.0479	0.0479	0.0507	0.0496	0.0505
	-0.7	0.0538	0.0481	0.0486	0.0471	0.0489	0.0542	0.0524
	-0.6	0.0506	0.0490	0.0569	0.0530	0.0480	0.0539	0.0491
	-0.5	0.0489	0.0512	0.0513	0.0466	0.0497	0.0456	0.0544
	-0.4	0.0475	0.0541	0.0487	0.0483	0.0544	0.0481	0.0498
	-0.3	0.0494	0.0501	0.0507	0.0486	0.0508	0.0506	0.0489
	-0.2	0.0498	0.0470	0.0473	0.0484	0.0503	0.0544	0.0517
	-0.1	0.0505	0.0493	0.0500	0.0487	0.0485	0.0521	0.0513
	0	0.0479	0.0489	0.0516	0.0496	0.0526	0.0481	0.0508
	0.1	0.0508	0.0510	0.0499	0.0536	0.0503	0.0495	0.0525
	0.2	0.0505	0.0482	0.0536	0.0500	0.0487	0.0491	0.0526
	0.3	0.0513	0.0460	0.0490	0.0478	0.0510	0.0509	0.0484
	0.4	0.0500	0.0517	0.0509	0.0490	0.0490	0.0527	0.0512
	0.5	0.0502	0.0472	0.0483	0.0502	0.0526	0.0501	0.0493
	0.6	0.0497	0.0495	0.0489	0.0469	0.0479	0.0491	0.0499
	0.7	0.0444	0.0474	0.0478	0.0469	0.0517	0.0496	0.0517
	0.8	0.0486	0.0545	0.0487	0.0502	0.0494	0.0500	0.0485
	0.9	0.0487	0.0514	0.0510	0.0503	0.0504	0.0472	0.0516

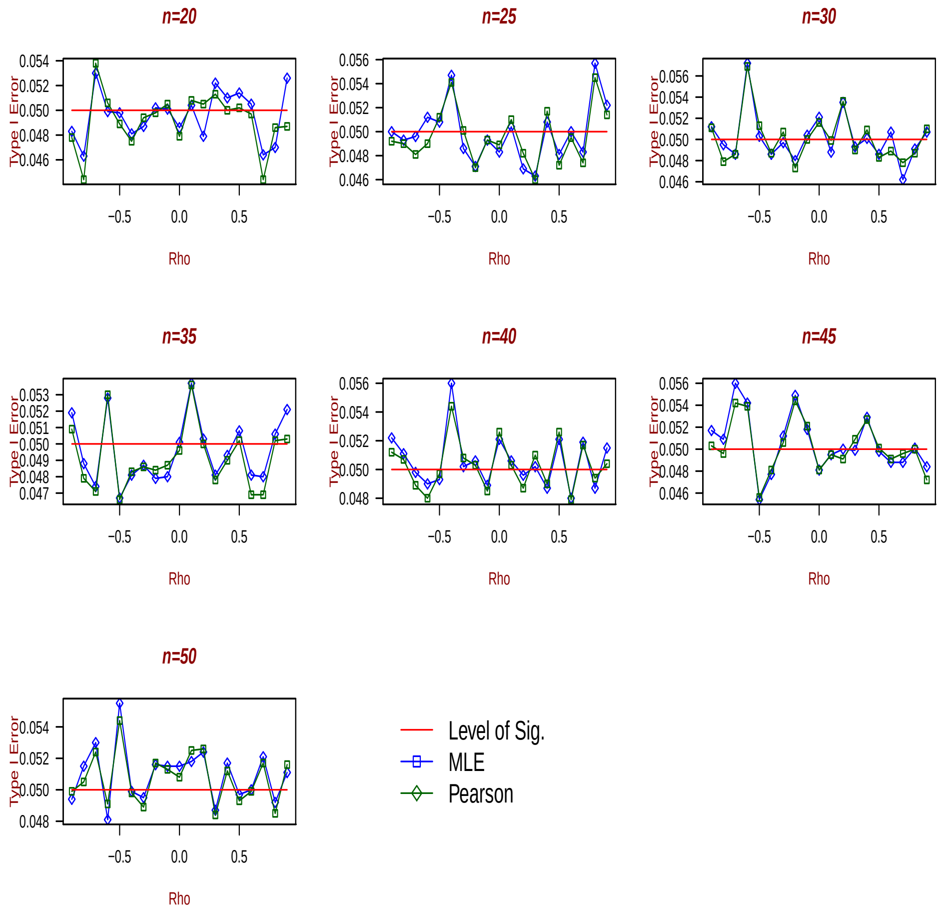

Figure 4.11: Comparison Type I error

4.3.2 Power of the Test

In order to calculate the power of testing the hypothesis in (4.8), different simulations were conducted.

Assuming $\rho = \rho_0$ take the values from -0.8 to 0.8 in 0.1 increments that were use in the null hypothesis, the values for ρ_1 in the alternative hypothesis were chosen as $\rho_1 = \rho \pm 0.10$. Table 4.15 presents the values of the power using the MLE and Table 4.16 using the Pearson Correlation Coefficient.

Table 4.15: Power MLE when $Ha : \rho_1 = \rho \pm 0.10$

	RHO	RHO1	20	25	30	35	40	45	50
(a)	-0.9	-0.8	0.3552	0.4321	0.5021	0.5669	0.6355	0.6856	0.7403
	-0.8	-0.9	0.3611	0.4275	0.5031	0.5694	0.6319	0.6954	0.7343
		-0.7	0.1628	0.1997	0.2265	0.2647	0.2985	0.3198	0.3592
	-0.7	-0.8	0.1657	0.1954	0.2266	0.2579	0.3053	0.3332	0.3729
		-0.7	0.1155	0.1277	0.1495	0.1622	0.1841	0.2092	0.2237
	-0.6	-0.07	0.1163	0.1314	0.1524	0.1731	0.1864	0.206	0.2288
		-0.5	0.0933	0.1079	0.1268	0.1274	0.1428	0.1543	0.1641
	-0.5	-0.6	0.0905	0.1093	0.1144	0.123	0.1347	0.1567	0.1625
		-0.4	0.0794	0.0927	0.0989	0.1047	0.1208	0.1277	0.1413
	-0.4	-0.5	0.0786	0.0988	0.0993	0.1068	0.1278	0.1245	0.1391
		-0.3	0.0751	0.0883	0.0905	0.0985	0.1041	0.1144	0.1221
	-0.3	-0.4	0.0767	0.09	0.0888	0.0989	0.1066	0.1154	0.1235
		-0.2	0.0733	0.0802	0.0865	0.0907	0.0997	0.1106	0.1163
	-0.2	-0.3	0.0722	0.0795	0.0867	0.0952	0.0963	0.1088	0.1125
		-0.1	0.0709	0.0761	0.0828	0.0893	0.101	0.1063	0.1109
	-0.1	-0.2	0.0691	0.0788	0.0876	0.0919	0.0967	0.1081	0.1053
		0	0.0677	0.0765	0.0843	0.0846	0.0933	0.1011	0.1023
	0	-0.1	0.0692	0.0738	0.0838	0.0859	0.0932	0.103	0.1047
		0.1	0.0684	0.0699	0.0808	0.091	0.0996	0.0964	0.1103
	0.1	0	0.0693	0.0765	0.082	0.0908	0.0912	0.1011	0.1092
		0.2	0.069	0.0781	0.0837	0.0908	0.0926	0.1001	0.1089
	0.2	0.1	0.0691	0.0736	0.0867	0.093	0.0959	0.1015	0.111
		0.3	0.0722	0.0803	0.0855	0.0906	0.0957	0.1086	0.1154
	0.3	0.2	0.0704	0.0769	0.0849	0.0934	0.102	0.1084	0.1137
		0.4	0.0776	0.0791	0.0881	0.1017	0.1074	0.1143	0.1239
	0.4	0.3	0.0745	0.0871	0.0914	0.1015	0.1088	0.121	0.1287
		0.5	0.0864	0.093	0.101	0.1089	0.1223	0.1323	0.1433
	0.5	0.4	0.0803	0.0882	0.1032	0.114	0.1248	0.1308	0.1419
		0.6	0.0919	0.1006	0.1219	0.1323	0.1427	0.1547	0.1652
	0.6	0.5	0.0952	0.1052	0.1169	0.1371	0.1454	0.1527	0.1721
		0.7	0.1073	0.1307	0.1505	0.1684	0.1847	0.2098	0.2272
	0.7	0.6	0.1098	0.1279	0.1496	0.1658	0.1862	0.2105	0.2271
		0.8	0.1631	0.1908	0.2338	0.2667	0.2959	0.326	0.3516
	0.8	0.7	0.1672	0.2052	0.2315	0.2588	0.295	0.3302	0.3602
		0.9	0.3547	0.4324	0.5077	0.5738	0.6347	0.6876	0.7329
	0.9	0.8	0.347	0.4278	0.5022	0.5754	0.6373	0.6839	0.7408

Table 4.16: Power Pearson when $Ha : \rho_1 = \rho \pm .10$

	RHO	RHO1	20	25	30	35	40	45	50
(b)	-0.9	-0.8	0.3771	0.4541	0.5228	0.5848	0.6527	0.7007	0.7537
	-0.8	-0.9	0.3105	0.3768	0.4595	0.529	0.5974	0.6647	0.7063
		-0.7	0.1758	0.2136	0.2429	0.2799	0.3121	0.3356	0.3742
	-0.7	-0.8	0.1389	0.1709	0.197	0.2306	0.2778	0.3072	0.3504
		-0.7	0.1235	0.1381	0.1591	0.1728	0.1958	0.2184	0.2342
	-0.6	-0.07	0.098	0.1146	0.1352	0.1537	0.1697	0.1893	0.2122
		-0.5	0.0984	0.1143	0.1341	0.136	0.15	0.162	0.1731
	-0.5	-0.6	0.0802	0.0984	0.1032	0.1114	0.1218	0.1429	0.1528
		-0.4	0.0834	0.0987	0.1034	0.111	0.1264	0.1329	0.1466
	-0.4	-0.5	0.0708	0.0904	0.0916	0.0986	0.1196	0.117	0.1318
		-0.3	0.0778	0.0908	0.0952	0.1033	0.1082	0.1175	0.1248
	-0.3	-0.4	0.0715	0.086	0.0836	0.0927	0.1014	0.1088	0.1184
		-0.2	0.0745	0.0823	0.0879	0.0929	0.1022	0.113	0.1182
	-0.2	-0.3	0.0687	0.0753	0.083	0.0924	0.0928	0.1064	0.1081
		-0.1	0.0717	0.0759	0.0839	0.0906	0.1026	0.108	0.1115
	-0.1	-0.2	0.0684	0.0764	0.0852	0.0901	0.0935	0.1057	0.102
		0	0.0674	0.0754	0.0842	0.0855	0.094	0.102	0.1019
	0	-0.1	0.0682	0.0722	0.0816	0.0856	0.093	0.1013	0.1037
		0.1	0.0673	0.0696	0.0808	0.0895	0.0976	0.0955	0.1082
	0.1	0	0.0697	0.0779	0.0827	0.0913	0.0916	0.1005	0.1102
		0.2	0.0676	0.0768	0.0801	0.0882	0.0904	0.0973	0.1074
	0.2	0.1	0.0719	0.0745	0.0879	0.093	0.0979	0.1029	0.1122
		0.3	0.067	0.0763	0.0819	0.0877	0.092	0.1032	0.113
	0.3	0.2	0.0737	0.0802	0.0884	0.0954	0.1038	0.1108	0.1159
		0.4	0.0719	0.0728	0.0824	0.096	0.1015	0.1086	0.1173
	0.4	0.3	0.0797	0.0916	0.0943	0.1064	0.1125	0.1239	0.1319
		0.5	0.0783	0.0848	0.0938	0.1014	0.1142	0.1254	0.1355
	0.5	0.4	0.0864	0.0934	0.1079	0.119	0.1311	0.1352	0.148
		0.6	0.0817	0.0919	0.1085	0.1218	0.1306	0.1438	0.1558
	0.6	0.5	0.1033	0.1138	0.1262	0.1436	0.1526	0.161	0.1783
		0.7	0.0921	0.1135	0.1327	0.1532	0.17	0.1919	0.2103
	0.7	0.6	0.1186	0.1375	0.1585	0.1754	0.1978	0.22	0.2378
		0.8	0.1337	0.1622	0.2054	0.2402	0.2718	0.3012	0.3253
	0.8	0.7	0.1804	0.219	0.2444	0.2754	0.3106	0.3464	0.3756
		0.9	0.3026	0.3837	0.4591	0.5349	0.5988	0.6541	0.7055
	0.9	0.8	0.3662	0.449	0.5218	0.5946	0.6553	0.698	0.7535

