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On Functional Central Limit Theorems for Dependent, Heterogeneous Tail Arrays with Applications to Tail Index and Tail Dependence Estimators

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Abstract

We establish functional central limit theorems for a broad class of dependent, heterogeneous tail arrays encountered in the extreme value literature, including extremal exceedances, tail empirical processes and tail empirical quantile processes. We trim dependence assumptions down to a minimum by constructing extremal versions of mixing and Near-Epoch-Dependence properties, covering mixing, ARFIMA, FIGARCH, bilinear, random coefficient autoregressive, nonlinear distributed lag and Extremal Threshold processes, and stochastic recurrence equations.

Of practical importance our theory can be used to characterize the functional limit distributions of sample means and covariances of tail arrays, including popular tail index estimators, the tail quantile function, and multivariate extremal dependence measures under substantially general conditions.

1. INTRODUCTION This paper presents functional central limit theorems of the form

$$\sum_{t=1}^{n(\xi)} X_{n,t}(u) \Rightarrow X(\xi, u), \text{ where } \xi, u \in [0,1],$$

for dependent, heterogenous *tail arrays* $\{X_{n,t}(u)\}$ in $D[0,1]$, with $X(\xi, u)$ a Gaussian process. $n(\xi)$ denotes a non-decreasing integer sequence, $n(\xi) \to \infty$

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as $n \to \infty$. We require the functional array $X_{n,t}(u)$ to be approximable by an "extremal" martingale difference array, and to satisfy

(1) $\quad \sup_t \sup_{u \geq 1}(E|X_{n,t}(u)|^r)^{1/r} = O(n^{-a(r)})$

for some non-stochastic function $a(r) \in (0, 1/2]$ and all $r \geq 1$. For non-stochastic sequences $b_{n(\xi)} \to \infty$ and $k_n(\xi) \to \infty$ property (1) covers for all $r \geq 1$ extreme threshold exceedances $\max\{\ln X_t/b_{n(\xi)}, 0\}$, tail empirical processes $1/k_n(\xi)^{1/2} \sum_{t=1}^{n(\xi)} I(X_t > b_{n(\xi)}u)$, and tail empirical quantile processes $X_{([k_n(\xi)])} \geq X_{([k_n(\xi)]+1)} \geq ...$. See Section 3.

Tail array sums are critical for risk management, damage and catastrophe modeling in the engineering, financial, actuarial and meteorological sciences, and the modeling of explosive economic events like asset market collapses, hyperinflation and cross-market contagion. See, e.g., Mittnik and Rachev (1993), Embrechts, Kluppelberg, and Mikosch (1997) and Finkenstadt and Rootzén (2003).

Moreover, exceedance and tail empirical processes provide a theoretical foundation from which to analyze tail index and tail dependence estimators. See, e.g., Hsing (1991).

Our research is motivated by the popular application of tail shape estimators and tail dependence measures on highly dependent and heterogeneous data, including equity, bond and exchange rate returns, prices, insurance claims, and climate extremes. See Brown Katz (1993), Hsieh (1999), Kysely (2002), and Galbraith and Zernov (2004), and Schmidt and Stadtmüller (2006) to name a very few. In both cases existing theory can only handle limited forms of dependence and heterogeneity.

1.1 Tail Index Estimation

Consider estimating the tail index parameter of a process $\{X_t\}$ on $[0, \infty)$ with a regularly varying tail:

(2) $\quad P(X_t > x) = x^{-\alpha}L(x), \quad$ as $x \to \infty$, $\quad$ where L is slowly varying.

B. Hill (1975) proposed a simple estimator of α^{-1} which we generalize to a functional sub-sample. If $X_{(i)} > 0$ denotes the i^{th} order statistic $X_{(1)} \geq X_{(2)} \geq ...$, and $[z]$ the nearest integer, $[z] \geq z$, the estimator is

$$\hat{\alpha}_n^{-1}(\xi) := 1/k_n(\xi) \sum_{t=1}^{n(\xi)} \left(\ln X_t/X_{([k_n(\xi)]+1)}\right)_+,$$

for some sequence $k_n(\cdot) : [0,1] \to \mathbb{N}$, $k_n(\xi) \to \infty$ as $n(\xi) \to \infty$, $k_n(\xi)/n(\xi) \to 0$. Fan, Quintos and Phillips (2001) deliver a functional distribution limit for $\hat{\alpha}_m^{-1}(\xi)$ under an *iid* assumption. In the non-functional case asymptotic normality has been established for *iid* and strong mixing processes (Hall 1982, Hall and Welsh 1985, Hsing 1991), and for processes with extremes that are Near-Epoch-Dependent on the extremes of a mixing process (Hill 2005b).

The Extremal Near Epoch Dependence [E-NED] property of Hill (2005b) has substantial practical advantages over mixing and conventional NED (Gallant and White 1988), and L_p-weak dependence (Wu 2006, Wu and Manli 2005) properties because it imposes memory and heterogeneity restrictions only on extremes: the Hill-estimator uses information strictly from the extreme sample tail, hence all assumptions governing non-extremes are superfluous. The E-NED property will be important for the analysis of many heavy-tailed time series in finance, macroeconomics and meteorology in which extremes appear to cluster, and for which the analyst does not want to specificy a model for the non-extremes.

The E-NED property only requires computation of a conditional probability, it is typically straightforward to verify, and it characterizes any mixing process, non-mixing processes, essentially any L_p-NED process, $p > 0$, linear distributed lags and linear conditional volatility processes with short or long memory, bilinear processes, random coefficient autoregressions, and many nonlinear distributed lags $X_t = g(X_{t-1}, \epsilon_t)$ where ϵ_t need only be L_p-bounded. In particular, the Extremal-NED property can characterize processes with highly dependent non-extremes (e.g. the non-extremes are a random walk) that are neither NED nor L_p-weakly dependent. See Hill (2005b).

1.2 Tail Dependence Estimation

Consider two stochastic processes $\{X_{1,t}, X_{2,t}\}$ with marginal distributions described by (2) with indices $\alpha_1 > 0$ and $\alpha_2 > 0$. The tail dependence coefficient for $X_{1,t}$ and $X_{2,t}$ is

$$\text{(3)} \quad \rho_{\alpha,n}(h) = (n/k_n)[P(X_{1,t} > b_{1,n},\ X_{2,t-h} > b_{2,n}) - P\left(X_{1,t} > b_{1,n}\right) P\left(X_{2,t-h} > b_{2,n}\right)],$$

for some sequences of thresholds $b_{i,n} \rightarrow \infty$, $k_n = o(n)$, and $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Tail dependence has been modeled extensively in the case of bivariate regularly varying processes that are marginally *iid*. See Ledford and Tawn (1996, 1997), Coles, Hefferman and Tawn (1999), Breymann, Dias and Embrechts (2003), Embrechts, Lindskog and McNeil (2003) and Hefferman and Tawn (2004). Schmidt and Stadtmüller (2006) analyze a non-parametric estimator of $\rho_{\alpha,n}(h)$ only for *iid* pairs $\{X_{1,t}, X_{2,t}\}$ with a parametric bivariate tail shape. The assumption of independence for the marginal distributions substantially restricts the types of bivariate environments in which tail dependence estimators may be applied, entirely omits the possibility of serial extremal dependence ($X_{1,t} = X_{2,t}$), and superfluously imposes structure on non-extremes.

1.3 Limit Theory for Tail Arrays

Rootzén, Leadbetter, and de Haan (1998) analyze general tail array sums under a mixing condition, and Hsing (1991, 1993) develops limit theory for a subset of such tail array sums in the strong mixing case. See also Rootzén, Leadbetter, and de Haan (1990) and Leadbetter and Rootzén (1993). Rootzén (1995), Drees (2002) and Eihnmal and Lin (2003) consider functional limit theory for *iid* and mixing tail empirical processes in $C[0, 1]$ and $D[0, \infty)$. Typically the imposed restrictions are rather abstract (Rootzén 1995; Eihnmal and Lin 2003), and have only been exemplified for linear non-extremal processes (Eihnmal and Lin 2003: asymptotically independent extremes, AR(1)). See also Csörgö, Csörgö, Horváth, and Mason (1986), Mason (1988), and Eihnmal (1990).

Moreover, the best extant central limit theory for dependent, heterogeneous arrays (de Jong 1997, Wu and Manli 2005) does not apply to tail arrays (see Hill 2006), and some processes with highly dependent non-extremes are neither NED nor L_p-weakly dependent.

In this paper we characterize sufficient conditions for functional central limit theorems for tail arrays under (reasonably) minimal assumptions regarding memory and heterogeneity. We use an extremal mixing property, and we exploit a functional version of the Extremal-Near-Epoch-Dependence property in Hill (2005b).

In the main result of the paper, Theorem 1, we deliver a functional limit theory for Extremal-NED arrays. In Section 3 we provide examples of limit theory for specific tail arrays, and Section 4 contains examples of processes that are Extremal-NED. The theory developed here essentially unifies and goes well beyond the disparate theory and environments treated in Rootzén *et al* (1990), Leadbetter and Rootzén (1993), Rootzén (1995), Drees (2002) and Eihnmal and Lin (2003). Proofs of the main results are in Appendix 1 and Appendix 2 contains supporting lemmata.

Our theory can equally handle sample means and covariances of tail arrays. Thus, we apply the theory to a complete asymptotic analysis of the Hill-estimator $\hat{\alpha}_n^{-1}(\xi)$ the tail empirical quantile estimator $X_{([k_n(\xi)])}$, and a tail dependence coefficient estimator $\hat{\rho}_{\alpha,n(\xi)}(h)$ in Section 5. Both results are the most general available in either literature, and we anticipate the applicability of the theory developed here for other tail-array based estimators.

Throughout $\rightarrow$ variously denotes convergence in probability or finite distributions; $\Rightarrow$ denotes weak convergence on some function space. $(z)_+ := \max\{z, 0\}$. Gaussian elements of function spaces have zero means. $K > 0$ denotes a finite constant that may change in value based on the context.

2. ASSUMPTIONS and MAIN RESULTS

We first state primitive tail array characteristics. We then define the memory properties used throughout this paper, and present the main results.

Let $\{X_t\} = \{X_t : -\infty < t < \infty\}$ be a stochastic process on some probability measure space $(\Omega, \Im, \mu)$, $\Im = \sigma(\cup_{t\in\mathbb{Z}}\Im_t)$, $\Im_{t-1} \subset \Im_t \equiv \sigma(X_\tau : \tau \leq t)$. We will work with the two dimensional *cadlag* space

$$D_2 := D([\underline{\xi}, 1] \times [0, 1]),\ \forall x \in D_2 : x(1, u) = x(1-, u),$$

for some $\underline{\xi} \in (0, 1]$. The theory of this paper is greatly expedited by bounding ξ away from 0, and $x(1, u) = x(1-, u)$ solves the right end-point problem with *cadlag* functions. Neither restriction reduces the generality of the main results by much.

Let $X_{n,t}(u)$ be a $D[0,1]$-valued stochastic array, and define the corresponding D_2-functional

$$X_n(\xi,u) := \sum\nolimits_{t=1}^{n(\xi)} X_{n,t}(u).$$

$n(\xi)$ is right continuous, non-decreasing in ξ, $n(\xi) \to \infty$ as $n \to \infty$, $n(\xi_1) - n(\xi_2) \to \infty$ $\forall \xi_1 \geq \xi_2$, $n(0) = 0$, and $n(1-) = n(1) \leq n$.

DEFINITION [Tail Array] *$\{X_{n,t}(u)\}$ is an L_r-Functional Tail Array of $\{X_t\}$ if $X_{n,t}(u) \in D[0,1]$, $\sup_t \sup_{u\in[0,1]} \|X_{n,t}(u)\|_r = O(n^{-a(r)})$ for some function $a : [1,\infty) \to (0, 1/2]$; and $\eta(\xi,u) := \lim_{n\to\infty} ||X_n(\xi,u)||_2$ exists for all ξ, where $\eta(\xi,u)$ is a non-decreasing function on $[0,1]^2$, $\eta(0,\cdot) = 0$, $\eta(1,\cdot) = 1$.*

Remark 1: We refer to $a(r)$ as the r^{th}-*moment index*. See Section 4 for examples of L_r-Tail Arrays.