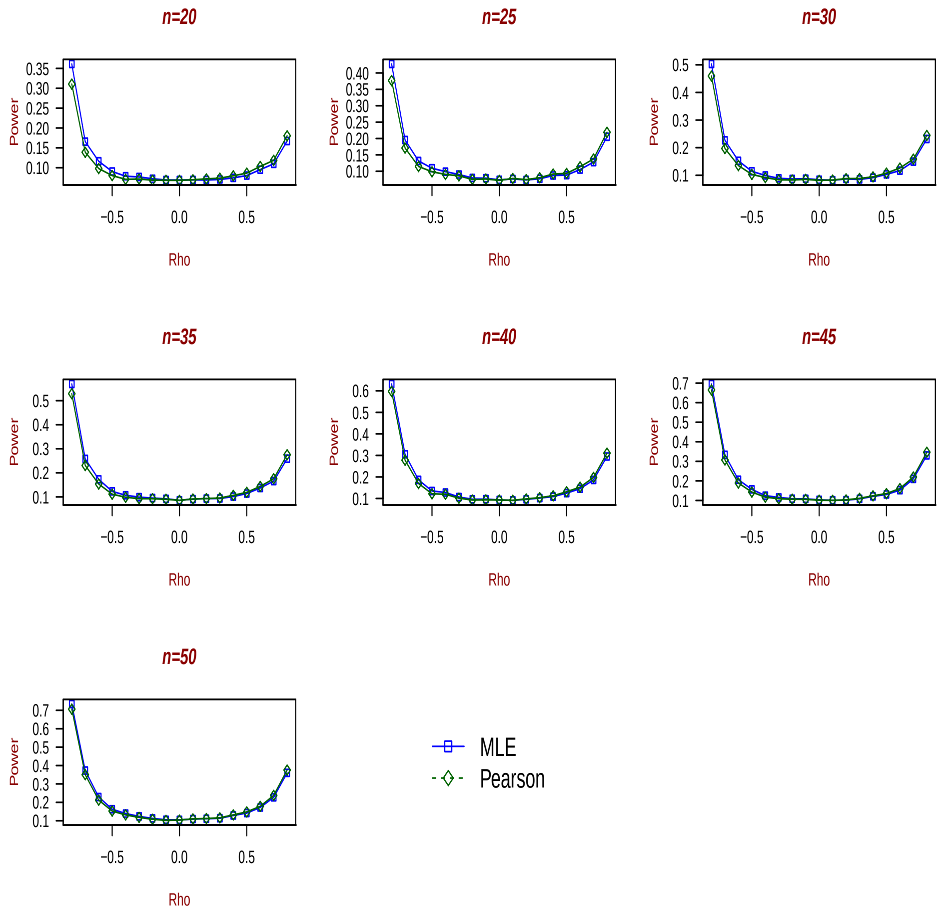

Figure 4.12: Comparison Power $Ha : \rho = \rho - 0.10$

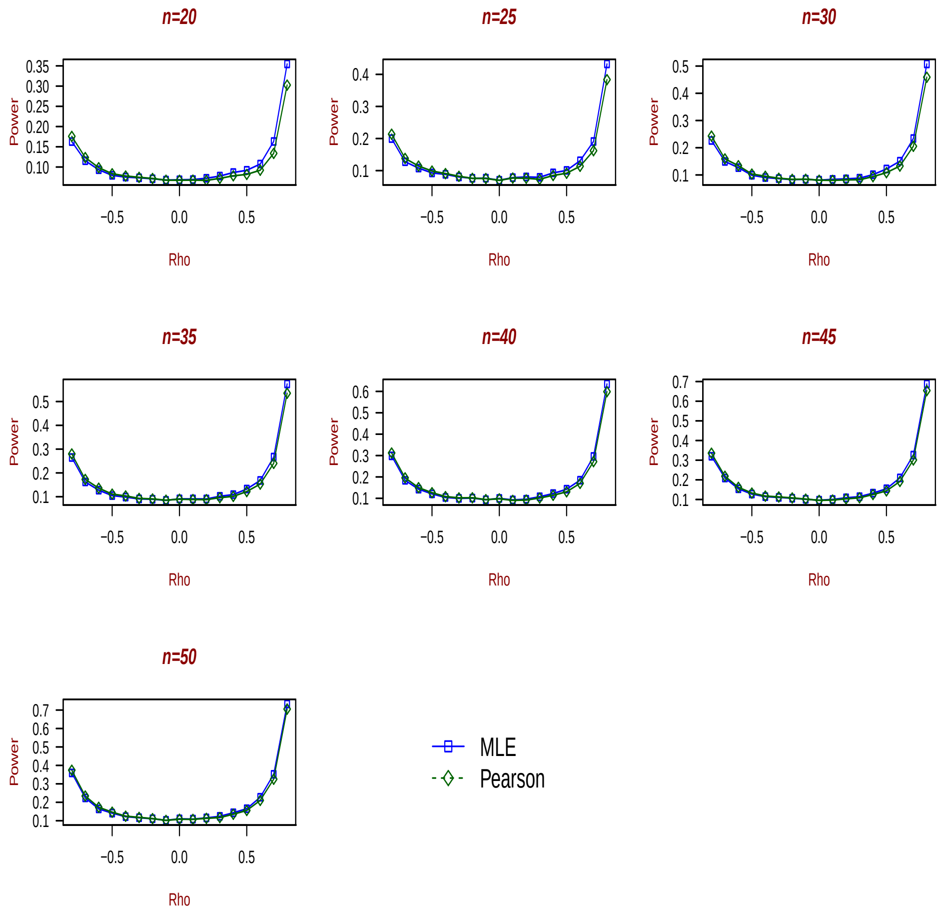

Figure 4.13: Comparison Power $Ha : \rho = \rho + 0.10$

It can be observed in Figures 4.12 and 4.13 the power of testing for the MLE and Pearson Correlation Coefficient. There is no difference between them. Moreover, the power of testing is much higher when ρ approches to its extreme values.

Additionally, $\rho_1 = \rho - 0.05$ and $\rho_1 = \rho + 0.05$ were also evaluated for $\rho = \rho_0$ ranging from -0.9 to 0.9. Tables 4.17 and 4.18 present the power of testing when using the (a) MLE and (b) the Pearson Correlation Coefficient, respectively and can be observed in Figures 4.14 and 4.15.

Table 4.17: Power MLE when $Ha : \rho_1 = \rho \pm .05$

	RHO	RHO1	20	25	30	35	40	45	50
(a)	-0.9	-0.95	0.3286	0.4057	0.4813	0.5511	0.6059	0.6613	0.6992
		-0.85	0.1543	0.1784	0.2094	0.2331	0.262	0.2862	0.3241
	-0.8	-0.85	0.1037	0.1127	0.1322	0.146	0.165	0.1793	0.1982
		-0.75	0.0774	0.0919	0.0952	0.1059	0.1224	0.1327	0.1383
	-0.7	-0.75	0.075	0.0789	0.0808	0.0882	0.1005	0.1114	0.1187
		-0.65	0.0699	0.0713	0.0785	0.0794	0.083	0.0914	0.0952
	-0.6	-0.65	0.0631	0.069	0.0778	0.077	0.0774	0.0864	0.0858
		-0.55	0.062	0.0653	0.0758	0.0721	0.0684	0.0804	0.0775
	-0.5	-0.55	0.0597	0.0637	0.0661	0.0617	0.0657	0.0671	0.0795
		-0.45	0.0581	0.0645	0.0655	0.0631	0.0688	0.07	0.0751
	-0.4	-0.45	0.055	0.0648	0.0623	0.061	0.0724	0.0674	0.0706
		-0.35	0.0555	0.0638	0.0591	0.0597	0.0653	0.0621	0.0687
	-0.3	-0.35	0.0562	0.0608	0.0595	0.06	0.0647	0.0681	0.067
		-0.25	0.0567	0.0554	0.06	0.0598	0.0628	0.0635	0.0668
	-0.2	-0.25	0.0544	0.055	0.0591	0.0616	0.0588	0.0656	0.0662
		-0.15	0.0547	0.0537	0.057	0.0607	0.0646	0.0664	0.066
	-0.1	-0.15	0.0564	0.0569	0.0581	0.0627	0.061	0.0658	0.0645
		-0.05	0.0516	0.0546	0.058	0.0547	0.0599	0.0636	0.0624
	0	-0.5	0.054	0.0545	0.0584	0.0564	0.0617	0.0618	0.0646
		0.5	0.0541	0.0518	0.0581	0.0605	0.0624	0.0635	0.0656
	0.1	0.5	0.0554	0.0567	0.056	0.062	0.0594	0.0619	0.0656
		0.15	0.055	0.0563	0.0564	0.0614	0.0609	0.0621	0.0658
	0.2	0.15	0.0555	0.0555	0.0599	0.0607	0.0592	0.0626	0.0665
		0.25	0.0531	0.0565	0.0596	0.062	0.0594	0.0652	0.0705
	0.3	0.25	0.0544	0.054	0.0591	0.0581	0.0625	0.0671	0.0651
		0.35	0.0574	0.0539	0.0586	0.0606	0.0667	0.0643	0.0636
	0.4	0.35	0.0569	0.0626	0.0602	0.0601	0.0684	0.0718	0.0702
		0.45	0.0593	0.0607	0.0609	0.0633	0.0655	0.0673	0.074
	0.5	0.45	0.0579	0.0572	0.0644	0.0669	0.0694	0.0709	0.0729
		0.55	0.0598	0.0596	0.0664	0.0698	0.0726	0.0705	0.0797
	0.6	0.55	0.0628	0.0632	0.0683	0.0724	0.0716	0.0752	0.0805
		0.65	0.0619	0.0688	0.0706	0.075	0.0756	0.0805	0.0876
	0.7	0.65	0.0624	0.0717	0.075	0.0828	0.0906	0.0939	0.0998
		0.75	0.068	0.0746	0.0864	0.0922	0.1029	0.1028	0.1143
	0.8	0.75	0.0847	0.0989	0.1015	0.1088	0.1188	0.1278	0.1411
		0.85	0.0983	0.1199	0.1353	0.1471	0.1642	0.172	0.1951
	0.9	0.85	0.1493	0.1766	0.2069	0.2384	0.2684	0.2871	0.3182
		0.95	0.3327	0.4097	0.4809	0.5438	0.6023	0.6574	0.7017

Table 4.18: Power Pearson when $Ha : \rho_1 = \rho \pm .05$

	RHO	RHO1	20	25	30	35	40	45	50
(b)	-0.9	-0.95	0.2769	0.3504	0.4344	0.5042	0.5652	0.6275	0.6688
		-0.85	0.1708	0.1958	0.2236	0.2489	0.2804	0.3045	0.3408
	-0.8	-0.85	0.0829	0.0967	0.1127	0.1273	0.1459	0.1601	0.1791
		-0.75	0.0866	0.1	0.1061	0.1169	0.1345	0.1433	0.1503
	-0.7	-0.75	0.0664	0.0674	0.0708	0.0786	0.0894	0.1003	0.1076
		-0.65	0.0762	0.0759	0.0825	0.0828	0.089	0.0978	0.1006
	-0.6	-0.65	0.0575	0.0624	0.0715	0.0707	0.0699	0.0796	0.0794
		-0.55	0.0649	0.0685	0.08	0.0761	0.0731	0.0834	0.0823
	-0.5	-0.55	0.0541	0.0583	0.0613	0.0573	0.0626	0.0643	0.0754
		-0.45	0.061	0.0683	0.0691	0.0652	0.072	0.0728	0.079
	-0.4	-0.45	0.0527	0.0604	0.0587	0.0588	0.0688	0.0644	0.0669
		-0.35	0.0569	0.0652	0.0623	0.0615	0.0677	0.065	0.0694
	-0.3	-0.35	0.0535	0.0586	0.0574	0.0579	0.062	0.0648	0.065
		-0.25	0.0583	0.0577	0.0611	0.062	0.0639	0.0653	0.0692
	-0.2	-0.25	0.0513	0.0539	0.0566	0.0594	0.0578	0.0636	0.0638
		-0.15	0.0546	0.054	0.0572	0.0614	0.0659	0.0673	0.0658
	-0.1	-0.15	0.0557	0.0559	0.0564	0.0608	0.0594	0.0649	0.0647
		-0.05	0.0536	0.0545	0.0581	0.0556	0.0603	0.0628	0.0627
	0	-0.5	0.0528	0.0533	0.0581	0.0563	0.0619	0.0617	0.0641
		0.5	0.0526	0.0523	0.0582	0.0599	0.0623	0.0628	0.0653
	0.1	0.5	0.0558	0.0562	0.0562	0.0626	0.0593	0.0617	0.0669
		0.15	0.0542	0.0548	0.0563	0.0592	0.06	0.062	0.0654
	0.2	0.15	0.0561	0.0564	0.0601	0.0612	0.0597	0.0634	0.0663
		0.25	0.0526	0.055	0.0586	0.0599	0.0583	0.0633	0.0686
	0.3	0.25	0.0573	0.0549	0.0612	0.0597	0.0639	0.0681	0.0663
		0.35	0.0556	0.0511	0.0559	0.0592	0.0645	0.0623	0.0609
	0.4	0.35	0.0572	0.0643	0.0624	0.0639	0.0702	0.0753	0.072
		0.45	0.0562	0.0569	0.0585	0.0589	0.0632	0.0645	0.0703
	0.5	0.45	0.0598	0.0604	0.0673	0.0694	0.0744	0.0752	0.0745
		0.55	0.0555	0.0548	0.0615	0.0645	0.0689	0.0658	0.0744
	0.6	0.55	0.0654	0.0662	0.0719	0.0752	0.0766	0.0799	0.0849
		0.65	0.055	0.0623	0.065	0.0698	0.0684	0.0735	0.0812
	0.7	0.65	0.067	0.0768	0.0802	0.0861	0.0965	0.1001	0.1067
		0.75	0.0588	0.0648	0.0758	0.0813	0.0927	0.0919	0.1046
	0.8	0.75	0.0913	0.1075	0.1106	0.1178	0.1275	0.1381	0.1527
		0.85	0.0774	0.1015	0.1129	0.1284	0.1453	0.1558	0.1753
	0.9	0.85	0.1624	0.1931	0.2226	0.2542	0.2827	0.3063	0.3349
		0.95	0.2748	0.3582	0.4316	0.4996	0.5633	0.6227	0.6746