Our main result uses a big block/little block argument, cf. Berstein (1927). Define the sequences $g_n(\underline{\xi})$, $l_n(\underline{\xi})$ and $r_n(\xi)$, $\xi \in [\underline{\xi}, 1]$, as follows: $g_n(\underline{\xi}) \to \infty$ as $n \to \infty$ and

$$1 \leq g_n(\underline{\xi}) = o(n(\underline{\xi})), \quad 1 \leq l_n(\underline{\xi}) \leq g_n(\underline{\xi}) - 1 \leq n(\underline{\xi}) - 1 \text{ where } l_n(\underline{\xi}) \to \infty;$$
$$l_n(\underline{\xi})/g_n(\underline{\xi}) \to 0, \quad r_n(\xi) = [n(\xi)/g_n(\underline{\xi})] \text{ where } r_n(\xi) \to \infty.$$

We omit the common argument $\underline{\xi}$ for clarity and write g_n and l_n. Notice $r_n(0) = 0$ and $r_n(\xi_1) \leq r_n(\xi_2)$ $\forall \xi_2 \geq \xi_1$. Define the blocks

$$Z_{n,i} = Z_{n,i}(u) := \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} X_{n,t}(u).$$

Under mild assumptions concerning memory and heterogeneity we will prove

$$X_n(\xi,u) = \sum\nolimits_{i=1}^{r_n(\xi)} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} Z_{n,i}(u) + o_p(1) \Rightarrow X(\xi,u).$$

Our main result relies on a new functional limit theorem for Tail Arrays that are approximable by martingale difference array. See Lemma A.1 of Appendix 1. In order to ensure the approximation condition holds for a wide array of processes we construct extremal versions of mixing and near-epoch-dependence properties. Let $\{\epsilon_t\}$ be a stochastic process with σ-algebra

$$G_t := \sigma(\epsilon_\tau : \tau \leq t).$$

Write $G_b^a := \sigma(\epsilon_t : a \leq t \leq b)$. Let $\{\pi_{n,t}\}$ be a sequence of constant real thresholds, $\pi_{n,t} \to \infty$ as $n \to \infty$, and denote by $\{E_{n,t}\}$ a G_t-measurable extremal process. Examples include the extreme event $I(|\epsilon_t| > \pi_{n,t})$, exceedance $(|\epsilon_t| - \pi_{n,t})_+$, or value $|\epsilon_t| \times I(|\epsilon_t| > \pi_{n,t})$.

Denote by $\digamma_{n,s}^{t} \in G_s^t$ the sigma sub-algebra induced by the extremal event

$$\digamma_{n,s}^{t} := \sigma(E_{n,\tau} : 1 \leq s \leq \tau \leq t \leq n),$$

and define the coefficients

$$\varepsilon_{q_n} \equiv \sup_{A_{n,t} \in \digamma_{n,1}^{t}, B_{n,t+q_n} \in \digamma_{n,t+q_n}^{n} : 1 \leq t \leq n-q_n} \left| P\left(A_{n,t} \cap B_{n,t+q_n}\right) - P(A_{n,t})P(B_{n,t+q_n}) \right|$$

$$\varpi_{q_n} \equiv \sup_{A_{n,t} \in \digamma_{n,1}^{t}, B_{n,t+q_n} \in \digamma_{n,t+q_n}^{n} : 1 \leq t \leq n-q_n} \left| P\left(B_{n,t+q_n} | A_{n,t}\right) - P(B_{n,t+q_n}) \right|,$$

where $\{q_n\}$ is a sequence of integers, $1 \leq q_n \leq n$, $q_n \to \infty$ as $n \to \infty$..

DEFINITION [E-Mixing] *If $q_n^{\lambda}\varepsilon_{q_n} \to 0$ as $n \to \infty$ for some all $\{q_n\}$ we say $\{\epsilon_{n,t}\}$ is* Extremal-Strong Mixing *with size $\lambda > 0$. If $h(q_n)^{\lambda}\varpi_{q_n} \to 0$ as $n \to \infty$ for all $\{q_n\}$ we say $\{\epsilon_t\}$ is* Extremal-Uniform mixing *with size $\lambda > 0$.*

Remark 1: The E-mixing property is simply a canonical uniform or strong mixing property assigned to the extremal process $\{E_{n,t}\}$ as $n \to \infty$, hence any measurable function of a finite sequence $\{E_{n,t}, E_{n,t-1}, ..., E_{n,t-h}\}$ is E-mixing, $h \geq 1$. Strong (uniform) mixing process implies E-strong (uniform) mixing, and well known inequalities hold for E-mixing processes. Cf. Ibragimov (1962) and Serfling (1968). See Hsing (1991: p.) for a similar construction.

Remark 2: The E-strong mixing property is related to Leadbetter's (1974) D-mixing property, cf. Leadbetter, Lindgren and Rootzén (1983). D-mixing is not uniform and therefore does not carry over to measurable functions of D-mixing random variables, a trait we explicitly exploit in the proof of the supporting Lemma A.4.

Let $\{\digamma_{n,t}\}$ be an arbitrary array of σ-fields.

DEFINITION [FE-NED]

1. $\{X_t\}$ *is* L_p-Functional-Extremal-NED, $p > 0$,*with size* $-1/2$, *on some* $\{\digamma_{n,t}\}$ *if some L_r-Functional-Tail Array $\{X_{n,t}(u)\}$ based on $\{X_t\}$ satisfies*

(4) $\left\| X_{n,t}(u) - E[X_{n,t}(u) | \digamma_{n,t-q}^{t+q} \right\|_p \leq d_{n,t}(u)\varphi_{q_n}.$

The Lebesgue measurable function array $\{d_{n,t}(u)\}$, $d_{n,t} : [0,1] \to \mathbb{R}_+$, satisfies $\sup_{u \in [0,1], t \geq 1} d_{n,t}(u) = O(n^{-a(r)})$. The coefficients $\{\varphi_{q_n}\}$ satisfy $n^{1/2-a(r)}q_n^{1/2}\varphi_{q_n} \to 0$.

2. *Define $Y_{n,t}(u_1, u_2) := X_{n,t}(u_1) - X_{n,t}(u_2)$. Then*

(5) $\left\| Y_{n,t}(u_1, u_2) - E[Y_{n,t}(u_1, u_2) | \digamma_{n,t-q}^{t+q}] \right\|_p \leq (\tilde{d}_{n,t} \times |u_2 - u_1|^{1/p}) \times \varphi_{q_n},$

$\forall u_1, u_1 \in [0,1]$, *where* $\sup_t \tilde{d}_{n,t} = O(n^{-a(r)})$.

Remark 1: The imposition of L_p-boundedness is irrelevant for popular tail arrays that are inherently L_r-bounded $\forall r \geq 1$. The second property (5) is imposed to handle tightness of sequences of distributions of $\{X_n(u, \xi)\}$.

Remark 2: On the surface the FE-NED property (4) is identical to the canonical NED property, cf. Gallant and White (1988) and Davidson (1994). Thus, the "constants" $d_{n,t}(u)$ effectively allow the "coefficients" φ_{q_n} to be scale-free. That we only impose the NED property on a Tail Array $X_{n,t}(u)$ of X_t is a crucial distinction. The FE-NED property is uniquely suited to characterize dependence in extremes simply by exploiting an appropriate Tail Array $X_{n,t}(u)$ of X_t, and will allow us to deliver invariance principles for processes with non-extremes that are too dependent to be NED or L_p-weakly dependent.

Remark 3: We say $\{X_t\}$ is FE-NED on $\{\epsilon_t\}$ if it is FE-NED on some array of σ-fields $\{\mathcal{F}_{n,t}\}$ induced by $\{\epsilon_t\}$.

Assumption 1

(a) $\{X_t\}$ is L_2-FE-NED of size $-1/2$ on an E-strong mixing base $\{\epsilon_{n,t}\}$ of size $\lambda = r/(r-2)$, $r > 2$, or E-uniform mixing base of size $\lambda = r/[2(r-1)]$, $r \geq 2$.

(b) For some finite function $\kappa(\xi, \delta) \geq 1$ and each $\xi \in [\underline{\xi}, 1-\delta]$, $\delta \in [0, 1]$,

$$|r_n(\xi + \delta)/r_n(\xi) - \kappa(\xi, \delta)| \to 0.$$

Moreover $2a(4) + a(2) > 1$, $2a(2r) > a(r)$. Furthermore

$$g_n = o(n^{\min\{[2a(4)+a(2)-1]/2,\ 2a(2r)-a(r)\}}).$$

THEOREM 1 *Under Assumption 1, $X_n(\xi, u) \Rightarrow X(\xi, u)$ on D_2 where $X(\xi, u)$ is Gaussian with independent increments and covariance function $E[X(\xi_i, u_i)X(\xi_j, u_j)]$.*

Remark 1: The rate $g_n = o(n^{\min\{[2a(4)+a(2)-1]/2,\ 2a(2r)-a(r)\}})$ is always possible, and merely expedites a proof that the Lindeberg condition holds. The assumption $2a(4) + a(2) > 1$ and $2a(2r) - a(r) > 0$ guarantee $g_n \to \infty$ as $n \to \infty$, and ensure a Lindeberg condition holds for Tail Arrays, cf. Lemma A.2. The restrictions imply, for example, $a(r) = 1/r$ for all $r \geq 1$ is not covered here. In Section 3 we demonstrate that at least three popular Tail Arrays satisfy all restrictions on $a(r)$.

Remark 2: The requirement $|r_n(\xi + \delta)/r_n(\xi) - \kappa(\xi, \delta)| \to 0$ further restricts g_n in order to ensure uniform tightness on D_2, cf. Lemma A.3. For example, if $n(\xi) = [n\xi]$ and $g_n = [(n\underline{\xi})^{\theta}]$ for some $\theta \in (0, [2a(4) + a(2) - 1]/2)$ then $\kappa(\xi, \delta) = (1 + \delta/\xi)^{1-\theta} < \infty\ \forall \xi \geq \underline{\xi} > 0$.

Remark 3: For any fixed $u \in [0, 1]$ if $\eta(\xi, \cdot) = \lim_{n\to\infty} ||X_n(\xi, \cdot)||_2 = \xi$ then $X(\xi, \cdot)$ is Brownian motion. Otherwise $X(\xi, \cdot)$ is the so-called transformed Brownian motion of Davidson (1994: p. 485) and de Jong and Davidson (2000).

3. INVARIANCE PRINCIPLES We now characterize invariance principles for specific extremal processes $\{X_{n,t}\}$. Assume $F_t(x) := P(X_t \leq x)$ has support on $[0, \infty)$, and $\bar{F}_t(x) := P(X_t > x)$ is regularly varying at ∞: for all t there exists some $\alpha > 0$ such that for all $\lambda > 0$,

$$(6) \quad \bar{F}_t(\lambda x)/\bar{F}_t(x) \to \lambda^{-\alpha} \text{ as } x \to \infty \quad \Leftrightarrow \quad \bar{F}_t(x) = x^{-\alpha}L(x), \quad x > 0,$$

for some slowly varying function $L(x)$. The class of distributions satisfying (6) includes the domain of attraction of the stable laws, coincides with the maximum domain of attraction of the extreme value distributions $\exp\{-x^{-\alpha}\}$, and characterizes many stochastic recurrence equations (e.g. GARCH). See de Haan (1970), Leadbetter *et al* (1983), Bingham, Goldie and Teugels (1987), Resnick (1987) and Basrak *et al* (2002).