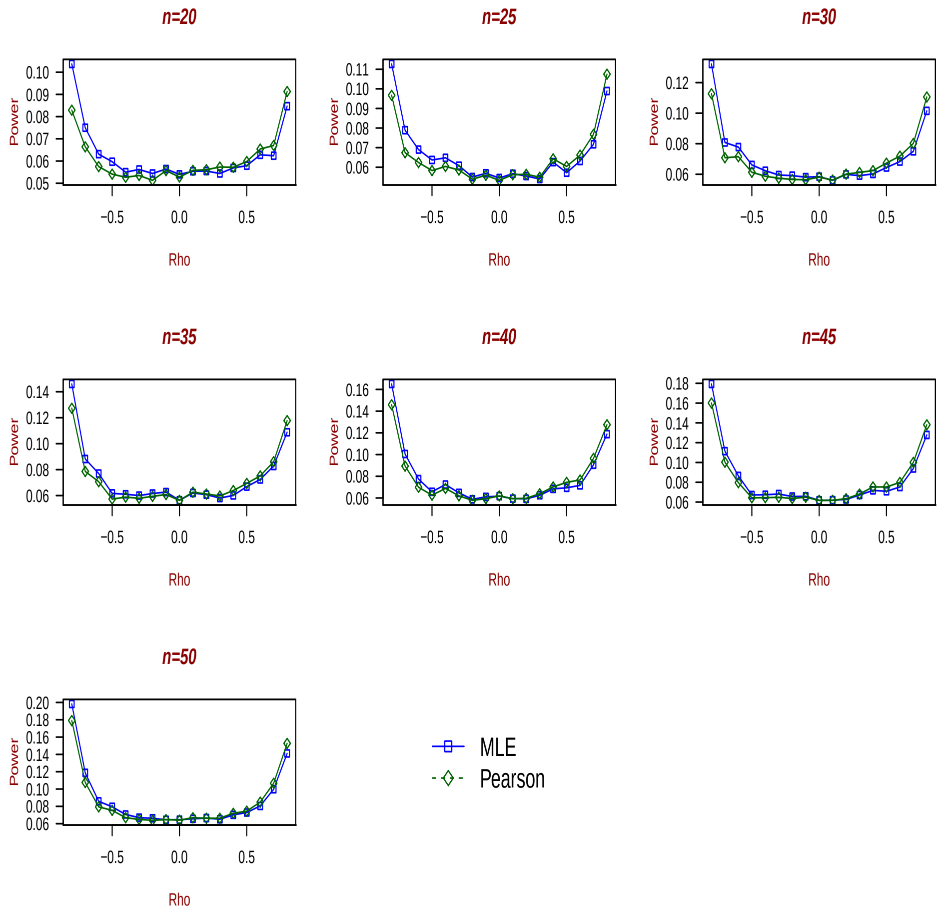

Figure 4.14: Comparison Power $Ha : \rho = \rho - 0.05$

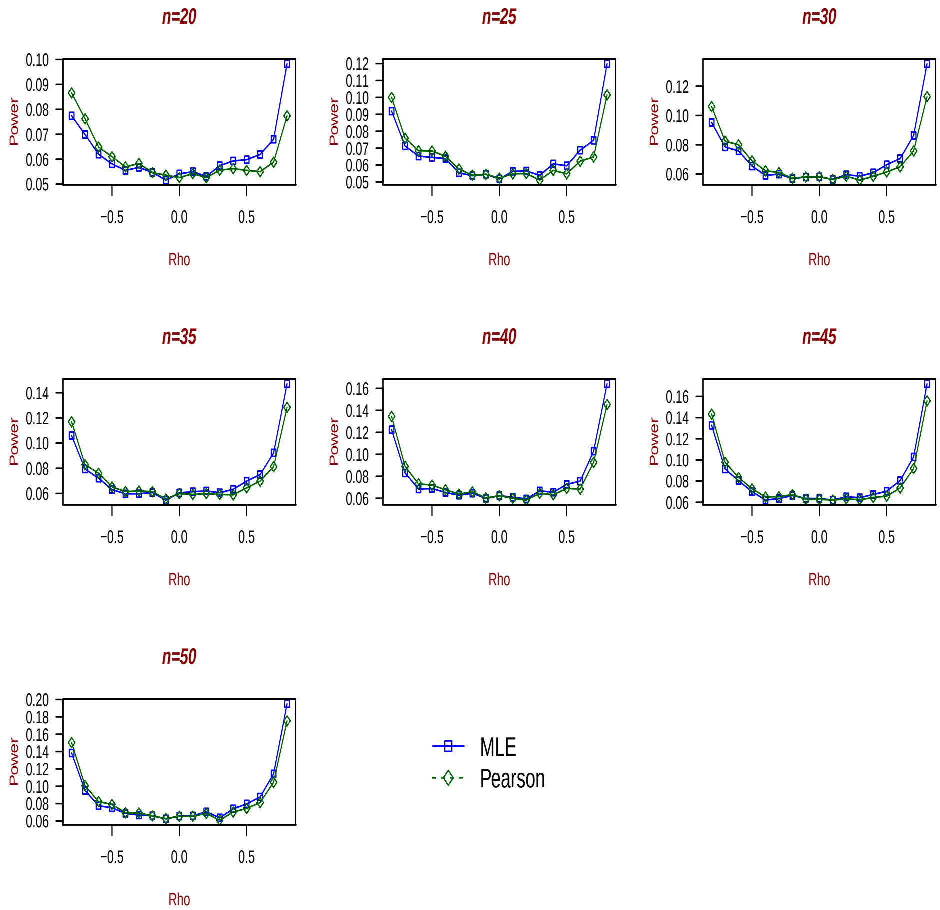

Figure 4.15: Comparison Power $Ha : \rho = \rho + 0.05$

It can be observed in Figures 4.14 and 4.15 the power of testing for the MLE and Pearson Correlation Coefficient. Like before, there is no difference when using both estimators, and the power of testing is much higher when ρ approches to its extreme values.

CHAPTER 5

NUMERICAL STUDY ON THE CASE OF TRIVARIATE NORMAL DISTRIBUTION

In Chapter 3 was found the MLE of the correlation coefficient in the case of equal variances in the Trivariate Normal Distribution. In this chapter simulations are made to evaluate its performance to estimate the true value of ρ.

For our simulations, we assumed $\mu^T = (0, 0, 0)$ and $\sigma^2 = 4$ and picked seven different sample sizes (n), with n ranging from 20 to 50 in increments of 5 and the 14 values ρ that ranges from -0.4 to 0.9 in 0.1 increments. One thousand of replicates were run.

The Chapter 5 is organized as follows: In Section 5.1 are discussed three estimators of ρ, and the MSE and Bias have been evaluated. Confidence intervals for the correlation coefficient are calculated in Section 5.2 and the probability of type I error and power when testing some hypotheses are found in Section 5.3.

5.1 Comparison Estimators of ρ

Three estimators of the correlation coefficient ρ have been considered: (a) MLE defined in Expression (3.33), (b) The Average Pearson and (c) the Average MLE. The Average Pearson and Average of the MLE's are found computing the average of the three pairwise Pearson correlation coefficients and Bivariate MLE of ρ, respectively, discussed in Chapter 2.

5.1.1 Point Estimators

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.4	-0.3955	-0.3969	-0.3974	-0.3981	-0.3984	-0.3983	-0.3981
	-0.3	-0.2945	-0.2957	-0.2952	-0.2971	-0.2965	-0.2974	-0.2976
	-0.2	-0.1967	-0.1961	-0.1958	-0.1968	-0.1972	-0.1994	-0.1975
	-0.1	-0.0958	-0.0967	-0.0993	-0.0991	-0.0982	-0.0989	-0.0989
	0	0.0004	0.0007	-0.0017	0.0005	0.0002	-0.0001	0.0004
	0.1	0.0983	0.0967	0.1001	0.0995	0.0992	0.0979	0.0998
	0.2	0.1935	0.1937	0.1947	0.1961	0.1956	0.1967	0.1964
	0.3	0.2903	0.2888	0.2922	0.2937	0.2946	0.2941	0.2954
	0.4	0.3848	0.3873	0.3904	0.3937	0.3940	0.3941	0.3935
	0.5	0.4826	0.4866	0.4886	0.4900	0.4915	0.4935	0.4943
	0.6	0.5806	0.5864	0.5875	0.5904	0.5907	0.5921	0.5930
	0.7	0.6820	0.6845	0.6879	0.6896	0.6906	0.6928	0.6923
	0.8	0.7848	0.7884	0.7901	0.7912	0.7932	0.7934	0.7941
	0.9	0.8905	0.8923	0.8940	0.8949	0.8957	0.8960	0.8963
(b)	-0.4	-0.3905	-0.3931	-0.3942	-0.3953	-0.3960	-0.3961	-0.3962
	-0.3	-0.2927	-0.2944	-0.2942	-0.2962	-0.2958	-0.2968	-0.2970
	-0.2	-0.1966	-0.1962	-0.1958	-0.1968	-0.1972	-0.1994	-0.1975
	-0.1	-0.0963	-0.0969	-0.0996	-0.0993	-0.0983	-0.0990	-0.0992
	0	0.0004	0.0006	-0.0019	0.0004	0.0001	-0.0001	0.0004
	0.1	0.0993	0.0977	0.1008	0.1000	0.0998	0.0984	0.1001
	0.2	0.1959	0.1957	0.1963	0.1977	0.1969	0.1979	0.1975
	0.3	0.2944	0.2922	0.2951	0.2962	0.2968	0.2961	0.2971
	0.4	0.3905	0.3919	0.3944	0.3971	0.3968	0.3967	0.3958
	0.5	0.4898	0.4923	0.4934	0.4941	0.4953	0.4968	0.4972
	0.6	0.5885	0.5930	0.5930	0.5951	0.5947	0.5956	0.5963
	0.7	0.6902	0.6910	0.6932	0.6941	0.6947	0.6965	0.6955
	0.8	0.7920	0.7940	0.7946	0.7951	0.7966	0.7965	0.7968
	0.9	0.8952	0.8960	0.8970	0.8974	0.8979	0.8979	0.8980
(c)	-0.4	-0.3822	-0.3865	-0.3886	-0.3905	-0.3918	-0.3924	-0.3928
	-0.3	-0.2860	-0.2890	-0.2898	-0.2924	-0.2924	-0.2938	-0.2944
	-0.2	-0.1919	-0.1925	-0.1926	-0.1941	-0.1948	-0.1973	-0.1956
	-0.1	-0.0939	-0.0950	-0.0980	-0.0979	-0.0971	-0.0980	-0.0982
	0	0.0004	0.0006	-0.0018	0.0004	0.0001	-0.0001	0.0004
	0.1	0.0969	0.0957	0.0991	0.0986	0.0985	0.0973	0.0991
	0.2	0.1912	0.1920	0.1932	0.1950	0.1945	0.1957	0.1956
	0.3	0.2878	0.2869	0.2905	0.2923	0.2935	0.2931	0.2943
	0.4	0.3823	0.3853	0.3890	0.3923	0.3927	0.3930	0.3925
	0.5	0.4803	0.4848	0.4871	0.4887	0.4905	0.4925	0.4935
	0.6	0.5788	0.5850	0.5864	0.5895	0.5898	0.5913	0.5923
	0.7	0.6808	0.6835	0.6871	0.6889	0.6901	0.6923	0.6919
	0.8	0.7841	0.7879	0.7897	0.7909	0.7929	0.7932	0.7939
	0.9	0.8903	0.8922	0.8939	0.8948	0.8956	0.8959	0.8962

Table 5.1: Comparison Point Estimators Trivariate

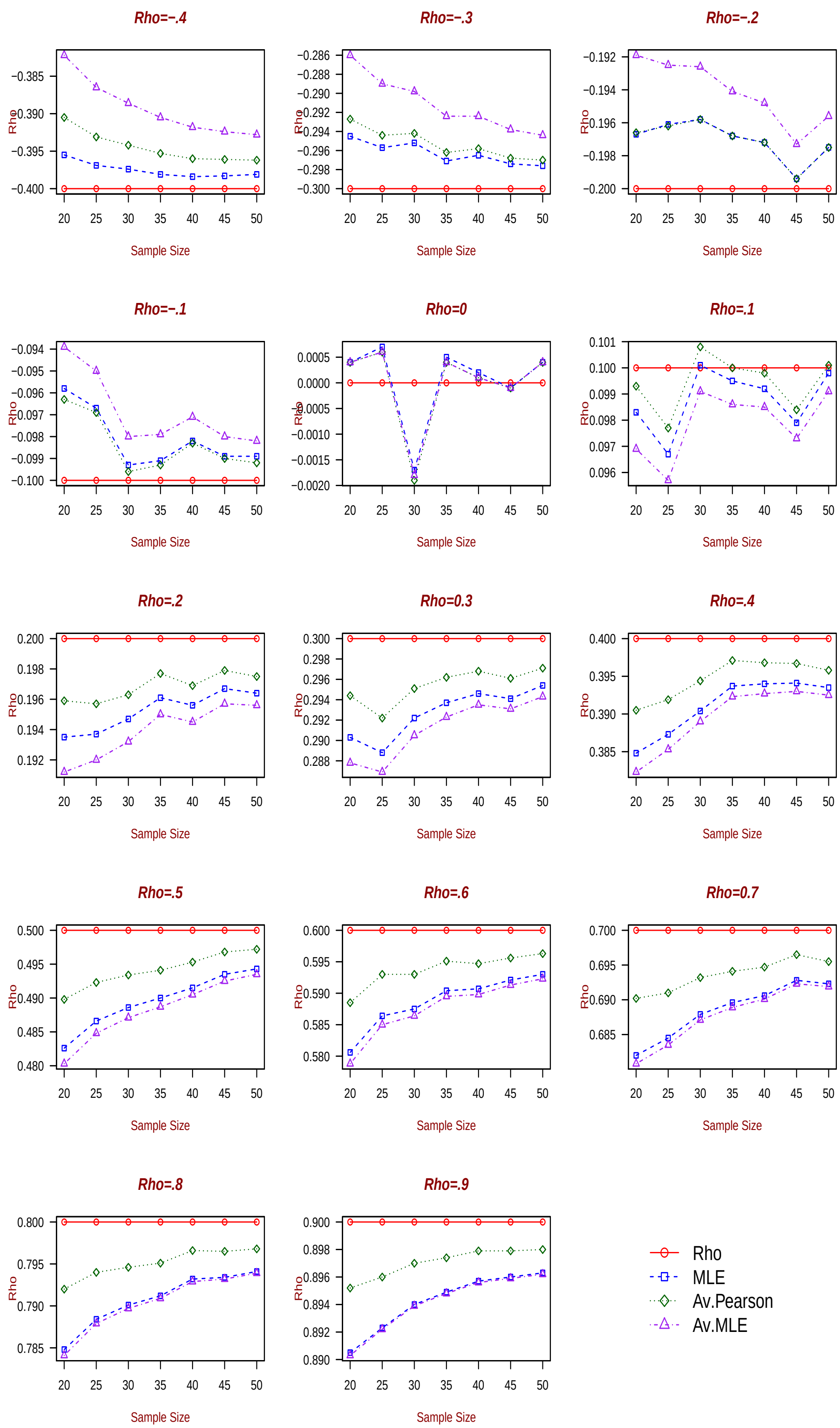

Figure 5.1: Comparison Point Estimators Trivariate Normal Distribution

In Figure 5.1 the three point estimators are presented. These tree estimators are close to the true value of ρ, however, the average of Pearsons is closer to it, followed by the MLE. Notice also that when $\rho = 0$, the three point estimators are the same.

5.1.2 Study on the performance of the estimators

In Table 5.2 is presented the Bias of the estimators and it have been visualized in Figure 5.2. It can be seen that the Average Pearson estimator has smaller bias except when $\rho = 0$.

The mean square error MSE is presented in Figure 5.3 and corresponding table, Table 5.3. The MSE for the tree estimators is small, but a difference between the Average Pearson and MLE can not be observable.

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.4	-0.0045	-0.0031	-0.0026	-0.0019	-0.0016	-0.0017	-0.0019
	-0.3	-0.0055	-0.0043	-0.0048	-0.0029	-0.0035	-0.0026	-0.0024
	-0.2	-0.0033	-0.0039	-0.0042	-0.0032	-0.0028	-0.0006	-0.0025
	-0.1	-0.0042	-0.0033	-0.0007	-0.0009	-0.0018	-0.0011	-0.0011
	0	-0.0004	-0.0007	0.0017	-0.0005	-0.0002	0.0001	-0.0004
	0.1	0.0017	0.0033	-0.0001	0.0005	0.0008	0.0021	0.0002
	0.2	0.0065	0.0063	0.0053	0.0039	0.0044	0.0033	0.0036
	0.3	0.0097	0.0112	0.0078	0.0063	0.0054	0.0059	0.0046
	0.4	0.0152	0.0127	0.0096	0.0063	0.0060	0.0059	0.0065
	0.5	0.0174	0.0134	0.0114	0.0100	0.0085	0.0065	0.0057
	0.6	0.0194	0.0136	0.0125	0.0096	0.0093	0.0079	0.0070
	0.7	0.0180	0.0155	0.0121	0.0104	0.0094	0.0072	0.0077
	0.8	0.0152	0.0116	0.0099	0.0088	0.0068	0.0066	0.0059
	0.9	0.0095	0.0077	0.0060	0.0051	0.0043	0.0040	0.0037
(b)	-0.4	-0.0095	-0.0069	-0.0058	-0.0047	-0.0040	-0.0039	-0.0038
	-0.3	-0.0073	-0.0056	-0.0058	-0.0038	-0.0042	-0.0032	-0.0030
	-0.2	-0.0034	-0.0038	-0.0042	-0.0032	-0.0028	-0.0006	-0.0025
	-0.1	-0.0037	-0.0031	-0.0004	-0.0007	-0.0017	-0.0010	-0.0008
	0	-0.0004	-0.0006	0.0019	-0.0004	-0.0001	0.0001	-0.0004
	0.1	0.0007	0.0023	-0.0008	0.0000	0.0002	0.0016	-0.0001
	0.2	0.0041	0.0043	0.0037	0.0023	0.0031	0.0021	0.0025
	0.3	0.0056	0.0078	0.0049	0.0038	0.0032	0.0039	0.0029
	0.4	0.0095	0.0081	0.0056	0.0029	0.0032	0.0033	0.0042
	0.5	0.0102	0.0077	0.0066	0.0059	0.0047	0.0032	0.0028
	0.6	0.0115	0.0070	0.0070	0.0049	0.0053	0.0044	0.0037
	0.7	0.0098	0.0090	0.0068	0.0059	0.0053	0.0035	0.0045
	0.8	0.0080	0.0060	0.0054	0.0049	0.0034	0.0035	0.0032
	0.9	0.0048	0.0040	0.0030	0.0026	0.0021	0.0021	0.0020
(c)	-0.4	-0.0178	-0.0135	-0.0114	-0.0095	-0.0082	-0.0076	-0.0072
	-0.3	-0.0140	-0.0110	-0.0102	-0.0076	-0.0076	-0.0062	-0.0056
	-0.2	-0.0081	-0.0075	-0.0074	-0.0059	-0.0052	-0.0027	-0.0044
	-0.1	-0.0061	-0.0050	-0.0020	-0.0021	-0.0029	-0.0020	-0.0018
	0	-0.0004	-0.0006	0.0018	-0.0004	-0.0001	0.0001	-0.0004
	0.1	0.0031	0.0043	0.0009	0.0014	0.0015	0.0027	0.0009
	0.2	0.0088	0.0080	0.0068	0.0050	0.0055	0.0043	0.0044
	0.3	0.0122	0.0131	0.0095	0.0077	0.0065	0.0069	0.0057
	0.4	0.0177	0.0147	0.0110	0.0077	0.0073	0.0070	0.0075
	0.5	0.0197	0.0152	0.0129	0.0113	0.0095	0.0075	0.0065
	0.6	0.0212	0.0150	0.0136	0.0105	0.0102	0.0087	0.0077
	0.7	0.0192	0.0165	0.0129	0.0111	0.0099	0.0077	0.0081
	0.8	0.0159	0.0121	0.0103	0.0091	0.0071	0.0068	0.0061
	0.9	0.0097	0.0078	0.0061	0.0052	0.0044	0.0041	0.0038

Table 5.2: Comparison Bias of Estimators Trivariate

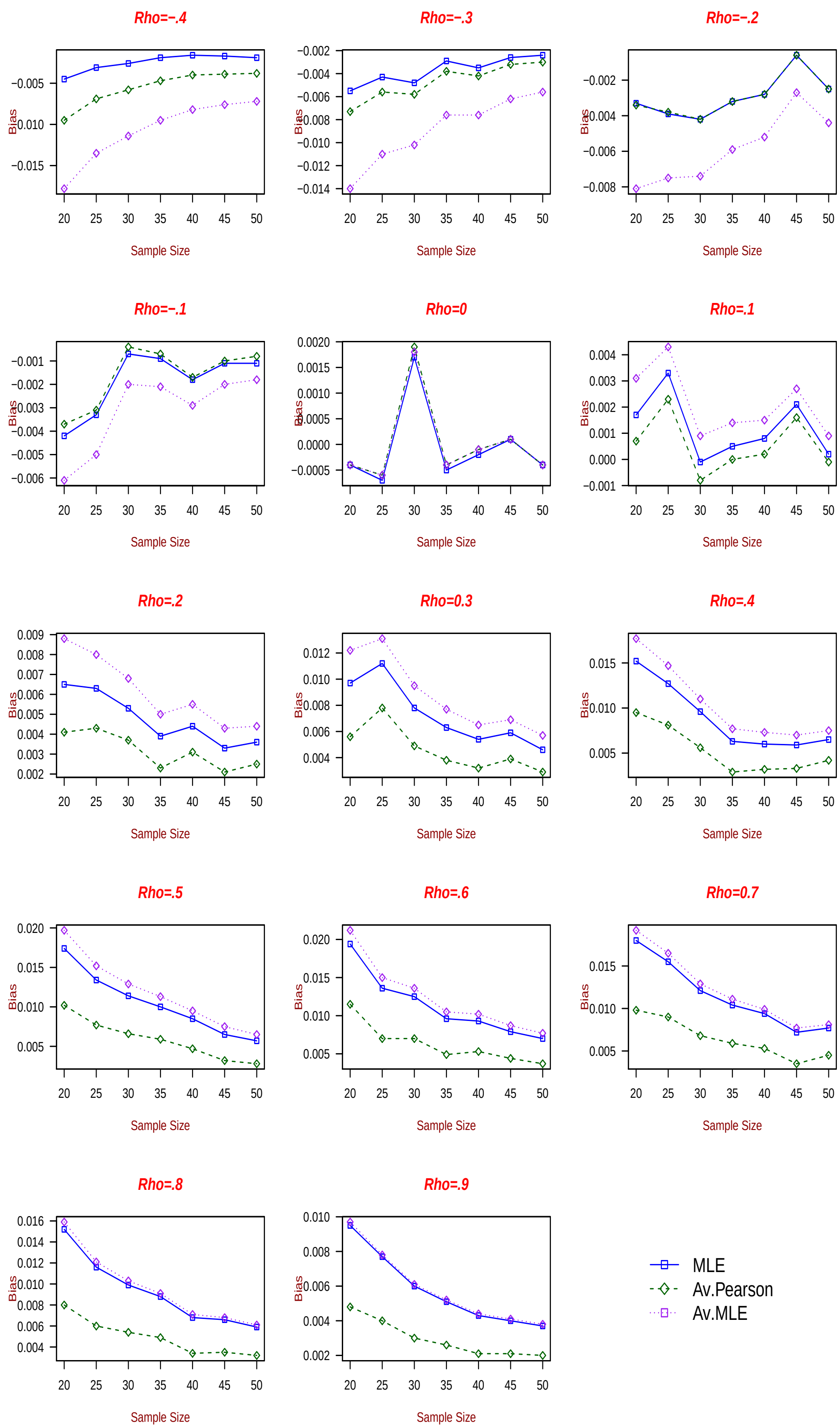

Figure 5.2: Comparison Bias Trivariate Normal Distribution

	Rho	SAMPLE SIZE						
		20	25	30	35	40	45	50
(a)	-0.4	0.0015	0.0012	0.0010	0.0008	0.0007	0.0006	0.0006
	-0.3	0.0050	0.0039	0.0032	0.0027	0.0024	0.0021	0.0019
	-0.2	0.0093	0.0073	0.0060	0.0051	0.0045	0.0040	0.0035
	-0.1	0.0134	0.0107	0.0087	0.0073	0.0066	0.0058	0.0051
	0	0.0168	0.0134	0.0114	0.0097	0.0083	0.0076	0.0067
	0.1	0.0195	0.0155	0.0130	0.0110	0.0099	0.0088	0.0075
	0.2	0.0211	0.0171	0.0146	0.0121	0.0107	0.0093	0.0085
	0.3	0.0212	0.0175	0.0144	0.0123	0.0106	0.0095	0.0086
	0.4	0.0204	0.0164	0.0134	0.0116	0.0101	0.0090	0.0080
	0.5	0.0186	0.0140	0.0119	0.0103	0.0087	0.0078	0.0068
	0.6	0.0153	0.0115	0.0095	0.0079	0.0069	0.0061	0.0056
	0.7	0.0106	0.0084	0.0065	0.0055	0.0050	0.0042	0.0037
	0.8	0.0060	0.0044	0.0037	0.0030	0.0026	0.0023	0.0020
	0.9	0.0019	0.0014	0.0011	0.0009	0.0008	0.0007	0.0006
(b)	-0.4	0.0017	0.0012	0.0010	0.0008	0.0007	0.0006	0.0006
	-0.3	0.0051	0.0040	0.0033	0.0027	0.0024	0.0021	0.0019
	-0.2	0.0093	0.0074	0.0061	0.0052	0.0045	0.0040	0.0036
	-0.1	0.0137	0.0109	0.0088	0.0074	0.0067	0.0058	0.0052
	0	0.0173	0.0138	0.0116	0.0099	0.0085	0.0077	0.0068
	0.1	0.0203	0.0161	0.0134	0.0113	0.0101	0.0089	0.0076
	0.2	0.0220	0.0177	0.0150	0.0124	0.0110	0.0095	0.0087
	0.3	0.0219	0.0181	0.0148	0.0126	0.0108	0.0097	0.0087
	0.4	0.0211	0.0169	0.0137	0.0118	0.0103	0.0091	0.0081
	0.5	0.0191	0.0143	0.0120	0.0104	0.0088	0.0078	0.0068
	0.6	0.0154	0.0116	0.0095	0.0079	0.0069	0.0061	0.0056
	0.7	0.0104	0.0082	0.0064	0.0054	0.0049	0.0041	0.0037
	0.8	0.0057	0.0042	0.0036	0.0029	0.0026	0.0023	0.0020
	0.9	0.0018	0.0014	0.0010	0.0009	0.0008	0.0007	0.0006
(c)	-0.4	0.0020	0.0014	0.0012	0.0009	0.0008	0.0007	0.0006
	-0.3	0.0051	0.0040	0.0033	0.0027	0.0024	0.0022	0.0019
	-0.2	0.0090	0.0071	0.0059	0.0051	0.0044	0.0040	0.0035
	-0.1	0.0131	0.0105	0.0086	0.0072	0.0066	0.0057	0.0051
	0	0.0165	0.0132	0.0113	0.0096	0.0083	0.0075	0.0066
	0.1	0.0193	0.0155	0.0130	0.0110	0.0099	0.0087	0.0075
	0.2	0.0210	0.0171	0.0146	0.0121	0.0107	0.0093	0.0085
	0.3	0.0212	0.0176	0.0144	0.0124	0.0106	0.0096	0.0086
	0.4	0.0207	0.0166	0.0135	0.0117	0.0102	0.0090	0.0081
	0.5	0.0190	0.0142	0.0120	0.0104	0.0087	0.0078	0.0068
	0.6	0.0156	0.0117	0.0096	0.0080	0.0070	0.0061	0.0057
	0.7	0.0108	0.0085	0.0066	0.0056	0.0050	0.0042	0.0037
	0.8	0.0061	0.0044	0.0038	0.0031	0.0026	0.0024	0.0020
	0.9	0.0020	0.0015	0.0011	0.0009	0.0008	0.0007	0.0006

Table 5.3: Comparison MSE of Estimators Trivariate

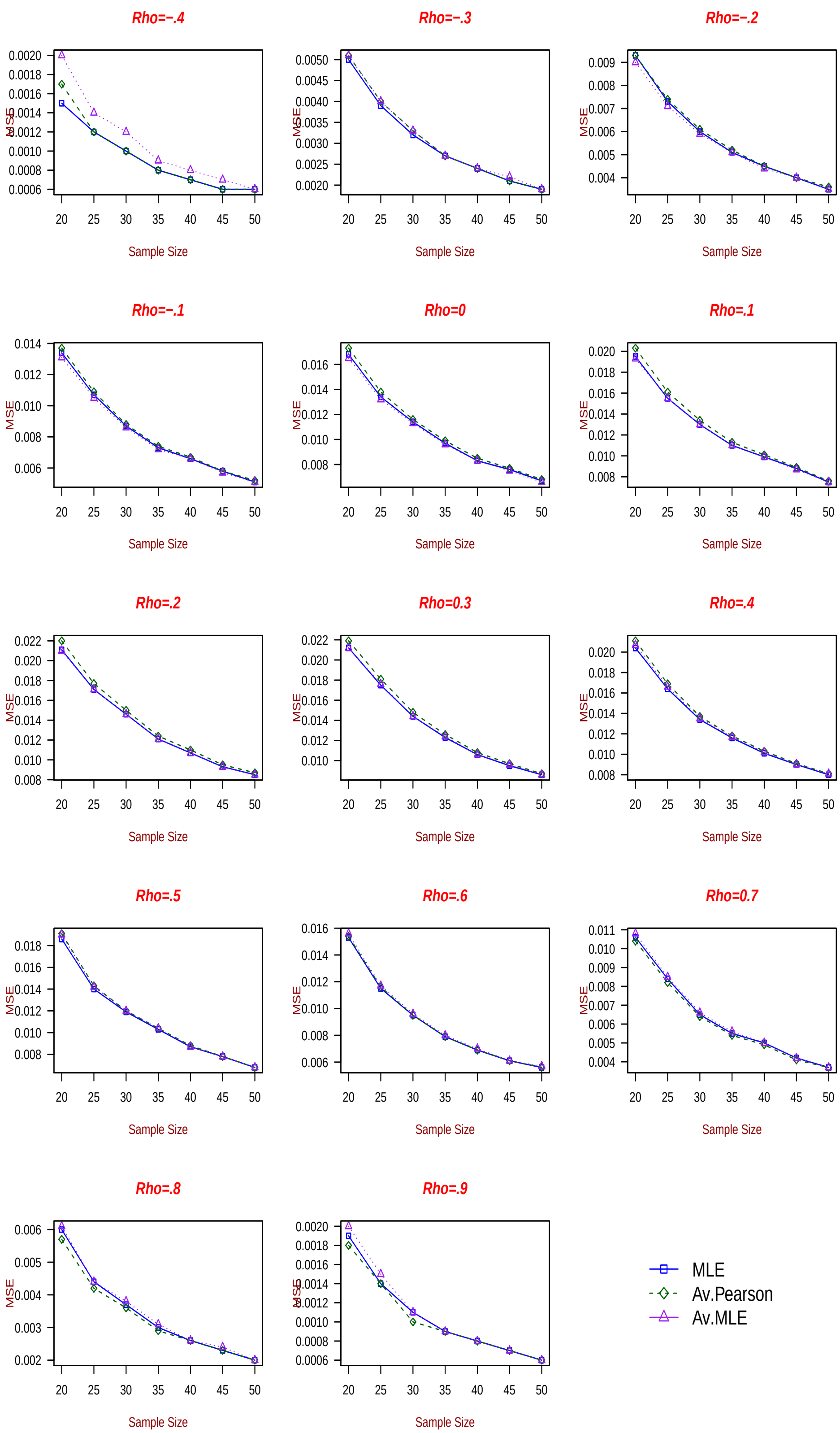

Figure 5.3: Comparison MSE Trivariate Normal Distribution

5.2 Confidence Intervals

In Section 3.3 was mentioned that

$$\hat{\rho} \sim N\bigg(\rho, \frac{(1-\rho^2)^2(1+2\rho)^2}{3n}\bigg),$$

therefore a $(1-\alpha)100\%$ confidence interval for ρ is given by

$$\hat{\rho} \pm z_{\alpha/2}\frac{(1-\hat{\rho})(1+2\hat{\rho})}{\sqrt{3n}} \tag{5.1}$$

Moreover, using the variance stabilizing transformation in Section 3.3.2 was found that

$$\frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\Big) \sim N\bigg(\frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\rho}{1-\rho}\Big), \frac{1}{n}\bigg)$$

thus, a $(1-\alpha)100\%$ confidence interval for $\frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\rho}{1-\rho}\Big)$ is expressed by

$$\frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\Big) \pm z_{\alpha/2}\frac{1}{\sqrt{n}}$$

Since our purpose is to find a confidence interval for ρ, it can be made by applying inverse transformation and solve for ρ, finding the $(1-\alpha)100\%$ given by

$$\Big(\frac{e^{\sqrt{3}l}-1}{2+e^{\sqrt{3}l}}, \frac{e^{\sqrt{3}u}-1}{2+e^{\sqrt{3}u}}\Big) \tag{5.2}$$

where

$$l = \frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\Big) - z_{\alpha/2}\frac{1}{\sqrt{n}}$$

and

$$u = \frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\Big) + z_{\alpha/2}\frac{1}{\sqrt{n}}$$

However, using the expressions in (5.1) and (5.2) the $(1-\alpha)100$ level of confidence is not achieved. For this reason, a study in the denominators of the sampling errors will be performed.