Let $\bar{F}_t(x)/\bar{F}_t(x-) \to 1$ as $x \to \infty$. Then there exists sequences $k_n = o(n)$ and $b_n(k_n) = o(n)$, $b_n(k_n) \to \infty$, satisfying (see Leadbetter *et al*, 1983)

$$(7) \quad \lim_{n\to\infty}(n/k_n)P(X_t > b_n) = 1.$$

Consider the *intermediate* case: $k_n \to \infty$ as $n \to \infty$, $n/k_n \to 0$, and define

$$(8) \qquad \begin{aligned} X_{n,t} &:= k_n^{-1/2}\left((\ln X_t/b_n(k_n))_+ - E[(\ln X_t/b_n(k_n))_+]\right), \\ X^*_{n,t}(u) &:= k_n^{-1/2}\left(I(x_t < v_n(k_n)u) - E[I(x_t < v_n(k_n)u)]\right), \ \ u \in [0,1] \end{aligned}$$

where

$$x_t := \bar{F}(X_t) \text{ and } v_n(k_n) := \bar{F}(b_n) \to 0 \text{ as } n \to \infty.$$

Notice (6) and (7) imply

$$v_n(k_n) \sim k_n/n, \quad \text{and as } n \to \infty \ x_t < v_n(k_n)u \ \Leftrightarrow \ X_t > b_n(k_n)u^{-1/\alpha}.$$

The sum $\sum_{t=1}^{n(\xi)} X^*_{n,t}(u)$ denotes a centered tail empirical distribution function, and $X_{n,t}$ simply represents a centered b_n-exceedance process. See Davis and Resnick (1984), Beirlant, Teugels and Vynckier (1994), Hsing (1991) and Rootzén *et al* (1998). Both processes have been used in damage modeling in the engineering, actuarial and meteorological sciences, and asset market risk and volatility analysis. Moreover, both are key to deriving an asymptotic theory for tail index estimators and tail quantile functions: see Theorem 6.

LEMMA 2 *Each $Y_{n,t}(u) \in \{X_{n,t}, X^*_{n,t}(u)\}$ is an L_r-Functional-Tail Array for all $r \geq 1$. In particular $\sup_t \sup_{u\geq 0} ||Y_{n,t}(u)||_r = O(k_n^{-(1/2-1/r)}n^{-1/r}) = O(n^{-a(r)})$ where $a(1) > 1/2$, $a(2) = 1/2$, $a(r) > 1/r$ $\forall r > 2$, and $2a(2r) > a(r)$. If $k_n \sim n^\delta$, $\delta \in (0,1]$, then $a(r) = 1/2 - (1-\delta)(1/2 - 1/r)$.*

Remark: Both properties $2a(4) + a(2) > 1$ and $2a(2r) > a(r)$ invoked in Assumption 1.b are satisfied for each $\{X_{n,t}, X^*_{n,t}(u)\}$.

The L_p-FE-NED properties (4)-(5) are particularly insightful for characterizing extremal dependence in $\{X_t\}$ itself if the associated Tail Array is the extremal event $X^*_{n,t}(u)$. Given $\Im^{t+q_n}_{t-q_n} := \sigma(X_\tau : t - q_n \le \tau \le t + q_n)$, properties (4)-(5) applied to $X^*_{n,t}(u)$ reduce to

$$
\begin{aligned}
(9) \quad & k_n^{-1/2} \left\| P\left(x_t < v_n u | \Im^{t+q_n}_{t-q_n}\right) - P\left(x_t < v_n u | F^{t+q_n}_{n,t-q_n}\right) \right\|_p \\
& \qquad \le d_{n,t}(u) \times \varphi_{q_n}
\end{aligned}
$$

$$
\begin{aligned}
(10) \quad & k_n^{-1/2} \left\| P\left(v_n u_1 < x_t \le v_n u_2 | \Im^{t+q_n}_{t-q_n}\right) - P\left(v_n u_1 < x_t \le v_n u_2 | F^{t+q_n}_{n,t-q_n}\right) \right\|_p \\
& \qquad \le (\tilde{d}_{n,t} \times |u_2 - u_1|^{1/p}) \times \varphi_{q_n}.
\end{aligned}
$$

for all $u_1 \le u_2$. The L_p-Extremal-NED property (9) was introduced in Hill (2005b).

Remark 1: Property (9) implies the scaled event $(n/k_n)^{1/2} I(X_t > b_n u^{-1/\alpha})$ can be almost surely perfectly approximated by $(n/k_n)^{1/2} P(X_t > b_n u^{-1/\alpha} | F^{t+q_n}_{n,t-q_n})$ as $n \to \infty$, given $\sqrt{n} d_{n,t}(u) q_n^\lambda \varphi_{q_n} \to 0$.

Remark 2: The above construction marks a fundamental distinction between canonical NED and FE-NED properties. The FE-NED properties restrict extreme events $X_t > b_n u^{-1/\alpha}$ to be Near-Epoch-Dependent on $\{F_{n,t}\}$, and $F_{n,t}$ need only be induced by the extremes of some ϵ_t. The FE-NED properties only characterize memory and heterogeneity for extremes: (9) and (10) say nothing about the event $X_t \le b_n u^{-1/\alpha} \to \infty$.

Remark 3: In Hill (2005b, 2006) we present a simple class of extremal processes that does not satisfy NED or L_p-weak dependence properties but does satisfy (9)-(10). Moreover, in Hill (2006) we prove that Tail Arrays like $X^*_{n,t}(u)$ do not satisfy key assumptions of the best extant central limit theory for NED arrays, cf. de Jong (1997).

If $\{X_t\}$ satisfies (9)-(10), then *both* tail arrays $\{X_{n,t}, X^*_{n,t}(u)\}$ are L_2-NED, a la Gallant and White (1988). In this sense (9)-(10) characterize a primitive tail memory property.

LEMMA 3 (Hill 2006) *Let $\{X_t\}$ satisfy (6), (9) and (10) with coefficients $\varphi_{q_n} = o(n^{a(r)-1/2} \times q_n^{-\lambda})$, and Lebesgue measurable functions $d_{n,t}(u) : [0,1] \to \mathbb{R}_+$, where $\sup_{u \in [0,1], t \ge 1} d_{n,t}(u) = O(n^{-a(r)})$, $(\int_0^1 u^{-1} d_{n,t}(u)^p du)^{1/p} = O(n^{-a(r)})$, and $\tilde{d}_{n,t} = O(n^{-a(r)})$. Then $\{X_{n,t}, X^*_{n,t}(u)\}$ are L_2-NED on $\{F_{n,t}\}$ with common coefficients φ_{q_n}. $\{X_{n,t}\}$ has constants*

$$
K \times \left(-\alpha^{-1} \int_0^1 u^{-1} d_{n,t}(u)^p du \right)^{1/p}.
$$

Let

$T_{n,t}(u) \in \{X_{n,t}, X^*_{n,t}(u)\}$.

The following result is a direct consequence of Theorem 1 and Lemmas 2 and 3.

Assumption 2

(a) $\{X_t\}$ satisfies (6) and is L_p-FE-NED *a la* (9) and (10) with coefficients $\{\varphi_{q_n}\}$ of size $-1/2$, constants $\{d_{n,t}(u), \tilde{d}_{n,t}\} = O(n^{-a(r)})$ and $(\int_0^\infty d_{n,t}(u)^p du)^{1/p} = O(n^{-a(r)})$. The base $\{\epsilon_t\}$ is either E-uniform mixing with size $-r/2(r-1)$, $r \geq 2$, or E-strong mixing base with size $-r/(r-2)$, $r > 2$.

(b) $|r_n(\xi+\delta)/r_n(\xi) - \kappa_\delta(\xi)| \to 0$ for some finite $\kappa_\delta(\xi) \geq 1$ and each $\xi \in [\underline{\xi}, 1-\delta]$ and $\{u, \delta\} \in [0,1]$, and $g_n = o(n^{\min\{[2a(4)+a(2)-1]/2,\ 2a(2r)-a(r)\}})$.

COROLLARY 4 *Under Assumption 2 with $p = 2$, $\sum_{t=1}^{n(\xi)} T_{n,t}(u) \Rightarrow T(\xi, u)$ on D_2, where $Y(\xi, u)$ is Gaussian with independent increments and covariance function $E[T(\xi_i, u_i)T(\xi_j, u_j)]$.*

If $k_n \sim n^\delta$ for some $\delta \in (0,1)$ then Lemma 2 can be used to verify $\min\{[2a(4) + a(2) - 1]/2, 2a(2r) - a(r)\} = \delta/4$. If additionally $n(\xi) = [n\xi]$ then Corollary 4 can be greatly simplified: all we ultimately need is Assumption 2.a.

COROLLARY 5 *Let Assumption 2.a hold with $p = 2$, and let $n(\xi) = [n\xi]$, $k_n \sim n^\delta$, $\delta \in (0,1)$ and $g_n = o(n^{\delta/4})$. Then $\sum_{t=1}^{[n\xi]} T_{n,t}(u) \Rightarrow T(\xi, u)$ on D_2, where $T(\xi, \cdot)$ is Brownian motion.*

4 FE-NED EXAMPLES

Any L_p-NED process X_t with tails (6) and $\lim_{x\to\infty} L(x) = c$ has the L_2-FE-NED property (9). This includes tails of the form $\bar{F}(x) = cx^{-\alpha}(1 + dx^{-\beta} + O(x^{-\theta}))$, $(c,d) \geq 0$, $(\beta, \theta) > 0$, and related expansions (e.g. Hall 1982, Hall and Welsh 1985). Thus, linear, conditional volatility (Davidson 2004), bilinear (Davidson 1994) and many nonlinear distributed lags (Gallant and White 1988) with *iid* innovations that satisfy (1) and $L(x) \to c$ are L_2-E-NED. For any slowly varying $L(x)$ in (6), (9) holds with a complicated expression for $d_{n,t}(u)$. In general, ARFIMA, FIGARCH, simple bilinear, random coefficient autoregression, and Extremal Threshold processes are L_2-FE-NED (9)-(10) if the underlying innovations are *iid* with tails (1). See Hill (2005b, 2006).

5. APPLICATIONS

In this section we develop a functional limit theory for a tail dependence estimator, the B. Hill (1975) tail index estimator, and an intermediate tail quantile function.

5.1 Functional Tail Index Estimation

Let $\{X_t\}$ satisfy (6). We derive the functional distribution limit of the Hill-estimator

$$\hat{\alpha}_n^{-1}(\xi) := 1/k_n(\xi) \sum_{t=1}^{n(\xi)} \left(\ln X_t / X_{([k_n(\xi)]+1)}\right)_+ ,$$

and tail quantile function $X_{([k_n(\xi)])}$ for L_2-FE-NED processes. Without introducing more notation, it is understood that $X_{([k_n(\xi)])}$ is the $[k_n(\xi)]^{th}$-largest observation from the $n(\xi)$-sample $\{X_1, ..., X_{n(\xi)}\}$.

The following limit theory is the most general available for the Hill-estimator, and for any other tail estimator (that we are aware of), including those suggested by Pickands (1975), Smith (1987), and Drees *et al (* 2004) to name a very few.

For brevity we restrict attention to

$$n(\xi) = [n\xi].$$

From (7) it is easily verified that

$$k_n(\xi) = [k_n \xi],$$

hence

$$(n(\xi)/k_n(\xi))P\left(X_t > b_{n(\xi)}(k_n(\xi))\right) \sim (n/k_n)P\left(X_t > b_{n(\xi)}(k_n(\xi))\right) \to 1$$

logically implies $b_{n(\xi)}(k_n(\xi)) \sim b_n(k_n)$.

We must restrict the tail shape in order to expedite asymptotic normality. Cf. Goldie and Smith (1987), Hsing (1991) and Hill (2005b).

Assumption 3 For some positive measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$

(11) $L(\lambda x)/L(x) - 1 = O(g(x))$ as $x \to \infty$.

We assume g has bounded increase: there exists $0 < D, z_0, \tau < \infty$ such that $g(\lambda z)/g(z) \leq D\lambda^{\tau}$ some for $\lambda \geq 1$, $z \geq z_0$. Assume $\tau \leq 0$. We require $\{k_n\}$and $g(\cdot)$ to satisfy

(12) $\sqrt{k_n} g(b_n(k_n)) \to 0$.

Remark: Tails satisfying (6), (11) and (12) include the popularly assumed forms $\bar{F}(x) = cx^{-\alpha}(1 + O((\ln x)^{-\theta}))$ and $\bar{F}(x) = cx^{-\alpha}(1 + O(x^{-\theta}))$. In the latter case (12) holds only if $k_n/n^{2\theta/(2\theta+\alpha)} \to 0$. See Haeusler and Teugels (1985).