5.2.1 Study on the denominators

A 95% confidence interval for ρ is desired. For this purpose, the coverage probability of the intervals

$$\hat{\rho} \pm 1.96 \frac{(1-\hat{\rho})(1+2\hat{\rho})}{\sqrt{3(n-k)}} \tag{5.3}$$

for $k = 0, 1, 2, 3, 4, 5$ is calculated and recorded in Tables 5.4 using the MLE.

RHO	SAMPLE SIZE	DENOMINADOR					
		n	n-1	n-2	n-3	n-4	n-5
-0.4	20	0.9093	0.9153	0.9202	0.9256	0.9297	0.9339
	25	0.9204	0.9253	0.9289	0.9321	0.9359	0.9396
	30	0.9249	0.9284	0.9313	0.9343	0.9374	0.9403
	35	0.9262	0.9287	0.9320	0.9348	0.9377	0.9401
	40	0.9301	0.9324	0.9346	0.9368	0.9397	0.9423
	45	0.9329	0.9355	0.9378	0.9392	0.9408	0.9430
	50	0.9359	0.9385	0.9406	0.9428	0.9453	0.9469
-0.3	20	0.9110	0.9156	0.9205	0.9265	0.9331	0.9390
	25	0.9162	0.9201	0.9243	0.9277	0.9309	0.9355
	30	0.9279	0.9318	0.9353	0.9388	0.9412	0.9444
	35	0.9311	0.9344	0.9380	0.9407	0.9438	0.9467
	40	0.9328	0.9354	0.9385	0.9408	0.9430	0.9450
	45	0.9302	0.9330	0.9361	0.9384	0.9411	0.9434
	50	0.9350	0.9366	0.9391	0.9413	0.9433	0.9456
-0.2	20	0.9051	0.9120	0.9167	0.9219	0.9283	0.9339
	25	0.9194	0.9239	0.9281	0.9328	0.9366	0.9412
	30	0.9247	0.9292	0.9334	0.9374	0.9413	0.9460
	35	0.9250	0.9289	0.9315	0.9346	0.9367	0.9389
	40	0.9324	0.9346	0.9378	0.9408	0.9430	0.9450
	45	0.9272	0.9296	0.9317	0.9349	0.9371	0.9393
	50	0.9348	0.9372	0.9394	0.9417	0.9438	0.9458
-0.1	20	0.9106	0.9159	0.9232	0.9295	0.9354	0.9420
	25	0.9180	0.9221	0.9268	0.9317	0.9373	0.9421
	30	0.9278	0.9315	0.9357	0.9394	0.9437	0.9467
	35	0.9295	0.9322	0.9348	0.9386	0.9419	0.9458
	40	0.9304	0.9331	0.9353	0.9379	0.9408	0.9435
	45	0.9350	0.9374	0.9396	0.9420	0.9445	0.9467
	50	0.9376	0.9401	0.9423	0.9442	0.9464	0.9490
0	20	0.9149	0.9199	0.9263	0.9316	0.9365	0.9422
	25	0.9202	0.9254	0.9301	0.9345	0.9402	0.9451
	30	0.9201	0.9248	0.9285	0.9326	0.9368	0.9414
	35	0.9271	0.9323	0.9365	0.9395	0.9439	0.9472
	40	0.9340	0.9380	0.9413	0.9445	0.9470	0.9501
	45	0.9296	0.9321	0.9348	0.9374	0.9396	0.9415
	50	0.9380	0.9398	0.9425	0.9450	0.9469	0.9484
0.1	20	0.9133	0.9194	0.9257	0.9323	0.9402	0.9468
	25	0.9228	0.9276	0.9325	0.9378	0.9425	0.9470
	30	0.9271	0.9313	0.9353	0.9393	0.9432	0.9483
	35	0.9326	0.9359	0.9397	0.9430	0.9462	0.9490
	40	0.9321	0.9348	0.9371	0.9405	0.9432	0.9457
	45	0.9318	0.9341	0.9368	0.9404	0.9430	0.9456
	50	0.9408	0.9436	0.9453	0.9470	0.9501	0.9522
0.2	20	0.9184	0.9240	0.9290	0.9359	0.9399	0.9453
	25	0.9198	0.9256	0.9305	0.9354	0.9404	0.9450
	30	0.9205	0.9247	0.9288	0.9338	0.9381	0.9411
	35	0.9282	0.9313	0.9355	0.9386	0.9423	0.9459
	40	0.9311	0.9343	0.9363	0.9386	0.9423	0.9449
	45	0.9372	0.9395	0.9428	0.9459	0.9489	0.9512
	50	0.9331	0.9345	0.9369	0.9398	0.9422	0.9458

RHO	SAMPLE SIZE	DENOMINADOR					
		n	n-1	n-2	n-3	n-4	n-5
0.3	20	0.9186	0.9244	0.9303	0.9369	0.9435	0.9502
	25	0.9190	0.9238	0.9289	0.9336	0.9388	0.9433
	30	0.9209	0.9260	0.9315	0.9368	0.9413	0.9468
	35	0.9297	0.9335	0.9367	0.9401	0.9451	0.9484
	40	0.9331	0.9366	0.9405	0.9435	0.9465	0.9504
	45	0.9338	0.9360	0.9394	0.9418	0.9449	0.9472
	50	0.9342	0.9369	0.9392	0.9421	0.9437	0.9460
0.4	20	0.9196	0.9265	0.9332	0.9396	0.9474	0.9529
	25	0.9241	0.9294	0.9330	0.9382	0.9430	0.9488
	30	0.9304	0.9334	0.9370	0.9399	0.9432	0.9479
	35	0.9319	0.9349	0.9380	0.9418	0.9457	0.9497
	40	0.9324	0.9361	0.9389	0.9413	0.9429	0.9461
	45	0.9350	0.9378	0.9408	0.9433	0.9456	0.9479
	50	0.9378	0.9399	0.9424	0.9444	0.9473	0.9489
0.5	20	0.9156	0.9218	0.9289	0.9346	0.9421	0.9493
	25	0.9296	0.9337	0.9376	0.9423	0.9486	0.9538
	30	0.9282	0.9328	0.9378	0.9426	0.9475	0.9515
	35	0.9325	0.9368	0.9397	0.9434	0.9462	0.9495
	40	0.9356	0.9391	0.9425	0.9451	0.9487	0.9516
	45	0.9396	0.9417	0.9435	0.9465	0.9485	0.9511
	50	0.9436	0.9464	0.9489	0.9506	0.9530	0.9543
0.6	20	0.9178	0.9240	0.9311	0.9388	0.9444	0.9504
	25	0.9293	0.9345	0.9402	0.9442	0.9490	0.9536
	30	0.9295	0.9329	0.9379	0.9408	0.9453	0.9486
	35	0.9353	0.9388	0.9415	0.9442	0.9475	0.9507
	40	0.9401	0.9428	0.9450	0.9483	0.9497	0.9528
	45	0.9403	0.9429	0.9457	0.9481	0.9510	0.9531
	50	0.9359	0.9383	0.9415	0.9442	0.9466	0.9486
0.7	20	0.9231	0.9295	0.9349	0.9410	0.9465	0.9531
	25	0.9322	0.9371	0.9414	0.9475	0.9528	0.9571
	30	0.9374	0.9427	0.9466	0.9489	0.9525	0.9559
	35	0.9371	0.9402	0.9436	0.9478	0.9507	0.9535
	40	0.9320	0.9347	0.9376	0.9410	0.9447	0.9475
	45	0.9396	0.9425	0.9456	0.9473	0.9494	0.9517
	50	0.9456	0.9479	0.9505	0.9529	0.9550	0.9566
0.8	20	0.9272	0.9322	0.9372	0.9419	0.9474	0.9540
	25	0.9356	0.9402	0.9451	0.9500	0.9535	0.9573
	30	0.9334	0.9369	0.9405	0.9446	0.9483	0.9511
	35	0.9369	0.9402	0.9438	0.9471	0.9501	0.9528
	40	0.9382	0.9407	0.9423	0.9450	0.9481	0.9509
	45	0.9393	0.9415	0.9440	0.9460	0.9481	0.9507
	50	0.9398	0.9427	0.9452	0.9473	0.9488	0.9509
0.9	20	0.9317	0.9368	0.9414	0.9466	0.9507	0.9556
	25	0.9344	0.9394	0.9427	0.9467	0.9504	0.9551
	30	0.9439	0.9462	0.9486	0.9503	0.9540	0.9567
	35	0.9376	0.9403	0.9425	0.9460	0.9484	0.9506
	40	0.9373	0.9393	0.9416	0.9443	0.9468	0.9496
	45	0.9371	0.9403	0.9423	0.9446	0.9479	0.9505
	50	0.9388	0.9409	0.9431	0.9443	0.9466	0.9480

Table 5.4: Comparison Denominators MLE Using Normal Trivariate

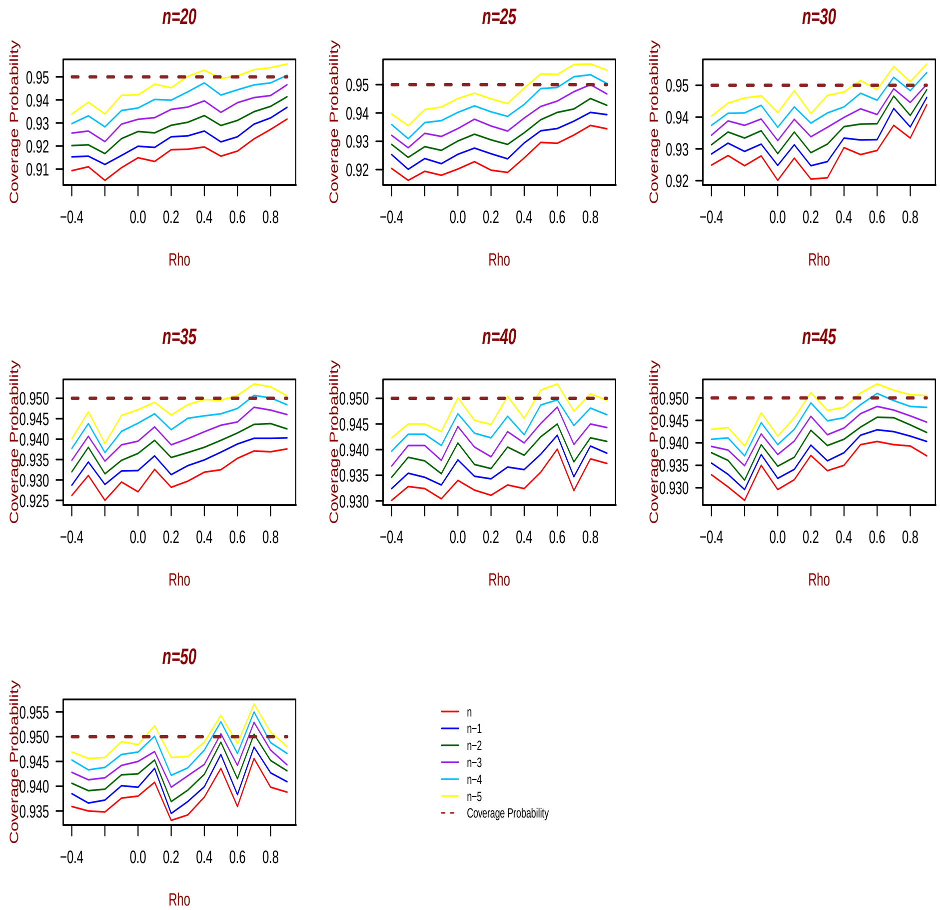

Figure 5.4: Comparison denominators using MLE in Expression 5.3

Figure 5.4 shows us that there is not an appropiate denominator to achieved the desired level of confidence when using the MLE as estimator of ρ.

For this reason, the confidence interval given in 5.1 will not be used further.

Now, using the expression in (5.2), a similar study on the denominators is performed; that is, l and u in this expression will be modified as

$$l = \frac{1}{\sqrt{3}} \ln\left(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\right) - z_{\alpha/2}\frac{1}{\sqrt{n-k}}$$

and

$$u = \frac{1}{\sqrt{3}} \ln\left(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\right) + z_{\alpha/2}\frac{1}{\sqrt{n-k}}$$

for k ranging from 0 to five, and finally, an inverse transformation is apply to find the 95% confidence inteval for ρ when $\hat{\rho}$ is the MLE.

		DENOMINADOR								DENOMINADOR					
RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5	RHO	SAMPLE SIZE	n	n-1	n-2	n-3	n-4	n-5
	20	0.9385	0.9451	0.9506	0.9562	0.9616	0.9684		20	0.9384	0.9455	0.9513	0.9571	0.9628	0.9675
	25	0.9452	0.9490	0.9536	0.9591	0.9631	0.9675		25	0.9357	0.9393	0.9441	0.9498	0.9560	0.9613
	30	0.9437	0.9476	0.9515	0.9562	0.9600	0.9639		30	0.9387	0.9433	0.9477	0.9525	0.9574	0.9617
-0.4	35	0.9470	0.9508	0.9536	0.9572	0.9602	0.9632	0.3	35	0.9425	0.9457	0.9501	0.9529	0.9558	0.9592
	40	0.9484	0.9505	0.9535	0.9557	0.9587	0.9616		40	0.9447	0.9479	0.9516	0.9547	0.9585	0.9611
	45	0.9446	0.9479	0.9501	0.9527	0.9571	0.9592		45	0.9436	0.9457	0.9483	0.9510	0.9537	0.9566
	50	0.9461	0.9475	0.9500	0.9527	0.9550	0.9585		50	0.9431	0.9457	0.9472	0.9492	0.9512	0.9543
	20	0.9396	0.9457	0.9516	0.9582	0.9641	0.9701		20	0.9407	0.9465	0.9515	0.9585	0.9630	0.9694
	25	0.9382	0.9437	0.9484	0.9528	0.9584	0.9615		25	0.9388	0.9443	0.9493	0.9539	0.9584	0.9625
	30	0.9425	0.9467	0.9512	0.9551	0.9594	0.9631		30	0.9417	0.9454	0.9493	0.9516	0.9551	0.9595
-0.3	35	0.9484	0.9519	0.9554	0.9586	0.9613	0.9641	0.4	35	0.9445	0.9485	0.9514	0.9545	0.9583	0.9616
	40	0.9452	0.9481	0.9516	0.9544	0.9578	0.9608		40	0.9410	0.9442	0.9475	0.9506	0.9530	0.9556
	45	0.9450	0.9468	0.9490	0.9523	0.9544	0.9572		45	0.9429	0.9454	0.9480	0.9510	0.9532	0.9555
	50	0.9445	0.9463	0.9486	0.9511	0.9527	0.9549		50	0.9444	0.9466	0.9484	0.9509	0.9534	0.9561
	20	0.9315	0.9385	0.9453	0.9516	0.9577	0.9648		20	0.9351	0.9413	0.9471	0.9534	0.9597	0.9655
	25	0.9378	0.9423	0.9485	0.9529	0.9587	0.9639		25	0.9427	0.9481	0.9533	0.9564	0.9613	0.9667
	30	0.9445	0.9491	0.9531	0.9574	0.9612	0.9641		30	0.9410	0.9453	0.9496	0.9540	0.9576	0.9613
-0.2	35	0.9408	0.9443	0.9476	0.9509	0.9538	0.9573	0.5	35	0.9426	0.9457	0.9488	0.9520	0.9549	0.9581
	40	0.9432	0.9468	0.9496	0.9519	0.9558	0.9581		40	0.9455	0.9481	0.9510	0.9541	0.9562	0.9589
	45	0.9387	0.9414	0.9443	0.9477	0.9498	0.9529		45	0.9470	0.9499	0.9526	0.9546	0.9584	0.9602
	50	0.9448	0.9470	0.9487	0.9516	0.9533	0.9553		50	0.9469	0.9495	0.9522	0.9550	0.9572	0.9590
	20	0.9378	0.9445	0.9503	0.9571	0.9616	0.9685		20	0.9337	0.9400	0.9460	0.9510	0.9584	0.9649
	25	0.9416	0.9462	0.9503	0.9549	0.9593	0.9633		25	0.9406	0.9462	0.9514	0.9572	0.9616	0.9665
	30	0.9433	0.9471	0.9513	0.9550	0.9580	0.9618		30	0.9415	0.9455	0.9498	0.9534	0.9575	0.9617
-0.1	35	0.9470	0.9502	0.9531	0.9563	0.9597	0.9633	0.6	35	0.9426	0.9465	0.9496	0.9530	0.9562	0.9591
	40	0.9420	0.9455	0.9490	0.9526	0.9554	0.9587		40	0.9460	0.9476	0.9517	0.9541	0.9571	0.9603
	45	0.9440	0.9473	0.9493	0.9516	0.9552	0.9588		45	0.9459	0.9486	0.9508	0.9539	0.9568	0.9597
	50	0.9506	0.9522	0.9544	0.9571	0.9593	0.9612		50	0.9443	0.9467	0.9488	0.9517	0.9537	0.9552
	20	0.9399	0.9451	0.9505	0.9569	0.9631	0.9671		20	0.9382	0.9445	0.9516	0.9565	0.9624	0.9680
	25	0.9407	0.9462	0.9526	0.9584	0.9621	0.9658		25	0.9416	0.9456	0.9500	0.9544	0.9590	0.9645
	30	0.9391	0.9434	0.9484	0.9527	0.9580	0.9622		30	0.9436	0.9484	0.9523	0.9563	0.9592	0.9624
0	35	0.9418	0.9450	0.9483	0.9525	0.9574	0.9601	0.7	35	0.9419	0.9457	0.9490	0.9533	0.9565	0.9598
	40	0.9480	0.9511	0.9541	0.9571	0.9593	0.9620		40	0.9378	0.9410	0.9447	0.9486	0.9518	0.9553
	45	0.9421	0.9437	0.9465	0.9483	0.9511	0.9536		45	0.9459	0.9482	0.9506	0.9537	0.9568	0.9591
	50	0.9478	0.9494	0.9520	0.9544	0.9569	0.9592		50	0.9497	0.9515	0.9538	0.9566	0.9589	0.9606
	20	0.9384	0.9436	0.9493	0.9561	0.9611	0.9662		20	0.9374	0.9443	0.9502	0.9555	0.9608	0.9665
	25	0.9421	0.9470	0.9526	0.9566	0.9607	0.9653		25	0.9435	0.9480	0.9523	0.9563	0.9613	0.9653
	30	0.9435	0.9472	0.9505	0.9540	0.9578	0.9614		30	0.9416	0.9454	0.9491	0.9530	0.9568	0.9600
0.1	35	0.9462	0.9494	0.9534	0.9563	0.9589	0.9617	0.8	35	0.9432	0.9463	0.9498	0.9532	0.9563	0.9599
	40	0.9440	0.9465	0.9497	0.9514	0.9542	0.9569		40	0.9443	0.9472	0.9498	0.9527	0.9562	0.9592
	45	0.9436	0.9466	0.9491	0.9515	0.9544	0.9569		45	0.9387	0.9421	0.9447	0.9475	0.9499	0.9536
	50	0.9493	0.9514	0.9543	0.9561	0.9590	0.9613		50	0.9452	0.9473	0.9491	0.9513	0.9539	0.9554
	20	0.9374	0.9422	0.9487	0.9554	0.9602	0.9654		20	0.9375	0.9442	0.9504	0.9569	0.9622	0.9667
	25	0.9393	0.9442	0.9481	0.9536	0.9588	0.9626		25	0.9408	0.9459	0.9516	0.9568	0.9612	0.9650
	30	0.9351	0.9388	0.9442	0.9491	0.9520	0.9561		30	0.9447	0.9487	0.9523	0.9560	0.9594	0.9636
0.2	35	0.9419	0.9454	0.9480	0.9508	0.9540	0.9573	0.9	35	0.9391	0.9429	0.9463	0.9495	0.9536	0.9574
	40	0.9406	0.9439	0.9475	0.9502	0.9527	0.9560		40	0.9408	0.9440	0.9469	0.9495	0.9526	0.9567
	45	0.9484	0.9508	0.9535	0.9550	0.9578	0.9607		45	0.9419	0.9453	0.9478	0.9502	0.9527	0.9552
	50	0.9419	0.9444	0.9464	0.9503	0.9528	0.9551		50	0.9397	0.9418	0.9444	0.9458	0.9485	0.9514

Table 5.5: Comparison Denominators MLE Using Variance Stabilizing Transformation Trivariate

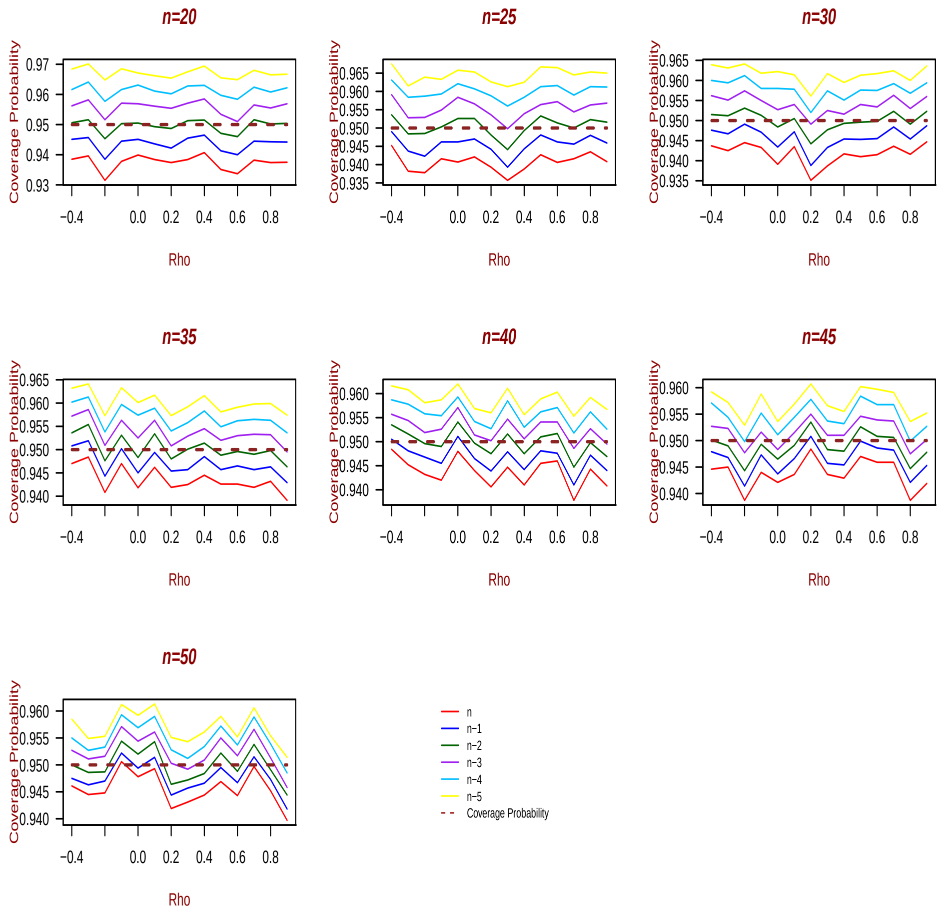

Figure 5.5: Comparison denominator MLE using variance stabilizing transformation

The coverage probability of the confidence interval using the MLE is presented in Table 5.5 and Figure 5.5.

It can be notice is Figure 5.5 that $n-2$ is the best denominator to achieve the desired level of confidence. Therefore, it can be conclude that

$$\frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\hat{\rho}}{1-\hat{\rho}}\Big) \sim N\left(\frac{1}{\sqrt{3}}\ln\Big(\frac{1+2\rho}{1-\rho}\Big), \frac{1}{n-2}\right) \tag{5.4}$$

is the more appropiate when using the MLE.

5.2.2 Confidence Intervals

Once we have the more appropiate distributions of $\frac{1}{\sqrt{3}}\ln\left(\frac{1+2\hat{\rho}_{AP}}{1-\hat{\rho}_{AP}}\right)$ when using the MLE, a confidence interval for ρ can be found.

In this section the coverage probability of the confidence intervals is made. Moreover, the average width of those intervals are also measured. The results are presented in Tables 5.6 and 5.7, respectively.

Rho	SAMPLE SIZE						
	20	25	30	35	40	45	50
-0.4	0.9506	0.9536	0.9515	0.9536	0.9535	0.9501	0.9500
-0.3	0.9516	0.9484	0.9512	0.9554	0.9516	0.9490	0.9486
-0.2	0.9453	0.9485	0.9531	0.9476	0.9496	0.9443	0.9487
-0.1	0.9503	0.9503	0.9513	0.9531	0.9490	0.9493	0.9544
0	0.9505	0.9526	0.9484	0.9483	0.9541	0.9465	0.9520
0.1	0.9493	0.9526	0.9505	0.9534	0.9497	0.9491	0.9543
0.2	0.9487	0.9481	0.9442	0.9480	0.9475	0.9535	0.9464
0.3	0.9513	0.9441	0.9477	0.9501	0.9516	0.9483	0.9472
0.4	0.9515	0.9493	0.9493	0.9514	0.9475	0.9480	0.9484
0.5	0.9471	0.9533	0.9496	0.9488	0.9510	0.9526	0.9522
0.6	0.9460	0.9514	0.9498	0.9496	0.9517	0.9508	0.9488
0.7	0.9516	0.9500	0.9523	0.9490	0.9447	0.9506	0.9538
0.8	0.9502	0.9523	0.9491	0.9498	0.9498	0.9447	0.9491
0.9	0.9504	0.9516	0.9523	0.9463	0.9469	0.9478	0.9444

Table 5.6: Coverage Probability Confidence Interval for ρ Trivariate

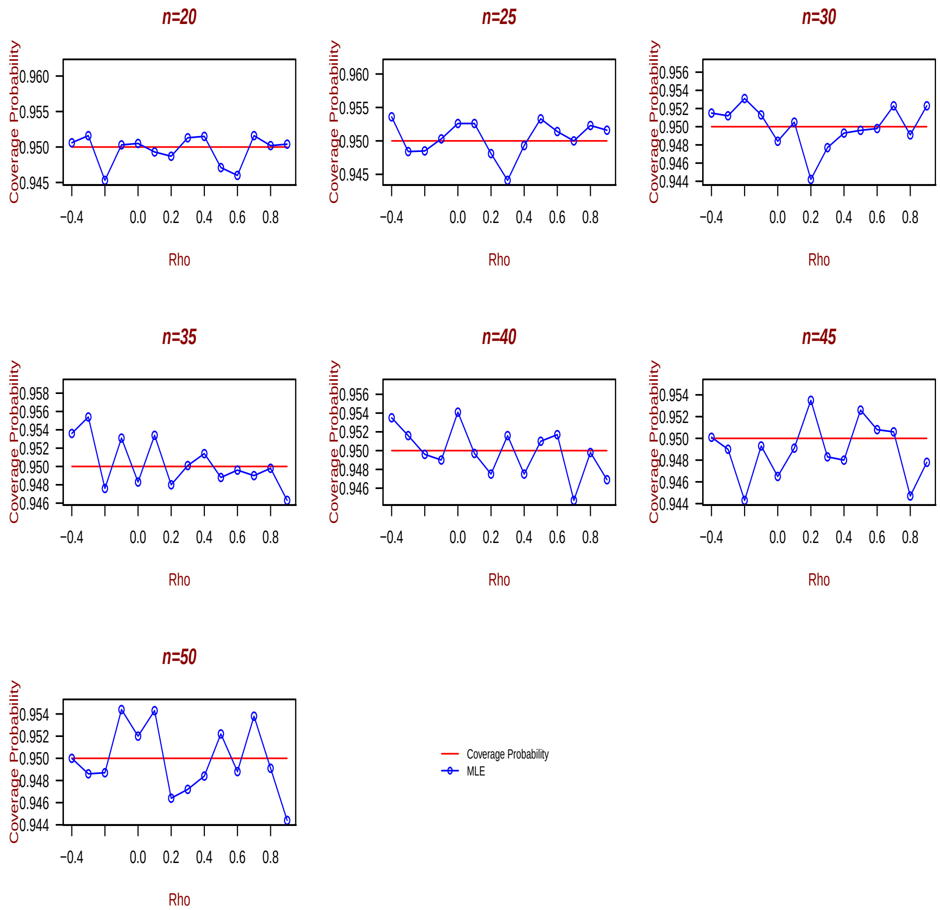

Figure 5.6: Comparison Coverage Probability of CI using Fisher

In Figure 5.6 it can be observed that the coverage probability of the confidence intervals are close to 0.95.