Define

$$\begin{aligned} \tilde{\sigma}_n^2(\xi) &:= E\left(\sum_{t=1}^{[n\xi]} X_{n,t}^*\left(u^{1/\sqrt{k_n}}\right)\right)^2 \\ \sigma_n^2(\xi) &:= E\left(k_n^{1/2}\left(\hat{\alpha}_n^{-1}(\xi) - \alpha^{-1}\right)\right)^2. \end{aligned}$$

THEOREM 6 *Let $\{X_t\}$ satisfy Assumption 2 with $p = 2$ and Assumption 3. Then*

$$k_n^{1/2}(\hat{\alpha}_n^{-1}(\xi) - \alpha^{-1}) \Rightarrow X(\xi)$$

where $X(\xi)$ *is Brownian motion with variance* $\lim_{n\to\infty}\sigma_n(\xi) < \infty$. *Moreover,*

$$k_n^{1/2}\left(\ln X_{([k_n\xi])}/b_n\right) \;\Rightarrow\; Y(\xi)$$

where $Y(\xi)$ *is Brownian motion of* $D[\underline{\xi},1]$ *with variance* $\lim_{n\to\infty}\breve{\sigma}_n^2(\xi)$ $< \infty$.

Remark 1: Hill (2005b: Theorem 6) proves a kernel variance estimator of $\sigma_n^2(1)$ is consistent for Extremal-NED processes. Extending the proof to $\sigma_n^2(\xi)$ is trivial.

Remark 2: A non-functional limit for B. Hill's estimator is immediate, cf. Theorem 5 of Hill (2005b): $k_n^{1/2}(\hat{\alpha}_n^{-1}(1) - \alpha^{-1})/\sigma_n(1) \Rightarrow N(0,1)$ provided $\liminf_{n\geq 1}\sigma_n^2(1) > 0$.

5.2 Tail Dependence

Consider two stochastic processes $\{X_{1,t}, X_{2,t}\}$ with marginal distributions F_i with support on $[\infty, 0)$. Assume each F_i has a regularly varying tails (6) with index $\alpha_i > 0$.

Define

$$Z_{i,n,t}(u_i) := I(x_{i,t} \leq v_{i,n}u_i) - E\left[I(x_{i,t} \leq v_{i,n}u_i)\right],\; i = 1,2,$$

where $x_{i,t} = \bar{F}_i(X_{i,t})$, $v_{i,n} = \bar{F}_i(b_{i,n}(k_{i,n}))$, and $b_{i,n}(k_{i,n})$ and $k_{i,n}$ satisfy (7) for each i. In most applications $u_i = 1$. Let $n(\xi) = [n\xi]$. We obtain from (6)-(7) and some sequence $k_{i,n}$

$$\begin{aligned}&\lim_{n\to\infty}(n/k_{i,n})E[Z_{i,n,t}^2(u_i)]\\ &\quad = \lim_{n\to\infty}(n/k_{i,n})\left[P(X_{i,t} > b_{i,n}(k_{i,n})u_i^{-1/\alpha_i}) - P(X_{i,t} > b_{i,n}(k_{i,n})u_i^{-1/\alpha_i})^2\right]\\ &\quad = u_i.\end{aligned}$$

This suggests a simple functional tail dependence coefficient:

$$\begin{aligned}\rho_{\alpha,n}(h,u) &= \frac{E\left[Z_{1,n,t}(u_1)\times Z_{2,n,t-h}(u_2)\right]}{\sqrt{u_1 k_{1,n}/n}\sqrt{u_2 k_{2,n}/n}}\\ &= (n/k_n(u))\times[P(X_{1,t} > b_{1,n}(k_{1,n})\tilde{u}_1,\; X_{2,t-h} > b_{2,n}(k_{2,n})\tilde{u}_2)\\ &\qquad - P\left(X_{1,t} > b_{1,n}(k_{1,n})\tilde{u}_1\right)P\left(X_{2,t-h} > b_{2,n}(k_{2,n})\tilde{u}_2\right)],\end{aligned}$$

where

$$k_n(u) := (k_{1,n}u_1k_{1,n}u_2)^{1/2} \;\text{ and }\; \tilde{u}_i = u_i^{-1/\alpha_i}.$$

If $\lim_{n\to\infty}\sup_{u\in[0,1]}|\rho_{\alpha,n}(h,u)| = 0$ then $\{X_{1,t}, X_{2,t}\}$ are *asymptotically extremal-independent* at displacement $h \geq 0$.

A natural non-parametric estimator of the tail dependence coefficient for an arbitrary $n(\xi)$-subsample is

$$\hat{\rho}_{\alpha,n}(h,\xi,u) := 1/k_n(u)\sum_{t=1}^{n(\xi)}\left[Z_{1,n,t}(u_1)\times Z_{2,n,t-h}(u_2)\right].$$

In order to exploit Correlation 4 we must show the centered product

$$ZZ_{n,t}(u,h) := \frac{1}{\sqrt{k_n(u)}} \{Z_{1,n,t}(u_1)Z_{2,n,t-h}(u_2) - E[Z_{1,n,t}(u_1)Z_{2,n,t-h}(u_2)]\}$$

is L_2-FE-NED. We do this by assuming each $\{X_{i,t}\}$ is L_4-FE-NED.

For arbitrary $\lambda \in \mathbb{R}^h$, $h \geq 1$ write

$$ZZ_{n,t}(\lambda, u, h) := \sum\nolimits_{i=1}^{h} \lambda_i ZZ_{n,t}(u,i).$$

LEMMA 7 (Hill, 2006) *Let $\{X_{1,t}, X_{2,t}\}$ satisfy the L_p-FE-NED property (9)-(10), with $p = 4$. Then $\{ZZ_{n,t}(\lambda, u, h)\}$ is L_2-NED on $\{\digamma_{n,t}\}$ with constants $dd_{n,t}(\lambda, u) = O(n^{-a(r)})$ and coefficients $\varphi\varphi_{q_n} = o(n^{a(r)-1/2} \times q_n^{-1/2})$. Moreover, $\{ZZ_{n,t}(\lambda, u, h), \digamma_{n,t}\}$ forms an L_2-mixingale sequence with size $-1/2$ and constants $cc_{n,t}(\lambda, u) = O(n^{-1/2})$. Finally, $E(\sum_{t=1}^{n} ZZ_{n,t}(\lambda, h, u))^2 = O(1)$.*

Lemmas 2, 3 and 7 imply Theorem 1 holds for $\{ZZ_{n,t}(\lambda, u, h)\}$ for any $\lambda \in \mathbb{R}^h$. Along with a Cramér-Wold device this proves the following claim. Write $\rho_\alpha^{(h)}(\xi, u) = [\rho_\alpha(1, \xi, u), ..., \rho_\alpha(h, \xi, u)]'$, etc.

THEOREM 8 *Let $\{X_{1,t}, X_{2,t}\}$ satisfy Assumption 2 with $p = 4$, coefficients $\{\varphi_{i,q_n}\}$ and constants $\{d_{i,n,t}(u), \tilde{d}_{i,n,t}\}$, $i = 1, 2$. Then*

$$\sqrt{k_n(u)} \left(\hat{\rho}_{\alpha,n}^{(h)}(\xi, u) - \rho_\alpha^{(h)}(\xi, u) \right) \Rightarrow X(\xi, u)$$

where $X(\xi, u)$ is a Gaussian h-vector with independent scalar increments, covariance matrix function $E[X(\xi_i, u_i)X(\xi_j, u_j)'] \in \mathbb{R}^{h \times h}$, and variances $E[X_i(\xi, u)^2] < \infty$.

Remark 1: The above result trivially applies to the serial extremal dependence case for any $h \geq 1$. Thus, Theorem 8 is the most general of its kind in the tail dependence literature.

Remark 2: In order to make $\hat{\rho}_{\alpha,n}^{(h)}(\xi, u)$ operational, the tail quantile estimator $X_{i,([k_{i,n}\xi]+1)}$ will have to be used as an estimate of the approximate $k_{i,n}/n^{th}$-quantile $b_{i,n}(k_{i,n})$. Theorem 6 implies $X_{([k_n\xi])} = b_n + O_p(1/\sqrt{k_n})$. We are currently investigating this issue in related work.

Appendix 1: Proofs of Main Results

Proof of Theorem 1. In Step 1 we prove $X_n(\xi, u) \Rightarrow X(\xi, u)$ by showing conditions (a)-(f) of Lemma A.1 in Appendix 2 hold under the maintained assumptions. In Step 2 we prove the increments of the limiting process $X(\xi, u)$

are independent.

Step 1 (weak convergence) By Lemma A.4 $\{X_{n,t}(u), \mathcal{F}_{n,t}\}$ forms an L_2-mixingale sequence with constants $c_{n,t}(u)$, $\sup_{u\in[0,1],t\geq 1} c_{n,t}(u) = O(n^{-1/2})$, and coefficients $\psi_{q_n} = o(q_n^{-1/2})$.

Define

$$\tilde{F}_{n,i} := \bigcup_{\tau\leq ig_n} \mathcal{F}_{n,\tau}, \quad \mathcal{F}_{n,s}^{t} := \sigma(E_{n,\tau} : 1 \leq s \leq \tau \leq t \leq n).$$

Condition (a): Using a standard bound for L_2-mixingales with size $-1/2$, cf. McLeish (1975),

$$\begin{aligned} E&\left(\sum_{i=1}^{r_n}\sum_{t=(i-1)g_n+1}^{(i-1)g_n+l_n} X_{n,t}(u)\right)^2 \\ &= O\left(\sum_{i=1}^{r_n}\sum_{t=(i-1)g_n+1}^{(i-1)g_n+l_n} c_{n,t}^2(u)\right) \\ &= O\left(r_n l_n n^{-1}\right) = O\left(l_n g_n^{-1}\right) = o(1), \end{aligned}$$

hence $\sum_{i=1}^{r_n}\sum_{t=(i-1)g_n+1}^{(i-1)g_n+l_n} X_{n,t}(u) = \sum_{i=1}^{r_n} Z_{n,i}(u) \to 0$ by Chebyshev's inequality.

Condition (b): Define the index sets

$$\begin{aligned} &A_i(t) := \{t : (i-1)g_n + l_n + 1 \leq t \leq ig_n\} \\ &A_{n,t} = \cup_{i=1}^{r_n} A_i(t) \end{aligned}$$

Analogous to de Jong's (1997: A.7-A.12) argument, because $\{X_{n,t}(u), \mathcal{F}_{n,t}\}$ forms an L_2-mixingale sequence, for $t \in A_{n,t}$ it can be shown $\{E[X_{n,t}(u)|\tilde{F}_{n,i-1}], \mathcal{F}_{n,t}\}$ forms an L_2-mixingale sequence with constants $c_{n,t}(u)\psi_{l_n}^{\eta}$ and coefficients $\psi_{l_n}^{1-\eta}$ $= o(l_n^{-1/2})$ for some sufficiently tiny $\eta > 0$. McLeish's (1975) bound now gives

$$\begin{aligned} E\left(\sum_{i=1}^{r_n} E\left[Z_{n,i}(u)|\tilde{F}_{n,i-1}\right]\right)^2 &= O\left(\sum_{i=1}^{r_n}\sum_{t=(i-1)g_n+l_n+1}^{ig_n} c_{n,t}^2(u)\psi_{l_n}^{2\eta}\right) \\ &= O(r_n g_n n^{-1} l_n^{-\eta}) = O(l_n^{-\eta}) = o(1). \end{aligned}$$

Condition (c): The proof mimics (b).