Rho	SAMPLE SIZE						
	20	25	30	35	40	45	50
-0.4	0.1631	0.1413	0.1267	0.1155	0.1070	0.1004	0.0949
-0.3	0.2853	0.2511	0.2279	0.2084	0.1945	0.1821	0.1722
-0.2	0.3770	0.3361	0.3061	0.2820	0.2631	0.2463	0.2345
-0.1	0.4492	0.4011	0.3648	0.3379	0.3162	0.2979	0.2826
0	0.4990	0.4477	0.4083	0.3794	0.3553	0.3350	0.3183
0.1	0.5322	0.4779	0.4384	0.4068	0.3810	0.3595	0.3421
0.2	0.5486	0.4934	0.4520	0.4201	0.3938	0.3722	0.3534
0.3	0.5500	0.4945	0.4532	0.4207	0.3945	0.3726	0.3539
0.4	0.5361	0.4812	0.4403	0.4079	0.3823	0.3609	0.3430
0.5	0.5052	0.4525	0.4131	0.3824	0.3579	0.3370	0.3198
0.6	0.4574	0.4066	0.3707	0.3419	0.3197	0.3008	0.2848
0.7	0.3885	0.3443	0.3113	0.2866	0.2668	0.2501	0.2373
0.8	0.2950	0.2580	0.2322	0.2131	0.1970	0.1849	0.1746
0.9	0.1697	0.1466	0.1303	0.1187	0.1095	0.1024	0.0965

Table 5.7: Average Width of Confidence Interval for ρ Trivariate

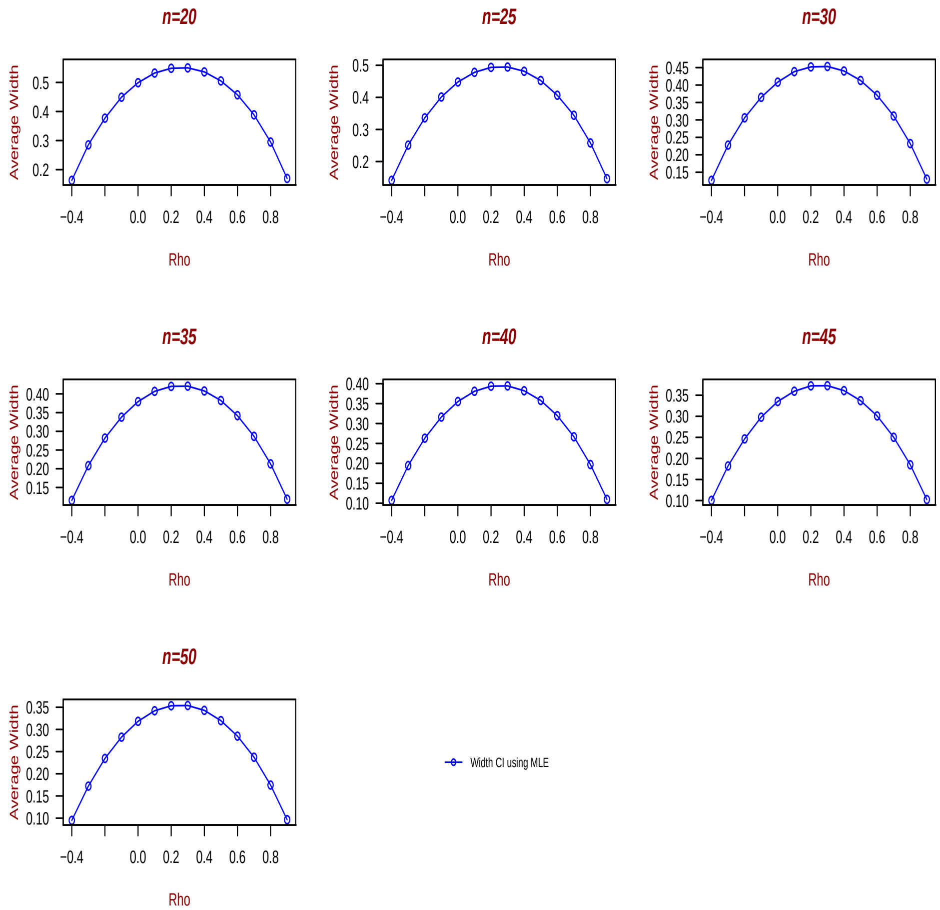

Figure 5.7: Comparison Width 95% CI using Fisher

Figure 5.7 shows that the average width of the confidence intervals of ρ is small when rho approaches to the extreme values.

5.3 Test of Hypothesis

5.3.1 Probability Type I error

A comparison of the probability of type I error for testing

$$H_0 : \rho = \rho_0 \quad vs. \quad Ha : \rho \neq \rho_0 \tag{5.5}$$

where ρ_0 is the true value of ρ are made using the MLE in Table 5.8 using a significance level of $\alpha = 0.05$

Rho	SAMPLE SIZE						
	20	25	30	35	40	45	50
-0.4	0.0494	0.0464	0.0485	0.0464	0.0465	0.0499	0.0500
-0.3	0.0484	0.0516	0.0488	0.0446	0.0484	0.0510	0.0514
-0.2	0.0547	0.0515	0.0469	0.0524	0.0504	0.0557	0.0513
-0.1	0.0497	0.0497	0.0487	0.0469	0.0510	0.0507	0.0456
0	0.0495	0.0474	0.0516	0.0517	0.0459	0.0535	0.0480
0.1	0.0507	0.0474	0.0495	0.0466	0.0503	0.0509	0.0457
0.2	0.0513	0.0519	0.0558	0.0520	0.0525	0.0465	0.0536
0.3	0.0487	0.0559	0.0523	0.0499	0.0484	0.0517	0.0528
0.4	0.0485	0.0507	0.0507	0.0486	0.0525	0.0520	0.0516
0.5	0.0529	0.0467	0.0504	0.0512	0.0490	0.0474	0.0478
0.6	0.0540	0.0486	0.0502	0.0504	0.0483	0.0492	0.0512
0.7	0.0484	0.0500	0.0477	0.0510	0.0553	0.0494	0.0462
0.8	0.0498	0.0477	0.0509	0.0502	0.0502	0.0553	0.0509
0.9	0.0496	0.0484	0.0477	0.0537	0.0531	0.0522	0.0556

Table 5.8: Type I error Trivariate

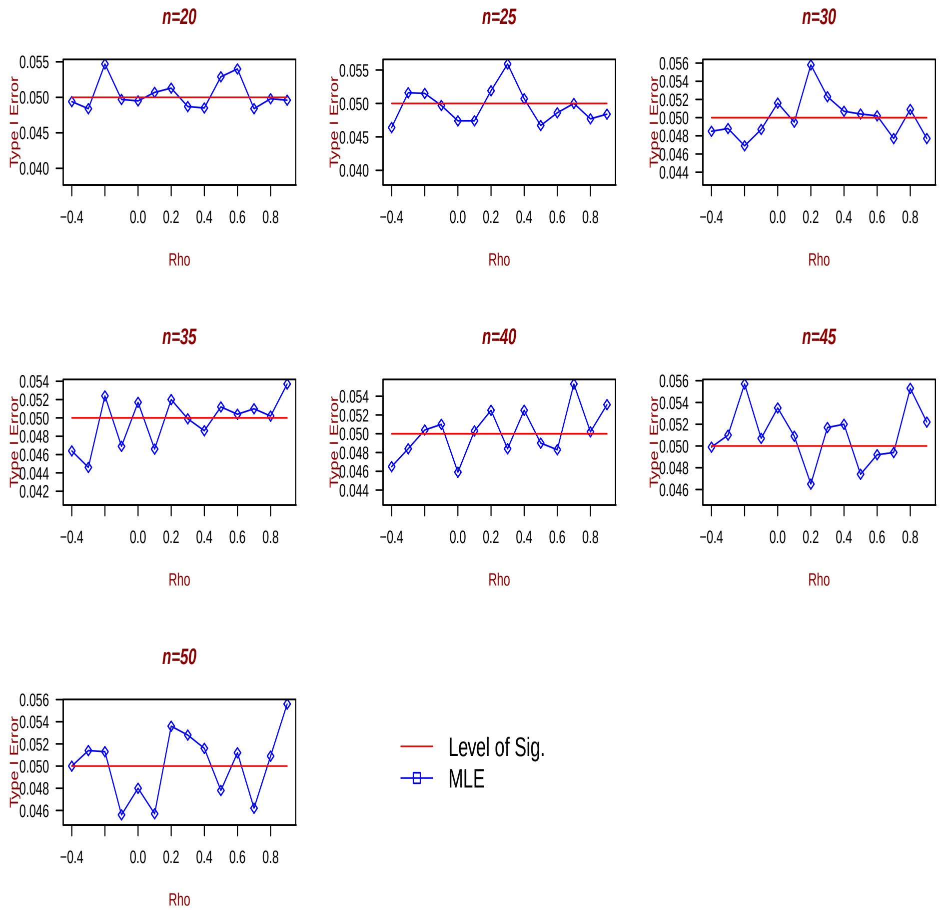

Figure 5.8: Comparison Type I error

Figure 5.8 shows that the probability of the Type I error committed for the MLE is around 0.05, ranging between 0.044 and 0.056.

5.3.2 Power of the test

In order to calculate the power when testing the hypotheses

$$H_0 : \rho = \rho_0 \quad vs. \quad H_a : \rho = \rho_1$$

when ρ_0 is the true value of ρ and $\rho_1 \neq \rho_0$ different simulations are conducted. Assuming that $\rho = \rho_0$ take the values from -0.3 to 0.9 in 0.1 increments, the values for ρ_1 in the alternative hypothesis were chosen as $\rho_1 = \rho_0 \pm 0.10$. Table 5.9 presents the values of the power using the MLE.

	RHO	RHO_1	20	25	30	35	40	45	50
(a)	-0.4	-0.3	0.4731	0.5837	0.6673	0.7357	0.7982	0.8397	0.8725
	-0.3	-0.4	0.4468	0.5508	0.6460	0.7100	0.7787	0.8189	0.8575
		-0.2	0.2334	0.2836	0.3205	0.3740	0.4114	0.4648	0.5067
	-0.2	-0.3	0.1966	0.2502	0.3126	0.3547	0.4011	0.4334	0.4839
		-0.1	0.1653	0.1876	0.2166	0.2431	0.2761	0.3112	0.3284
	-0.1	-0.2	0.1386	0.1681	0.1871	0.2205	0.2545	0.2811	0.3089
		0	0.1336	0.1552	0.1749	0.1928	0.2144	0.2368	0.2553
	0	-0.1	0.1072	0.1286	0.1490	0.1721	0.1925	0.2138	0.2356
		0.1	0.1165	0.1312	0.1566	0.1729	0.1837	0.2022	0.2200
	0.1	0	0.0945	0.1129	0.1345	0.1496	0.1689	0.1818	0.1996
		0.2	0.1100	0.1261	0.1342	0.1492	0.1705	0.1899	0.1903
	0.2	0.1	0.0888	0.1057	0.1239	0.1367	0.1566	0.1657	0.1803
		0.3	0.1114	0.1273	0.1400	0.1517	0.1697	0.1838	0.1988
	0.3	0.2	0.0854	0.1021	0.1186	0.1365	0.1469	0.1614	0.1802
		0.4	0.1091	0.1324	0.1427	0.1596	0.1719	0.1873	0.2016
	0.4	0.3	0.0852	0.1047	0.1174	0.1432	0.1529	0.1677	0.1808
		0.5	0.1171	0.1383	0.1502	0.1632	0.1829	0.2026	0.2219
	0.5	0.4	0.0959	0.1075	0.1325	0.1490	0.1629	0.1891	0.1994
		0.6	0.1380	0.1517	0.1737	0.1985	0.2127	0.2369	0.2532
	0.6	0.5	0.1071	0.1263	0.1464	0.1701	0.1913	0.2155	0.2411
		0.7	0.1654	0.1900	0.2181	0.2424	0.2773	0.3060	0.3346
	0.7	0.6	0.1369	0.1628	0.1885	0.2170	0.2600	0.2834	0.3064
		0.8	0.2366	0.2896	0.3303	0.3785	0.4270	0.4553	0.5142
	0.8	0.7	0.1982	0.2492	0.3073	0.3453	0.3992	0.4398	0.4764
		0.9	0.4867	0.5860	0.6599	0.7397	0.7919	0.8362	0.8786
	0.9	0.8	0.4526	0.5460	0.6395	0.7078	0.7698	0.8179	0.8533

Table 5.9: Comparison Power when $Ha : \rho_1 = \rho_0 \pm 0.10$ using MLE Trivariate

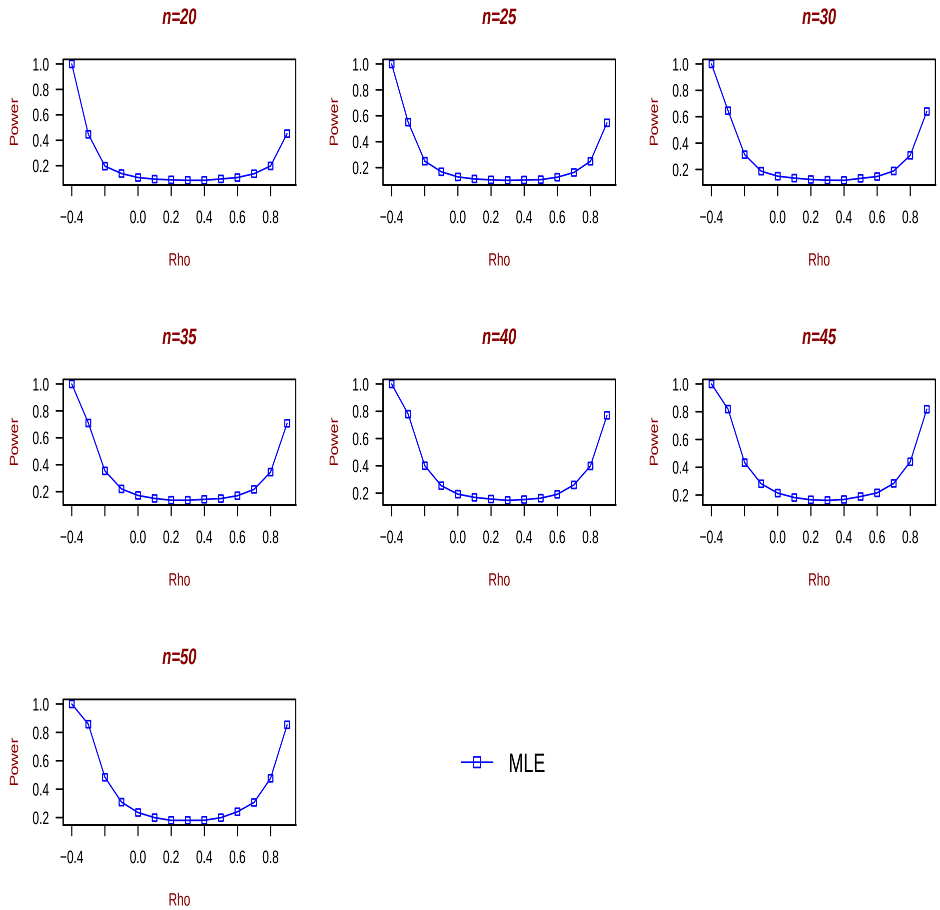

Figure 5.9: Comparison Power when Ha: $\rho_1 = \rho_0 - .10$

In Figure 5.9 we can observed that there the power of testing is higher when ρ approaches to its extreme values and increases as the sample size increases.