Condition (d): Recall $W_{n,i} := E[Z_{n,i}|\tilde{F}_{n,i}] - E[Z_{n,i}|\tilde{F}_{n,i-1}]$, write

$$Z_{n,i}(j) = Z_{n,i}(\xi_j, u_j), \quad W_{n,i}(j) = W_{n,i}(\xi_j, u_j),$$

and note

$$\sum_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}^2(\pi) = \sum_{l=1}^{k}\sum_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)}\left(\sum_{j=l}^{k}\pi_j W_{n,i}(j)\right)^2, \ \pi'\pi = 1.$$

Analogous to de Jong (1997: A.13-A.17) we obtain

$$\sup_{\pi'\pi=1} \left\| \sum_{l=1}^{k} \sum_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)} \left(\sum_{j=l}^{k} \pi_j Z_{n,i}(u_j) \right)^2 - \sum_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}^2(\pi) \right\|_1$$

$$= \sup_{\pi'\pi=1} \left\| \sum_{l=1}^{k} \sum_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)} \left[\left(\sum_{j=l}^{k} \pi_j Z_{n,i}(u_j) \right)^2 - \left(\sum_{j=l}^{k} \pi_j W_{n,i}(u_j) \right)^2 \right] \right\|_1$$

$$\leq K \sum_{l=1}^{k} \sum_{j_1,j_2=l}^{k} \sum_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)} \left\| (Z_{n,i}(u_{j_1}) - W_{n,i}(u_{j_1})) \right\|_2 \times \left\| Z_{n,i}(u_{j_2}) \right\|_2$$

$$= O\left(\sum_{l=1}^{k} \sum_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)} \left(\sum_{t\in A_i(t)} c_{n,t}^2 \psi_{l_n}^{2\eta} \right)^{1/2} \left(\sum_{t\in A_i(t)} c_{n,t}^2 \right)^{1/2} \right)$$

$$= O\left(\sum\nolimits_{l=1}^{k} r_n(\xi_l) \times [g_n n^{-1} l_n^{-\eta}]^{1/2} \times \left[g_n n^{-1}\right]^{1/2} \right)$$

$$= O(l_n^{-\eta/2}) = o(1).$$

Along with Lemma A.4 $\sup_{\pi'\pi=1} |\sum_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}^2(\pi) - 1| \to 0$.

Condition (e): For any $\{\xi, (u, \delta)\} \in [\underline{\xi}, 1] \times [0, 1]^2$ and some integer sequence $\{r_n^*(\xi)\}$ satisfying $\forall n \geq 1$

$$0 \leq [r_n^*(\xi)\delta] \leq r_n(\xi + \delta) - r_n(\xi)$$

we have

$$\text{(13)} \quad \left\| \sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} W_{n,i}(u) \right\|_2$$

$$\leq \left\| \sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} X_{n,t}(u) \right\|_2$$

$$+ \left\| \sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} \left(X_{n,t}(u) - E[X_{n,t}(u)|\tilde{F}_{n,i}] \right) \right\|_2$$

$$+ \left\| \sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} E[X_{n,t}(u)|\tilde{F}_{n,i-1}] \right\|_2.$$

Under the maintained assumptions $\{X_{n,t}(u), \digamma_{n,t}\}$ forms an L_2-mixingale sequence with size $-1/2$ and constants $c_{n,t}(u) = O(n^{-1/2})$. Similarly, for each $t \in A_{n,t}$, $\{E[X_{n,t}(u)|\tilde{F}_{n,i-1}], \digamma_{n,t}\}$ and $\{X_{n,t}(u) - E[X_{n,t}(u)|\tilde{F}_{n,i}], \digamma_{n,t}\}$ form L_2-mixingale sequences with size $-1/2$ and constants $\{c_{n,t}(u)\psi_{n,l_n}^{\eta}\}$ for some tiny $\eta > 0$: see de Jong (1997: p. 360-361). Applying McLeish's (1975) bound to each right-hand-side term of (13), and noting $\psi_{n,l_n}^{\eta} = O(l_n^{-\eta/2}) = o(1)$,

$$\begin{aligned}
E\left(\sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} W_{n,i}(u)\right)^2 &= O\left(\sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} c_{n,t}^2(u)\right) \\
&= \delta \times O\left(r_n^*(\xi) g_n / n\right).
\end{aligned}$$

By Assumption 1.b there exists for each $\{\xi, \delta\} \in [\underline{\xi}, 1] \times [0,1]$ a finite $\kappa(\xi, \delta) \geq 1$ satisfying

$$\begin{aligned}
[r_n^*(\xi)\delta] g_n / n &\leq [r_n^*(\xi)\delta] g_n / n(\xi) \leq \left(r_n(\xi+\delta)/r_n(\xi) - 1\right) \times (1 + o(1)) \\
&\qquad \rightarrow \kappa(\xi, \delta) - 1 < \infty.
\end{aligned}$$

But if $[r_n^*(\xi)\delta] g_n / n$ is bounded then so is $r_n^*(\xi) g_n / n$. Hence, for sufficiently large n and some $\kappa \geq 1$

$$\sum\nolimits_{i=1}^{[r_n^*(\xi)\delta]} E\left(W_{n,i}(u)\right)^2 \leq \delta \times \kappa.$$

Condition (f): Define $Y_{n,t}(u_1, u_2) := X_{n,t}(u_1) - X_{n,t}(u_2)$. Mimicking (13) and the subsequent logic and, exploiting the fact that linear functions of mixingales are mixingales, we deduce

$$\begin{aligned}
&\|W_n(\xi, u_1) - W_n(\xi, u_2)\|_2 \qquad (14) \\
&\quad \leq \left\| \sum\nolimits_{i=1}^{r_n(\xi)} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} Y_{n,t}(u_1, u_2) \right\|_2 \\
&\qquad + \left\| \sum\nolimits_{i=1}^{r_n(\xi)} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} \left(Y_{n,t}(u_1, u_2) - E[Y_{n,t}(u_1, u_2) | \tilde{F}_{n,i}] \right) \right\|_2 \\
&\qquad + \left\| \sum\nolimits_{i=1}^{r_n(\xi)} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} E[Y_{n,t}(u_1, u_2) | \tilde{F}_{n,i-1}] \right\|_2 \\
&\quad = O\left(\sum\nolimits_{i=1}^{r_n(\xi)} \sum\nolimits_{t=(i-1)g_n+l_n+1}^{ig_n} \tilde{c}_{n,t}^2 \times |u_1 - u_2|^{1/2} \right) \\
&\quad = |u_1 - u_2|^{1/2} \times O\left(r_n(\xi) g_n / n\right) \leq K \times |u_1 - u_2|^{1/2}.
\end{aligned}$$

Step 2 (increments)

For any points $0 < \xi_k < \xi_l < 1$ and $0 < \{u_k, u_l\} < 1$ we need only show

$$E\left(X_n(\xi_k, u_k) - X_n(\xi_{k-1}, u_{k-1})\right)\left(X_n(\xi_l, u_l) - X_n(\xi_{l-1}, u_{l-1})\right) = o_p(1).$$

Write

$$\begin{aligned}
&X_n(\xi_k, u_k) - X_n(\xi_{k-1}, u_{k-1}) \\
&\quad = \left[X_n(\xi_k, u_k) - X_n(\xi_{k-1}, u_k)\right] + \left[X_n(\xi_{k-1}, u_k) - X_n(\xi_{k-1}, u_{k-1})\right] \\
&\quad = A_{n,k} + B_{n,k}.
\end{aligned}$$

Analogous to de Jong and Davidson (2000: pp. 635-636), by exploiting MCleish's (1975) bound for L_2-mixingales of size $-1/2$ and the assumption $\sup_{u\in[0,1],t\geq 1} c_{n,t}(u) = O(n^{-1/2})$, for arbitrary $\delta \in [0,1]$ we obtain

$$
\begin{aligned}
|E[A_{n,k}A_{n,l}]| &\leq \left\| \sum_{t=n(\xi_{k-1})+1}^{n(\xi_k)} X_{n,s}(u_k) \right\|_2 \left\| \sum_{t=n(\xi_{l-1})+1}^{n(\xi_{l-1}+\delta)} X_{n,t}(u_l) \right\|_2 \\
&\quad + \sum_{t,s=1}^{n(1)} E\left| X_{n,t}(u_k) X_{n,s}(u_l) \right| I\left(|s-t| \geq n(\xi_{l-1}+\delta) - n(\xi_k) \right) \\
&= O\left(\sup_{0<\xi_i<1} \left[\left(n(\xi_i) - n(\xi_{i-1}) \right)/n \right] \right) + o_p(1) = o_p(1).
\end{aligned}
$$

The last line follows from Lemma A.3 of de Jong and Davidson (2000), and the assumptions $n(\xi) - n(\xi') \to \infty \ \forall \xi > \xi'$ and $n(1) \leq n$.

Next, for arbitrary $\delta > 0$

$$
\begin{aligned}
&|E[A_{n,k}B_{n,l}]| \\
&\leq \left\| \sum_{t=n(\xi_{k-1})+1}^{n(\xi_{k-1}+2\delta)} X_{n,t}(u_k) \right\|_2 \left\| \sum_{t=1}^{n(\xi_{l-1})} \left(X_{n,t}(u_l) - X_{n,t}(u_{l-1}) \right) \right\|_2 \\
&\quad + \left\| \sum_{t=n(\xi_{k-1}+2\delta)+1}^{n(\xi_k)} X_{n,t}(u_k) \right\|_2 \left\| \sum_{t=n(\xi_{k-1}+\delta)+1}^{n(\xi_{l-1})} \left(X_{n,t}(u_l) - X_{n,t}(u_{l-1}) \right) \right\|_2 \\
&\quad + \sum_{s,t=1}^{n(1)} E[\left| X_{n,t}(u_k) \left(X_{n,s}(u_l) - X_{n,s}(u_{l-1}) \right) \right| \\
&\qquad\qquad \times I\left(|s-t| \geq n(\xi_{k-1}+2\delta) - n(\xi_{k-1}+\delta) \right)] \\
&= O\left(\left(\sum_{t=n(\xi_{k-1})+1}^{n(\xi_{k-1}+2\delta)} c_{n,t}^2(u_k) \right)^{1/2} \left(\sum_{t=1}^{n(\xi_{l-1})} \tilde{c}_{n,t}^2 \right)^{1/2} \right) \\
&\quad + O\left(\left(\sum_{t=n(\xi_{k-1}+2\delta)+1}^{n(\xi_k)} c_{n,t}^2(u_k) \right)^{1/2} \left(\sum_{t=n(\xi_{k-1}+\delta)+1}^{n(\xi_{l-1})} \tilde{c}_{n,t}^2 \right)^{1/2} \right) + o_p(1) \\
&= o_p(1),
\end{aligned}
$$

because $\sup_{u\in[0,1],t\geq 1} c_{n,t}(u)$ and $\sup_{t\geq 1} \tilde{c}_{n,t}$ are $O(n^{-1/2})$, $0 < \xi_k < \xi_l < 1$ and $n(1) \leq n$.

Finally, because $0 < \xi_k < \xi_l < 1$ and $n(\xi_k) < n(\xi_l) < n$,

$$\begin{aligned}
|E\left[B_{n,k}B_{n,l}\right]| &\leq \left\| \sum_{t=1}^{n(\xi_{k-1})} (X_{n,s}(u_k) - X_{n,s}(u_{k-1})) \right\|_2 \\
&\qquad \times \left\| \sum_{s=1}^{n(\xi_{l-1})} (X_{n,s}(u_l) - X_{n,s}(u_{l-1})) \right\|_2 \\
&= O\left(\left(\sum\nolimits_{t=1}^{n(\xi_{k-1})} \tilde{c}_{n,t}^2 \right)^{1/2} \left(\sum\nolimits_{t=1}^{n(\xi_{l-1})} \tilde{c}_{n,t}^2 \right)^{1/2} \right) \\
&= O\left(\left(n(\xi_{k-1})/n \right)^{1/2} \left(n(\xi_{l-1})/n \right)^{1/2} \right) = o(1).
\end{aligned}$$

■

Proof of Lemma 2. From (6) and (7), $\forall u \in [0,1]$

(15) $\bar{F}(b_n)u = \bar{F}(u^{-1/\alpha}b_n) \times L(b_n)/L(u^{-1/\alpha}b_n)$

where $u^{-1/\alpha} \geq 1$ and $L(b_n)/L(u^{-1/\alpha}b_n) \to 1$ by the slow variation property. Hence, for any $r \geq 1$ and $u \in [0,1]$

$$\begin{aligned}
&\lim_{n\to\infty} (n/k_n)^{1/r} k_n^{1/2} ||X_{n,t}^*(u)||_r \\
&\qquad \leq 2 \lim_{n\to\infty} \left[(n/k_n) P\left(\bar{F}(X_t) < \bar{F}(b_n)u \right) \right]^{1/r} \\
&\qquad = 2 \lim_{n\to\infty} \left[(n/k_n) P\left(\bar{F}(X_t) < \bar{F}(b_n u^{-1/\alpha}) \right) \right]^{1/r} \\
&\qquad = 2 \lim_{n\to\infty} \left[(n/k_n) P\left(X_t > b_n u^{-1/\alpha} \right) \right]^{1/r} \\
&\qquad = 2 \lim_{n\to\infty} \left[(n/k_n) P\left(X_t > b_n \right) \frac{P\left(X_t > b_n u^{-1/\alpha} \right)}{P\left(X_t > b_n \right)} \right]^{1/r} \\
&\qquad = 2u^{1/r} < \infty.
\end{aligned}$$

Similarly, arguments in Hsing (1991: p. 1554) imply

$$\begin{aligned}
&\lim\nolimits_{n\to\infty} (n/k_n)^{1/r} k_n^{1/2} ||X_{n,t}||_r \\
&\qquad \leq 2 \lim\nolimits_{n\to\infty} (n/m)^{1/r} ||(\ln X_t/b_n)_+||_r = 2 \left(\int_0^\infty e^{-\alpha y^{1/r}} dy \right)^{1/r} < \infty.
\end{aligned}$$

Therefore $X_{n,t}$ and $X_{n,t}^*(u)$ satisfy

(16) $\left\{ \|X_{n,t}\|_r , \left\|X_{n,t}^*(u)\right\|_r \right\} = O(k_n^{-(1/2-1/r)} n^{-1/r}).$

As long as $k_n \to \infty$ as $n \to \infty$, then trivially $k_n^{-(1/2-1)} n^{-1/2} > n^{-1/2}$, $k_n^{-(1/2-1/2)} n^{-1/2} = n^{-1/2}$, and $k_n^{-(1/2-1/r)} n^{-1/r} < n^{-1/r}$ $\forall r > 2$. Hence for some $a(r)$ where $a(1) > 1/2$, $a(2) = 1/2$, and $a(r) > 1/r\ \forall r > 2$, each $\{||X_{n,t}||_r, ||X^*_{n,t}(u)||_r\}$ $= O(n^{-a(r)})$. Similarly,

$$\begin{aligned} O(n^{-2a(2r)}) &= O\left(k_n^{-(1/2-1/2r)}n^{-1/2r}\right)^2 = O\left(k_n^{-(1-1/r)}n^{-1/r}\right) \\ &= O\left(k_n^{-(1/2-1/r)}n^{-1/r}\right)k_n^{-1/2} = O(k_n^{-1/2}n^{-a(r)}) \leq O(n^{-a(r)}) \end{aligned}$$

and $O(n^{-2a(4)}) = k_n^{-1/2}n^{-1/2}$. Hence $2a(2r) > a(r)$ and $2a(4) + a(2) > 1/2$.

If $k_n \sim n^\delta$, $\delta \in (0,1]$, then (16) implies $a(r) = 1/2 - (1-\delta)(1/2 - 1/r)$. ■

Proof of Theorem 6.

Step 1 ($\hat{\alpha}_n^{-1}(\xi)$): By Lemma A.7 for arbitrary $u \in [0,1]$

$$k_n^{1/2}\left(\hat{\alpha}_n^{-1}(\xi) - \alpha^{-1}\right) = \sum\nolimits_{t=1}^{[n\xi]}\left(X_{n,t} - \alpha^{-1}X^*_{n,t}(u^{k_n^{-1/2}})\right) + o_p(1),$$

where $\{X_{n,t}, X^*_{n,t}(\cdot)\}$ are defined in (8). Define

$$\check{\sigma}_n^2(\xi) := E\left(\sum\nolimits_{t=1}^{[n\xi]}\left(X_{n,t} - \alpha^{-1}X^*_{n,t}(u^{k_n^{-1/2}})\right)\right)^2.$$

By Lemma 4 $\{X_{n,t}, X^*_{n,t}(u)\}$ are L_2-FE-NED with constants $K(-\alpha^{-1}\int_0^1 u^{-1}d_{n,t}(u)^2du)^{1/2}$ $= O(n^{-a(r)})$ and $d_{n,t}(u) = O(n^{-a(r)})$ respectively, and common coefficients $\varphi_{q_n} = o(n^{a(r)-1/2} \times q_n^{-\lambda})$. An argument identical to Lemma 3 of Hill (2005b) shows $\{X_{n,t}, X^*_{n,t}(u)\}$ are L_2-mixingales with size $-1/2$ and constants of order $O(n^{-1/2})$, hence $\check{\sigma}_n^2(\xi) = O(1)$ follows from McLeish's (1975a) bounds.

The continuous mapping theorem and Corollary 4 now imply

$$\sum\nolimits_{t=1}^{[n\xi]}\left(X_{n,t} - \alpha^{-1}X^*_{n,t}(u^{k_n^{-1/2}})\right) \Rightarrow X(\xi)$$

for some Gaussian element $X(\xi)$ of $D[\underline{\xi},1]$ with variance $\lim_{n\to\infty}\check{\sigma}_n(\xi) < \infty$. Therefore, $|\sigma_n^2(\xi) - \check{\sigma}_n^2(\xi)| \to 0$ implying $\sigma_n^2(\xi) = O(1)$.

Step 2 ($X_{([\xi k_n])}$): By Lemma A.8 $\sum_{t=1}^{[n\xi]}X^*_{n,t}(u/\sqrt{k_n}) \Rightarrow Y(\xi)$ for some Gaussian element $Y(\xi)$ of $D[\underline{\xi},1]$ with variance

$$\lim_{n\to\infty}\tilde{\sigma}_n^2(\xi) = \lim_{n\to\infty}E\left(\sum\nolimits_{t=1}^{[n\xi]}X^*_{n,t}(u^{k_n^{-1/2}})\right)^2 < \infty.$$

The limit $\lim_{n\to\infty}\tilde{\sigma}_n^2(\xi) < \infty$ follows from Step 1. An argument identical to Theorem 2.2 of Hsing (1991) now completes the proof: $\sum_{t=1}^{[n\xi]}X^*_{n,t}(u^{k_n^{-1/2}}) \Rightarrow$ $Y(\xi)$ implies $k_n^{1/2}\ln X_{([k_n\xi])}/b_n \Rightarrow Y(\xi)$. ■

Appendix 2: Supporting Lemmata

We require the following notation. Let $\{\mathcal{F}_{n,t}\}$ be an arbitrary array of σ-fields and define

$$\tilde{F}_{n,i} := \bigcup\nolimits_{\tau \le i g_n} \mathcal{F}_{n,\tau}.$$

For any point $\{\xi, u\}$ and $\pi \in \mathbb{R}^k$, $\pi'\pi = 1$, write

$$W_n(\xi, u) := \sum\nolimits_{i=1}^{r_n(\xi)} W_{n,i}(u) = \sum\nolimits_{i=1}^{r_n(\xi)} \left(E[Z_{n,i}|\tilde{F}_{n,i}] - E[Z_{n,i}|\tilde{F}_{n,i-1}] \right)$$

$$\tilde{W}_{n,i}(\pi) := \sum\nolimits_{j=l}^{k} \pi_j W_{n,i}(u_j),\ i = r_n(\xi_{l-1}) + 1 ... r_n(\xi_l),\ l = 1...k.$$

LEMMA A.1 *Let $\{X_{n,t}(u)\}$ be an L_r-Functional Tail Array with r^{th}-moment index $a(r)$, $2a(4) + a(2) > 1$. Let $g_n = o(n^{[2a(4)+a(2)-1]/2})$. If*

$$\text{(a)} \quad \sum\nolimits_{i=1}^{r_n(\xi)} Z_{n,i}(u) \to 0,\ \forall j = 1...k$$

$$\text{(b)} \quad \sum\nolimits_{i=1}^{r_n(\xi)} E[Z_{n,i}(u)|\tilde{F}_{n,i-1}(\xi)] \to 0,$$

$$\text{(c)} \quad \sum\nolimits_{i=1}^{r_n(\xi)} \left(Z_{n,i}(u) - E[Z_{n,i}(u)|\tilde{F}_{n,i}] \right) \to 0,$$

$$\text{(d)} \quad \sup_{\pi'\pi=1} \left| \sum\nolimits_{i=1}^{r_n(\xi_k)} \tilde{W}^2_{n,i}(\pi) - 1 \right| \to 0,$$

$$\text{(e)} \quad \sum\nolimits_{i=1}^{[r^*_n(\xi)\delta]} E\left(W_{n,i}(u)\right)^2 \le \delta \times \kappa, \quad \forall \delta \in [0, 1],$$

for some some $\kappa \ge 1$ and sequence $\{r^*_n(\xi)\}$ satisfying $0 \le [\delta r^*_n(\xi)] \le r_n(\xi + \delta) - r_n(\xi)$, and $\forall u_1, u_2 \in [0, 1]$

$$\text{(f)} \quad \sup_{\xi \in [\underline{\xi}, 1]} \|W_n(\xi, u_1) - W_n(\xi, u_2)\|_2 \le K|u_2 - u_1|^{1/2},$$

then $X_n(\xi, u) \Rightarrow X(\xi, u)$ on D_2 where $X(\xi, u)$ is Gaussian with covariance function $E[X(\xi_i, u_i)X(\xi_j, u_j)]$.

Remark: Conditions (a)-(c) imply $X_n(\xi, u)$ is approximable by a partial sum of martingale differences $E[Z_{n,i}|\tilde{F}_{n,i}] - E[Z_{n,i}|\tilde{F}_{n,i-1}]$. Condition (d) ensures convergence of finite dimensional distributions of $\{W_n(\xi, u)\}$. Conditions (e) and (f) ensure the sequence $\{W_n(\xi, u)\}$ is uniformly tight with respect to ξ and u, respectively.

LEMMA A.2 *Under conditions (a)-(d) of Lemma A.1, $W_n(\xi, u) \to W(\xi, u)$ with respect to finite dimensional distributions, where $W(\xi, u)$ is Gaussian with covariance function $E[W(\xi_i, u_i)W(\xi_j, u_j)]$.*

LEMMA A.3 *Under conditions (e) and (f) of Lemma A.1 the sequence $\{W_n(\xi, u)\}$ is uniformly tight in D_2.*

LEMMA A.4

1. *Under Assumption 1.a* $\{X_{n,t}(u), \digamma_{n,t}\}$ *forms an* L_2*-mixingale sequence* (cf. McLeish 1975) *with size* $-1/2$ *and constants* $\sup_{u\in[0,1],t\geq 1} c_{n,t}(u) = O(n^{-1/2})$. *If particular,*

$$\begin{aligned} &\|X_{n,t}(u) - E[X_{n,t}(u)|\digamma_{n,t+q}]\|_2 \leq c_{n,t}(u)\psi_{q_n+1} \\ &\|E\left[X_{n,t}(u)|\digamma_{n,t-q}\right]\|_2 \leq c_{n,t}(u)\psi_{q_n}, \end{aligned}$$

and if $Y_{n,t}(u_1, u_2) := X_{n,t}(u_1) - X_{n,t}(u_2)$ *then*

$$\begin{aligned} &\|Y_{n,t}(u_1,u_2) - E[Y_{n,t}(u_1,u_2)|\digamma_{n,t+q}]\|_2 \leq (\tilde{c}_{n,t} \times |u_2 - u_1|) \times \psi_{q_n+1} \\ &\|E\left[Y_{n,t}(u_1,u_2)|\digamma_{n,t-q}\right]\|_2 \leq (\tilde{c}_{n,t} \times |u_2 - u_1|^{1/2}) \times \psi_{q_n}, \end{aligned}$$

$\forall u_1, u_1 \in [0,1]$, *where* $\sup_t \tilde{c}_{n,t} = O(n^{-1/2})$.