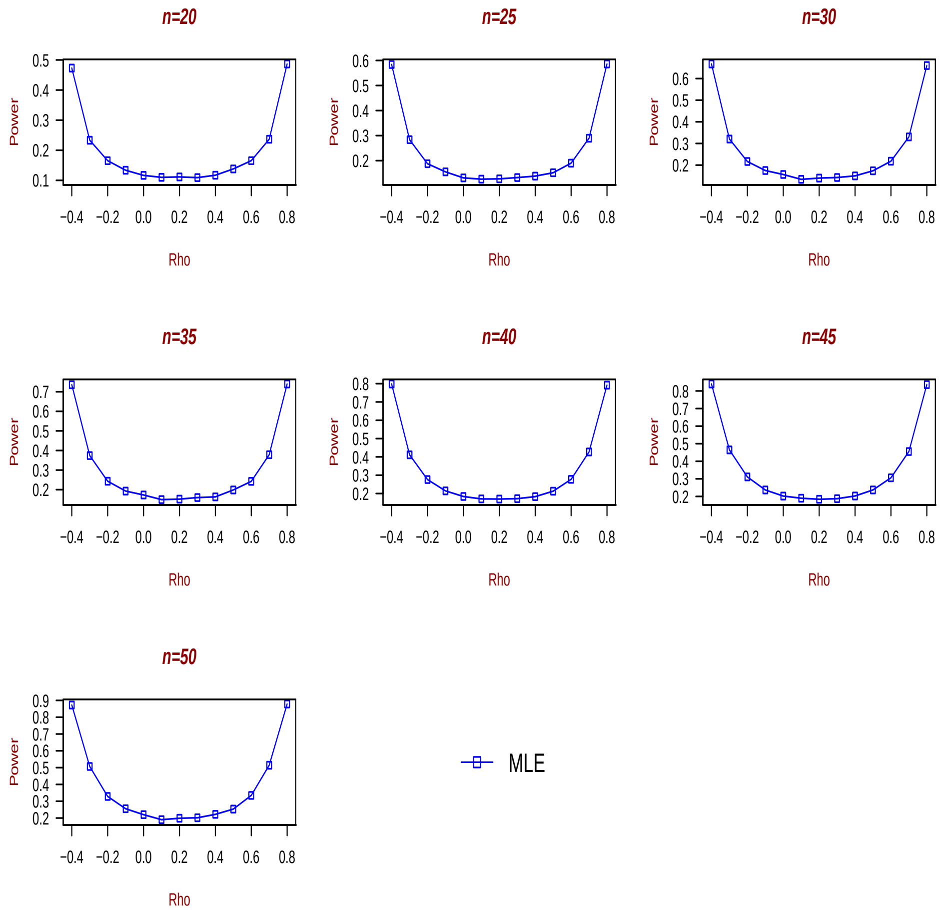

Figure 5.10: Comparison Power when $Ha : \rho_1 = \rho_0 + .10$

Figure 5.10 occurs the same behaviour as before, that is that the power of testing is higher when ρ approaches to its extreme values and increases as the sample size increases.

	RHO	RHO_1	20	25	30	35	40	45	50
(a)	-0.4	-0.45	0.4222	0.5063	0.5929	0.6600	0.7192	0.7795	0.8235
		-0.35	0.1983	0.2400	0.2818	0.3268	0.3677	0.3963	0.4315
	-0.3	-0.35	0.1105	0.1308	0.1563	0.1722	0.2028	0.2256	0.2441
		-0.25	0.1053	0.1228	0.1237	0.1462	0.1561	0.1764	0.1880
	-0.2	-0.25	0.0724	0.0835	0.0933	0.1082	0.1161	0.1191	0.1355
		-0.15	0.0905	0.0901	0.0992	0.1061	0.1154	0.1329	0.1283
	-0.1	-0.15	0.0623	0.0699	0.0726	0.0782	0.0902	0.0920	0.0986
		-0.05	0.0755	0.0826	0.0815	0.0881	0.0949	0.1015	0.1054
	0	-0.05	0.0565	0.0642	0.0720	0.0749	0.0784	0.0834	0.0875
		0.05	0.0702	0.0755	0.0842	0.0859	0.0839	0.0959	0.0949
	0.1	0.05	0.0557	0.0585	0.0681	0.0676	0.0724	0.0760	0.0779
		0.15	0.0686	0.0725	0.0733	0.0759	0.0843	0.0868	0.0839
	0.2	0.15	0.0544	0.0594	0.0687	0.0664	0.0724	0.0714	0.0781
		0.25	0.0661	0.0733	0.0791	0.0816	0.0837	0.0850	0.0959
	0.3	0.25	0.0555	0.0603	0.0648	0.0691	0.0700	0.0729	0.0756
		0.35	0.0653	0.0760	0.0787	0.0792	0.0852	0.0881	0.0927
	0.4	0.35	0.0538	0.0591	0.0625	0.0713	0.0739	0.0769	0.0795
		0.45	0.0685	0.0746	0.0770	0.0817	0.0826	0.0898	0.0955
	0.5	0.45	0.0613	0.0592	0.0681	0.0711	0.0745	0.0815	0.0804
		0.55	0.0726	0.0744	0.0805	0.0934	0.0912	0.0968	0.1003
	0.6	0.55	0.0636	0.0653	0.0738	0.0774	0.0780	0.0855	0.0998
		0.65	0.0803	0.0825	0.0910	0.0933	0.1066	0.1075	0.1190
	0.7	0.65	0.0688	0.0718	0.0772	0.0879	0.0987	0.1072	0.1086
		0.75	0.0927	0.1096	0.1130	0.1201	0.1359	0.1419	0.1563
	0.8	0.75	0.0831	0.0965	0.1153	0.1295	0.1452	0.1602	0.1676
		0.85	0.1299	0.1566	0.1790	0.2014	0.2199	0.2455	0.2652
	0.9	0.85	0.1706	0.2133	0.2541	0.2972	0.3428	0.3766	0.4114
		0.95	0.4387	0.5356	0.6184	0.6910	0.7438	0.7969	0.8380

Table 5.10: Comparison Power when $Ha : \rho = \rho \pm 0.05$ using MLE Trivariate

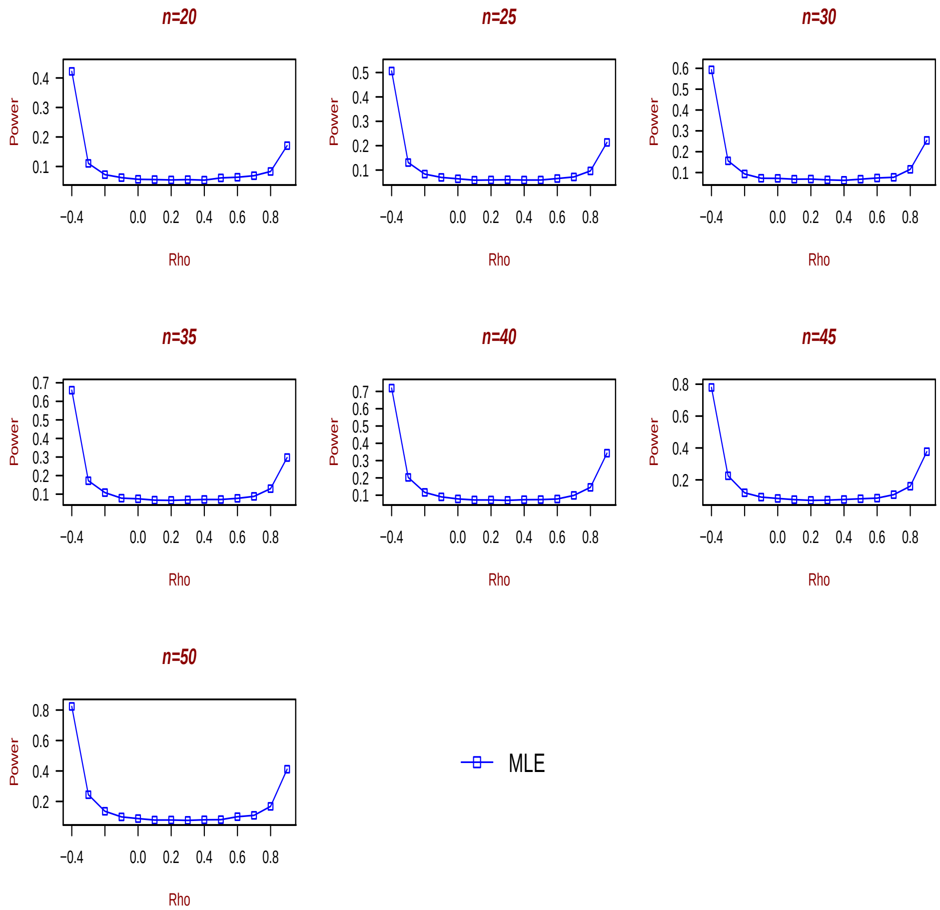

Figure 5.11: Comparison Power when $Ha : \rho_1 = \rho_0 - .05$

It can be noticed in Figure 5.11 that the higher power is obtain when $\rho = -0.40$.

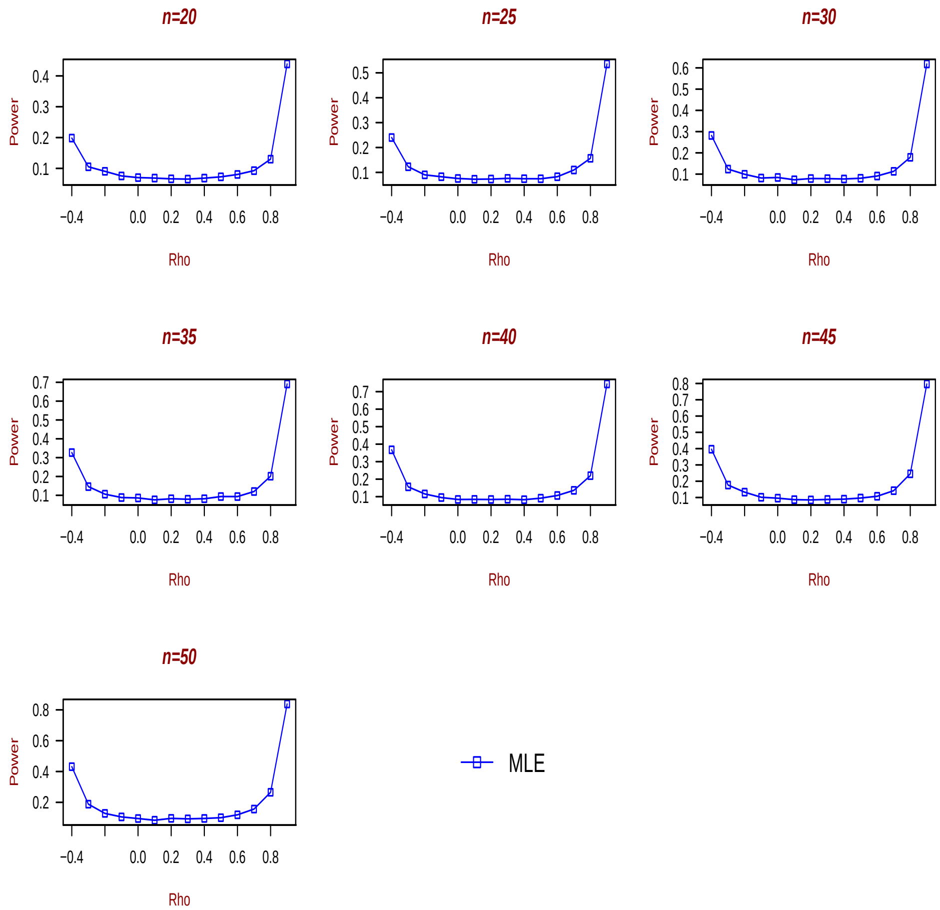

Figure 5.12: Comparison Power when $Ha : \rho_1 = \rho_0 + .05$

In Figure 5.12 we can see that the higher power is achieved when $\rho = 0.9$.

5.4 Conclusion

It can be concluded from the numerical study that, the MLE is a good estimator of ρ in both cases, bivariate and trivariate normal distribution with equal variances and convariances. Moreover, the MLE would generally lead to better performance in the following aspects: higher power of testing and narrower confidence intervals, while keeping the coverage probabilities and type I errors close to the desired level.

BIBLIOGRAPHY

Y. Sun 2007 and A.C.M. Wong. Interval estimation for the normal correlation coefficient. *Statistics And Probability Letters*, 77(17):1652 – 1661, 2007. ISSN 0167-7152. doi: http://dx.doi.org/10.1016/j.spl.2007.04.004.

Brian E. Clauser, 2007. The life and labors of francis galton: A review of four recent books about the father of behavioral statistics. *Journal of Educational and Behavioral Statistics*, 32(4):pp. 440–444, 2007. ISSN 10769986.

R. A. Fisher, 1921. On the probable error of a coefficient of correlation deduced from a small sample. *Metron*, 1:3–32, 1921.

Ronald Aylmer. Fisher, 2009. Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika*, 10(4): 507–521, 1915.

M. Goria, 1980. On testing the correlation coefficient of a bivariate normal distribution. *Metrika*, 27(1):189–194, 1980.

Harold Hotelling, 1953. New light on the correlation coefficient and its transforms. *Journal of the Royal Statistical Society. Series B (Methodological)*, 15(2):pp. 193–232, 1953. ISSN 00359246.

Elias Masry, 2011. The estimation of the correlation coefficient of bivariate data under dependence: Convergence analysis. *Statistics And Probability Letters*, 81 (8):1039–1045, 2011.

Jie. Mie. *Private communication.*

S. R. Paul, 1988. Estimation of and testing significance for a common correlation coefficient. *Communications in Statistics - Theory and Methods*, 17(1):pp. 39–53, 1988. ISSN 03610926.

Harold Ruben, 1966. Some new results on the distribution of the sample correlation coefficient. *Journal of the Royal Statistical Society. Series B (Methodological)*, 28(3):pp. 513–525, 1966. ISSN 00359246. URL http://www.jstor.org/stable/2984445.