2. *Let* $\{(\xi_i, u_i)\}_{i=1}^{k,}$ *be arbitrary,* $k \geq 1$, $(\xi_i, u_i) \in [\underline{\xi}, 1] \times [0,1]$. *If additionally Assumption 1.b holds, then*

$$\sup_{\pi'\pi=1} \left| \sum_{l=1}^{k} \sum_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)} \left(\sum_{j=l}^{k} \pi_j Z_{n,i}(u_j) \right)^2 - 1 \right| \to 0, \quad \forall \{(\xi_i, u_i)\}_{i=1}^{k,}.$$

LEMMA A.5 (de Jong, Lemma 4) *If* $\{X_{n,t}, \digamma_{n,t}\}$ *is an* L_2*-mixingale with size* $-1/2$ *and constants* $\sup_t c_{n,t} = O(n^{-1/2})$ *then*

$$\lim_{n\to\infty} \left| \sum_{i=1}^{r_n} \sum_{k=i+1}^{r_n} \sum_{t=(i-1)g_n+l_n+1}^{ig_n} \sum_{s=(k-1)g_n+l_n+1}^{kg_n} E\left[X_{n,s}X_{n,t}\right] \right| = 0.$$

LEMMA A.6 *Under Assumption 1,* $\sum_{i=1}^{r_n(\xi)} (Z_{n,i}^2(u) - E[Z_{n,i}^2(u)\ ]) \to 0\ \forall \xi, u \in [\underline{\xi}, 1] \times [0,1]$.

LEMMA A.7 *Let the conditions of Theorem 6 hold. For any* $u \in [0,1]$

$$k_n^{1/2}\left(\hat{\alpha}_m^{-1}(\xi) - \alpha^{-1}\right) = \sum_{t=1}^{n(\xi)} \left(X_{n,t} - \alpha^{-1} X_{n,t}^*(u^{1/\sqrt{k_n}}) \right) + o_p(1).$$

LEMMA A.8 *Let the conditions of Theorem 6 hold. For any* $\tilde{u} \in \mathbb{R}$

$$\left\{ \sum_{t=1}^{n(\xi)} X_{n,t},\ \sum_{t=1}^{n(\xi)} X_{n,t}^*(u^{1/\sqrt{k_n}}) \right\} \Rightarrow \{X_1(\xi), X_2(\xi)\}$$

jointly on $D[\underline{\xi}, 1]$, *where each* $X_i(\xi)$ *is Gaussian.*

Proof of Lemma A.1. Write $X_{n,i}(u) = X_{n,i}$, $X_n(\xi, u) = X_n$, etc., and decompose

$$
\begin{aligned}
(17)\quad X_n \;=\; & \sum\nolimits_{i=1}^{r_n(\xi)} W_{n,i} \\
& + \sum\nolimits_{i=1}^{r_n(\xi)} E[Z_{n,i}|\tilde{F}_{n,i-1}] + \sum\nolimits_{i=1}^{r_n(\xi)} \left(Z_{n,i} - E[Z_{n,i}|\tilde{F}_{n,i-1}] \right) \\
& + \sum\nolimits_{i=1}^{r_n(\xi)} \sum\nolimits_{t=(i-1)g_n+1}^{(i-1)g_n+l_n} X_{n,t} + \sum\nolimits_{t=r_n(\xi)g_n+1}^{n(\xi)} X_{n,t}.
\end{aligned}
$$

By the definition of an L_r-Tail Array and $r_n(\xi) = [n(\xi)/g_n]$

$$
\left\| \sum\nolimits_{t=r_n(\xi)g_n+1}^{n(\xi)} X_{n,t} \right\|_1 = O\left((n(\xi) - r_n(\xi)g_n) n^{-a(1)} \right) = o(1).
$$

The second-through-fourth terms in (17) are $o_p(1)$ by conditions (a)-(c). Hence

$$
(18)\quad X_n(\xi, u) = \sum\nolimits_{i=1}^{r_n(\xi)} W_{n,i}(u) + o_p(1) = W_n(\xi, u) + o_p(1).
$$

By Lemma A.2 $W_n(\xi, u) \to W(\xi, u)$ with respect to finite dimensional distributions, where $W(\xi, u)$ is Gaussian with covariance function $E[W(\xi_i, u_i)W(\xi_j, u_j)]$.

By Lemma A.3 the sequence $\{W_n(\xi, u)\}$ is uniformly tight on D_2. Therefore $W_n(\xi, u) \Rightarrow W(\xi, u)$ on D_2 by Corollary 13.4 of Billingsley (1999). We conclude $X_n(\xi, u) \Rightarrow X(\xi, u) = W(\xi, u)$ given (18). ■

Proof of Lemma A.2. Pick any $\pi \in \mathbb{R}^k$, $\pi'\pi = 1$. For any finite collection $\{(\xi_j, u_j)\}_{j=1}^k$, $\xi_1 \leq ... \leq \xi_k$, write

$$
\sum\nolimits_{j=1}^{k} \pi_j W_n(\xi_j, u_j) = \sum\nolimits_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}(u, \pi),
$$

where

$$
\begin{aligned}
& \tilde{W}_{n,i}(u, \pi) \;:=\; \sum\nolimits_{j=l}^{k} \pi_j W_{n,i}(u_j),\; i = r_n(\xi_{l-1}) + 1 ... r_n(\xi_l),\; l = 1...k. \\
& W_{n,i}(u) = \left(E[Z_{n,i}(u)|\tilde{F}_{n,i}] - E[Z_{n,i}(u)|\tilde{F}_{n,i-1}] \right) \\
& \tilde{F}_{n,i} := \bigcup\nolimits_{\tau \leq i g_n} F_{n,\tau} \;\text{ and }\; F_{n,t} = \sigma(E_{n,\tau} : 1 \leq s \leq \tau \leq t \leq n)
\end{aligned}
$$

By construction $\{\tilde{W}_{n,i}(u, \pi), \tilde{F}_{n,i}\}$ forms a martingale difference sequence. The limit

$$
\sum\nolimits_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}(u, \pi) \to N\left(0, \lim\nolimits_{n\to\infty} E(\sum\nolimits_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}(u, \pi))^2 \right),
$$

follows from Theorem 2.3 of McLeish (1974), where $\lim_{n\to\infty} || \sum_{i=1}^{r_n(\xi_k)} \tilde{W}_{n,i}(u, \pi)||_2 \leq k$ follows from (18) and $||X_n(\xi, u)||_2 = 1$.

McLeish's condition (c) is our condition (d). Moreover, McLeish's conditions (a) and (b) hold if the Lindeberg condition holds, $\sum_{i=1}^{r_n(\xi_k)} E[\tilde{W}^2_{n,i}(u, \pi) \times$

$I(|\tilde{W}_{n,i}(u,\pi)| > \varepsilon)] \to 0$ for any $\varepsilon > 0$. Note $|\pi| \le |\pi|_2 = 1$, $r_n(\xi_k) \le r_n(1)$, and $r_n(1)g_n \sim n(1) \le n$. Therefore, for any $\varepsilon > 0$

$$
\begin{aligned}
&r_n(\xi_k) \max_i E\left[\tilde{W}^2_{n,i}(u,\pi) I(|\tilde{W}_{n,i}(u,\pi)| > \varepsilon)\right] \\
&\quad \le r_n(1) \max_i ||\tilde{W}_{n,i}(u,\pi)||_4^2 ||\tilde{W}_{n,i}(u,\pi)||_2 \varepsilon^{-1} \\
&\quad \le r_n(1) \max_i |\pi| \times | \times \sup_{u\in[0,1]} ||W_{n,i}(u)||_4^2 \sup_{u\in[0,1]} ||W_{n,i}(u)||_2 \varepsilon^{-1} \\
&\quad \le K \times r_n(1) g_n^2 \sup_t \sup_{u\in[0,1]} \|X_{n,t}(u)\|_4^2 \, g_n \sup_t \sup_{u\in[0,1]} \|X_{n,t}(u)\|_2 \, \varepsilon^{-1} \\
&\quad = O\left(r_n g_n^2 n^{-2a(4)} g_n n^{-a(2)}\right) \\
&\quad = O\left(g_n^2 n^{1-2a(4)-a(2)}\right) = o(1),
\end{aligned}
$$

given $g_n = o(n^{[2a(4)+a(2)-1]/2})$. The first inequality is Hölder's and Markov's; the second inequality is Jensen's; the third inequality and first equality follow from the definition of a Functional Tail Array and Minkowski's inequality.

A Crámer-Wold device completes the proof. ■

Proof of Lemma A.3. Let $\delta \in [0,1]$. Define the metrics $w(\cdot)$ and $w''(\cdot)$, cf. Billingsley (1999):

$$
\begin{aligned}
w(W_n,\xi,u,\delta) &= \sup_{|\xi-\xi'|\le\delta,\ |u-u'|\le\delta} \left|W_n(\xi,u) - W_n(\xi',u')\right| \\
w''(W_n,\xi,u,\delta) &= \sup_{\substack{\xi_1\le\xi\le\xi_2\ u_1\le u\le u_2 \\ \xi_2-\xi_1\le\delta\ u_2-u_1\le\delta}} \left\{|W_n(\xi,u) - W_n(\xi_1,u_1)| \wedge |W_n(\xi,u) - W_n(\xi_2,u_2)|\right\}.
\end{aligned}
$$

If ξ is fixed at some $\tilde{\xi}$ we write

$$
w(W_n,\tilde{\xi},u,\delta) = \sup_{|u-u'|\le\delta} \left|W_n(\tilde{\xi},u) - W_n(\tilde{\xi},u')\right|
$$

For any triplet $\{\xi,u,\delta\}$ clearly we can find some $\tilde{\xi} \in [\underline{\xi},1]$ and $\tilde{u} \in [0,1]$ such that

(19) $\quad w''(W_n,\xi,u,\delta) \le w''(W_n,\tilde{\xi},u,\delta) + w''(W_n,\xi,\tilde{u},\delta).$

Moreover, by construction (see Billingsley, 1999),

(20) $\quad w''(W_n,\xi,u,\delta) \le w(W_n,\xi,u,2\delta) \quad \forall\delta \in [0,1/2].$

From (19)-(20), and equation (13.10) in Billingsley (1999), it suffices to show for every $\varepsilon > 0$ and $\eta > 0$ there exists some $\delta \in [0,1/2]$ and $N \in \mathbb{N}$ such that $\forall n \ge N$

$$
P(w''(W_n,\tilde{\xi},u,\delta) > \varepsilon/2) + P\left(w(W_n,\xi,\tilde{u},\delta) > \varepsilon/2\right) \le \eta.
$$

The proof now follows from Lemmas A.3.1-A.3.3, below, and Corollary 5.4 and Theorem 13.3 of Billingsley (1999). ■

LEMMA A.3.1 *Let $u \in [0,1]$. Then, $\forall \varepsilon,\eta > 0$ there exists some $\delta \in [0,1/2]$ and $N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$*

$$P\left(\sup_{\xi \leq \xi' \leq \xi+\delta} \left|W_n(\xi', u) - W_n(\xi, u)\right| > \varepsilon/2\right) \leq \eta\delta/2.$$

Proof of Lemma A.3.1. Drop the common argument u for clarity. Let $Z \sim N(0,1)$ be chosen below. Choose any

$$\lambda > \max\{\varepsilon/\sqrt{2},\ 8 \times E|Z|^3 \times \kappa/\eta\varepsilon^2\}$$

and fix

$$\delta = \varepsilon^2/\kappa 4\lambda^2 \leq 1/2$$

for some finite $\kappa \geq 1$ to be chosen below.

We can always find a sequence of positive integers $\{r_n^*(\xi)\}_{n\geq 1}$ satisfying $0 \leq [r_n^*(\xi)\delta] \leq r_n(\xi+\delta) - r_n(\xi)$ and $0 \leq r_n^*(\xi) \leq n(\xi)$ such that

$$\begin{aligned}
&P\left(\sup_{\xi \leq \xi' \leq \xi+\delta} \left|W_n(\xi') - W_n(\xi)\right| > \lambda v_n\right) \\
&\qquad \leq P\left(\sup_{1 \leq j \leq [r_n^*(\xi)\delta]} \left|\sum_{i=1}^{r_n(\xi)+j} W_{n,i} - \sum_{i=1}^{r_n(\xi)} W_{n,i}\right| > \lambda v_n\right) \\
&\qquad \leq E\left|\sum\nolimits_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*(\xi)\delta]} W_{n,i}/v_n\right|^3 \lambda^{-3}
\end{aligned}$$

where the second inequality is Kolmogorov's, and $v_n := ||\sum_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*\delta]} W_{n,i}||_2$. By construction

$$E\left|Z\right|^3/\lambda^3 < \eta\varepsilon^2/\kappa 8\lambda^2 = \eta\delta/2,$$

and from Lemma A.2

$$\sum\nolimits_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*(\xi)\delta]} W_{n,i}/v_n \rightarrow Z.$$

Thus, there exists a sufficiently large N_0 such that $\forall n \geq N_0$

$$E\left|\sum\nolimits_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*(\xi)\delta]} W_{n,i}/v_n\right|^3 \lambda^{-3} \leq \eta\varepsilon^2/8\lambda^2 = \eta\delta/2.$$

Furthermore, by condition (e) of Lemma A.1 and the martingale difference property we have for some finite $\kappa \geq 1$

$$v_n^2 = E\left(\sum\nolimits_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*(\xi)\delta]} W_{n,i}\right)^2 = \sum\nolimits_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*(\xi)\delta]} E\left(W_{n,i}\right)^2 \leq \delta \times \kappa.$$

Hence, for sufficiently large N_0, $\forall n \geq N_0$, and $\delta = \varepsilon^2/\kappa 4\lambda^2$,

$$\lambda v_n \leq \lambda \delta^{1/2}\kappa^{1/2} = \varepsilon/2.$$

We deduce for some $\delta \in [0, 1/2]$ and $\forall n \geq N_0$

$$P\left(\sup_{\xi \leq \xi' \leq \xi+\delta} \left|W_n(\xi') - W_n(\xi)\right| > \varepsilon/3\right) \leq E\left|\sum_{i=r_n(\xi)+1}^{r_n(\xi)+[r_n^*(\xi)\delta]} W_{n,i}/v_n\right|^3 \lambda^{-3} \leq \eta\delta/2.$$

■

LEMMA A.3.2 *Let $\tilde{\xi} \in [\underline{\xi}, 1]$ be arbitrary. Then $\forall \varepsilon, \eta > 0$ there exists some $\delta \in [0, 1/2]$ and $N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$*

$$P\left(w''(W_n, \tilde{\xi}, u, \delta) > \varepsilon/2\right) \leq \eta/2$$

Proof of Lemma A.3.2. Drop the common argument ξ. Let $0 \leq u_1 \leq u_2 \leq u_3 \leq 1$ be arbitrary. By condition (f) of Lemma A.1 and $r_n(\tilde{\xi}) := [n(\tilde{\xi})/g_n]$

$$\begin{aligned}
&E\left[|W_n(u_1) - W_n(u_2)|\,|W_n(u_2) - W_n(u_3)|\right] \\
&\quad \leq \|W_n(u_1) - W_n(u_2)\|_2 \,\|W_n(u_2) - W_n(u_3)\|_2 \\
&\quad = \left\|\sum_{i=1}^{r_n(\tilde{\xi})} (W_{n,i}(u_1) - W_{n,i}(u_2))\right\|_2 \\
&\qquad\quad \times \left\|\sum_{i=1}^{r_n(\tilde{\xi})} (W_{n,i}(u_2) - W_{n,i}(u_3))\right\|_2 \\
&\quad = |u_2 - u_1|^{1/2} \times |u_3 - u_2|^{1/2} \times O\left(r_n(\tilde{\xi})g_n/n\right) \\
&\quad \leq K \times |u_3 - u_1|,
\end{aligned}$$

for some finite $K > 0$. Now apply (13.13)-(13.14) of Billingsley (1999). ■

LEMMA A.3.3 For every $\varepsilon > 0$, $\lim_{\delta \to 0} P(|W_n(1,1) - W_n(1-\delta, 1-\delta)| > \varepsilon) = 0$.

Proof of Lemma A.3.3. Using conditions (e) and (f) of Lemma A.1, clearly there exists some $\{\tilde{\xi}, \tilde{u}\} \in [\underline{\xi}, 1] \times [0, 1]$ such that

$$\begin{aligned}
&P(|W_n(1,1) - W_n(1-\delta, 1-\delta)| > \varepsilon) \\
&\quad \leq P(|W_n(\tilde{\xi}, 1) - W_n(\tilde{\xi}, 1-\delta)| > \varepsilon/2) + P(|W_n(1, \tilde{u}) - W_n(1-\delta, \tilde{u})| > \varepsilon/2) \\
&\quad \leq 4\varepsilon^{-2}\left[\left\|W_n(\tilde{\xi}, 1) - W_n(\tilde{\xi}, 1-\delta)\right\|_2^2 + (r_n(1) - r_n(1-\delta))^2 \sup\nolimits_{i \geq 1} \|W_{n,i}(\tilde{u})\|_2^2\right] \\
&\quad \leq 4\varepsilon^{-2}\left[K \times \delta^2 + (r_n(1) - r_n(1-\delta))^2 \sup\nolimits_{i \geq 1} \|W_{n,i}(\tilde{u})\|_2^2\right].
\end{aligned}$$

As $\delta \to 0$ the right-hand-side vanishes due to $r_n(1-) = r_n(1)$ given $n(1-) = n(1)$. ■

Proof of Lemma A.4.

Step 1: If the base $\{\epsilon_{n,t}\}$ is E-strong mixing then Theorem 17.5 of Davidson (1994) implies

$$\begin{aligned} &\left\| X_{n,t}(u) - E\left[X_{n,t}(u)|\digamma_{n,t-q}\right]\right\|_2 \\ &\qquad \le \max\left\{\left\|X_{n,t}(u)\right\|_r,\ d_{n,t}(u)\right\} \times \max\left(6\varepsilon_{q_n}^{1/2-1/r},\ \varphi_{q_n}\right). \end{aligned}$$

By assumption $n^{a(r)-1/2}q_n^{r/(r-2)}\varepsilon_{q_n} = o(1)$, $\sup_t \sup_{u\geq 0}\left\|X_{n,t}(u)\right\|_r = O(n^{-a(r)})$, $\sup_t \sup_{u\geq 0} d_{n,t}(u) = O(n^{-a(r)})$, and $\varphi_{q_n} = o(n^{a(r)-1/2} \times q_n^{-1/2})$. We may therefore write for sufficiently large $K > 0$,

$$\begin{aligned} &\left\| X_{n,t}(u) - E\left[X_{n,t}(u)|\digamma_{n,t-q}\right]\right\|_2 \\ &\quad \le K \times n^{-1/2}\max\left\{\left(n^{[1/2-a(r)]2r/[r-2]}\varepsilon_{q_n}\right)^{1/2-1/r}, n^{1/2-a(r)}\varphi_{q_n}\right\} \\ &\quad = c_{n,t}(u) \times \psi_{q_n}, \end{aligned}$$

say, where $\sup_t \sup_{u\geq 0} c_{n,t}(u) = O(n^{-1/2})$ is trivial and $\psi_{q_n} = o(q_n^{-1/2})$ follows from the properties of E-Mixing and FE-NED coefficients. A similar argument holds for the remaining mixingale bound, $||X_{n,t}(u) - E\left[X_{n,t}(u)|\digamma_{n,t+q}\right]||_2 \le c_{n,t}(u) \times \psi_{q_n+1}$, and in the E-uniform mixing case. Consult Davidson (1994).

An identical argument can be applied to $Y_{n,t}(u_1, u_2) = X_{n,t}(u_1) - X_{n,t}(u_2)$.

Step 2: The limit

$$\sup_{\pi'\pi=1}\left|\sum\nolimits_{l=1}^{k}\sum\nolimits_{i=r_n(\xi_{l-1})+1}^{r_n(\xi_l)}\left(\sum\nolimits_{j=l}^{k}\pi_j Z_{n,i}(\xi_j, u_j)\right)^2 - 1\right| \to 0$$

$\forall l = 1...k$ and each $\{(\xi_i, u_i)\}_{i=1}^{k,}$ now follows from Lemmas A.5 and A.6 and an argument identical to de Jong's (1997: A.39-A.41). ■

Proof of Lemma A.6. Because u and ξ are arbitrary, the claim follows from Lemma A.4 of Hill (2005b). ■

Proof of Lemma A.7. Write $b_{[n\xi]} = b_{[n\xi]}([k_n\xi])$ and

(21)

$$\begin{aligned} &k_n^{1/2}\left(\hat{\alpha}_m^{-1}(\xi) - \alpha^{-1}\right) \\ &\quad = k_n^{1/2}\left(1/[k_n\xi]\sum\nolimits_{i=1}^{[k_n\xi]}\ln X_{(i)}/X_{([k_n\xi]+1)} - \alpha^{-1}\right) \\ &\quad = k_n^{1/2}\left(1/[k_n\xi]\sum\nolimits_{i=1}^{[k_n\xi]}\ln X_{(i)}/b_{[n\xi]} - E\left(1/[k_n\xi]\sum\nolimits_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+\right)\right) \\ &\qquad\quad - k_n^{1/2}\ln X_{([k_n\xi]+1)}/b_{[n\xi]} + k_n^{1/2}\left[E\left(1/[k_n\xi]\sum\nolimits_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+\right) - \alpha^{-1}\right]. \end{aligned}$$

From (12) and arguments in Hsing (1991: p. 1554)

$$(22)\quad E\left(1/[k_n\xi]\sum\nolimits_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+ - \alpha^{-1}\right) = o(1/\sqrt{[k_n\xi]}).$$

Moreover, from Lemma A.8

$$\left\{\sum\nolimits_{t=1}^{[n\xi]} X_{n,t}, \sum\nolimits_{t=1}^{[n\xi]} X_{n,t}^*(u^{1/\sqrt{k_n}})\right\} \Rightarrow \{X_1(\xi), X_2(\xi)\}$$

jointly on $D[\underline{\xi}, 1]$, where each $X_i(\xi)$ is Gaussian. Furthermore, under the maintained assumptions $|\ln X_{([\rho[k_n\xi]])} - \ln b_n(\rho[k_n\xi])| \to 0$ for all ρ in an arbitrary neighborhood of 1 by Lemma 1 of Hill (2005b). Therefore an argument identical to Theorem 2.2 of Hsing (1991: eq. 2.4-2.7) applies:

(23)

$$k_n^{1/2}\left(\frac{1}{[k_n\xi]}\sum\nolimits_{i=1}^{[k_n\xi]}\ln X_{(i)}/b_{[n\xi]} - E\left(\frac{1}{[k_n\xi]}\sum\nolimits_{t=1}^{[n\xi]}\left(\ln X_t/b_{[n\xi]}\right)_+\right)\right) \Rightarrow X_1(\xi)$$

$$\alpha\times\sum\nolimits_{t=1}^{[n\xi]} X_{n,t}^*\left(u^{1/\sqrt{k_n}}\right) = \alpha\times\sum\nolimits_{t=1}^{[n\xi]} X_{n,t}^*\left(\tilde{u}/\sqrt{k_n}\right) \Rightarrow X_2(\xi),$$

where

$$X_{n,t}^*(\tilde{u}) := k_n^{-1/2}\left(I(X_t > b_n(k_n)e^{\tilde{u}}) - E[I(X_t > b_n(k_n)e^{\tilde{u}})]\right)$$
$$\tilde{u} = -(1/\alpha)\ln u.$$

Together, (21)-(23) imply

$$k_n^{1/2}\left(\hat{\alpha}_m^{-1}(\xi) - \alpha^{-1}\right) = \sum\nolimits_{t=1}^{[n\xi]}\left(X_{n,t} - \alpha^{-1}X_{n,t}^*(u^{k_n^{-1/2}})\right) + o_p(1).$$

■

Proof of Lemma A.8. If $\{X_{n,t}, X_{n,t}^*(u)\}$ are L_r-Tail Arrays with moment index $a(r)$ then by Minkowski's inequality so is $\{\pi_1 X_{n,t} + \pi_2 X_{n,t}^*(u)\}$ $\forall\pi \in \mathbb{R}^2$, $\pi'\pi = 1$. Moreover, under the maintained assumptions Lemma 4 and Theorem 17.8 of Davidson (199) imply $\{\pi_1 X_{n,t} + \pi_2 X_{n,t}^*(u)\}$ is L_2-NED on $\{\digamma_{n,t}\}$ with size $-1/2$ .The claim now follows by applying Corollary 4 to $\{\pi_1 X_{n,t} + \pi_2 X_{n,t}^*(u^{k_n^{-1/2}})\}$ and invoking a Cramér-Wold device. ■

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