8-9-2008

# Nash Demand Game and the Kalai-Smorodinsky Solution

Nejat Anbarci
*Department of Economics, Florida International University*

John H. Boyd III
*Department of Economics, Florida International University*, boydj@fiu.edu

# NASH DEMAND GAME AND THE KALAI-SMORODINSKY SOLUTION

NEJAT ANBARCI AND JOHN H. BOYD III

ABSTRACT. We introduce two new variations on the Nash demand game. One, like all known Nash-like demand games so far, has the Nash solution outcome as its equilibrium outcome. In the other, the range of solutions depends on an exogenous breakdown probability; surprisingly, the Kalai-Smorodinsky outcome proves to be the most robust equilibrium outcome. While the Kalai-Smorodinsky solution always finishes on top, there is no possible general ranking among the remaining solution concepts considered; in fact, the rest of the solution concepts take their turns at the bottom at various bargaining problems, depending on the specifics of the bargaining setup.

## 1. INTRODUCTION

More than half-a-century ago the publication of Nash's paper "Two Person Cooperative Games" (Nash, 1953) established a new research agenda, commonly referred to as the Nash program (see Binmore, 1998). It utilizes the strategic (non-cooperative) approach to provide non-cooperative foundations for cooperative bargaining solution concepts. The prototype is Nash's demand game (Nash, 1953). In it, two players simultaneously make demands; each player receives the payoff they demand if the demands are jointly feasible, and nothing otherwise.

While the simplicity of Nash's game is its great virtue, it has a major downside: every point on the Pareto frontier is a Nash equilibrium outcome. There have been several attempts to rectify this problem. The first attempt was by Nash (1953) himself. He used a "smoothing"

approach in which incompatible demand combinations did not necessarily lead to zero payoffs. Although this smoothing attempt uniquely provided non-cooperative foundations for the Nash solution, it was not deemed reasonable by game theorists since that time and several alternatives have been proposed.

In this paper, we introduce two new variations on the Nash demand game. The first is a simultaneous move procedure that yields the Nash solution as its unique equilibrium outcome. The second is based on an exogenous breakdown probability. It can have multiple equilibria, but they always include the Kalai-Smorodinsky solution, sometimes uniquely.

The first of our demand games is in the tradition of Howard (1992) and Rubinstein et al. (1992). These papers avoided the multiplicity of equilibria without resorting to smoothing. The common feature of these attempts was that they were sequential in nature—they did not involve simultaneous demands by players any more. Our demand game shows that a non-smoothing simultaneous version of the Nash demand game can also lead to the Nash solution outcome uniquely.

Moulin (1984) proposed an alternative procedure where the unique subgame-perfect equilibrium outcome coincides with that of the Kalai-Smorodinsky solution. However, his setup is very different from Nash demand games. Moulin's procedure has a first-price auction format that requires players to bid for the first-mover advantage.

The existing literature might lead one to think that all variations on the Nash demand game lead to the Nash solution outcome, and that other types of procedures, such as the auction framework of Moulin (1984), are necessary to obtain other outcomes. To think so would be a mistake. Our second demand game yields the Kalai-Smorodinsky solution.

This demand game takes its inspiration from the Rubinstein et al. (1992) version of the Nash demand game. Their original feature pertains to a player-induced endogenous break-down probability of the procedure. We instead consider an exogenous break-down probability, and further we do so in a simultaneous version of the Nash demand game.

In our game, two players simultaneously make demand proposals. Each player receives the payoff they demand if the demands are jointly feasible. Otherwise, the game continues with probability $p$; thus, it terminates with probability $(1-p)$, in which case players receive nothing. If the game continues, each player's proposal is selected at random with equal probabilities. Most importantly, we find that at the lowest level of the break-down probability that allows the players to come to an agreement, there is a unique Nash equilibrium where the players agree.[1] That equilibrium is always the Kalai-Smorodinsky solution outcome (which will sometimes coincide with other solution outcomes).

One may then wonder which solution concept(s) would next follow the Kalai-Smorodinsky solution as we raise the break-down probability. We examine this issue for several well-known solution concepts. All that can be said is that the Equal Sacrifice outcome is supported whenever the Egalitarian solution is supported. Beyond that, anything goes. We provide examples in which the (1) Egalitarian and Equal Sacrifice, (2) Average Payoff, (3) Nash, and (5) Equal Area solutions rank last. Thus, while the Kalai-Smorodinsky solution is always on top, there is no possible general ranking among the remaining solution concepts.

Section two sets up the bargaining problem and defines some standard solutions. Our two Nash demand games are examined in sections three and four. Some concluding remarks are in section five.

## 2. The Bargaining Problem and Solution Concepts

A two-person cooperative bargaining problem is described by a pair $(S, d)$ where $S \subset \mathbf{R}_+^2$ is the *utility possibility set* with *disagreement point* $d \in S$ being the utility allocation that results if no agreement is reached. For notational convenience, let $d = (d_1, d_2)$ be normalized such that each $d_i = 0$ for each $i = 1, 2$. With $d$ so defined, the bargaining problem is defined by $S$ alone.

We use the following notation for vector inequalities: $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2$; $x > y$ means $x \geq y$ and there is some $i$ with $x_i > y_i$; $x \gg y$ means $x_i > y_i$ for all $i$. The set $S$ is assumed to contain

---

[1]There are also two equilibria where the players fail to agree.

some $x \gg (0,0)$ and to be convex, compact, and comprehensive. The last means that if $x \in S$ and $(0,0) \le y \le x$, then $y \in S$.

Let $B$ be the set of all such bargaining problems $S$. A *solution* is a function $f \colon B \to \mathbf{R}_+^2$ with $f^S = (f_1^S, f_2^S) \in S$ for all $S \in B$. Let $\partial S$ denote the *Pareto frontier* (or *boundary*) of $S$. Thus $\partial S = \{x \in S : x' > x$ implies $x' \notin S\}$.

The *Nash solution* $N$ is defined such that its outcome for $S$ maximizes $N_1 N_2$ over the set $S$ (Nash, 1950).

Let $b_i^S = \max\{x_i : (x_1, x_2) \in S\}$. The *Kalai-Smorodinsky solution* $KS$ is defined such that its outcome for $S$ is the maximal point in $S$ with $KS_1/KS_2 = b_1^S/b_2^S$ (Kalai and Smorodinsky, 1975).

Let $l^S$ be the line through the origin dividing $S$ into equal areas. The *Equal Area solution* is defined such that its outcome for $S$ is $EA = l^S \cap \partial S$ (Anbarci and Bigelow, 1994; Anbarci, 1993).

Let $m^S$ be the center of gravity of $S$. The *Average Payoff solution* $AP$ is defined such that its outcome for $S$ is the intersection of the line through the origin and $m^S$ with the Pareto frontier $\partial S$ (Anbarci, 1995).

All of the above solution concepts satisfy the axioms of Symmetry, Weak Pareto Optimality, and Scale Invariance. As a consequence, their outcomes coincide not only in any symmetric $S$, but also in any $S$ with a linear frontier. In that case the outcome for all is the mid-point of the Pareto frontier.

Two prominent solution concepts fail to satisfy one of the three axioms mentioned above, the Egalitarian and Equal Sacrifice solutions. Both fail Scale Invariance. The *Egalitarian solution* $E$ has solution outcome for $S$ as the point $(E^S, E^S) \in S$ that maximizes $E^S$ (Kalai, 1977; Roth, 1977). Regardless of how asymmetric $S$ is, $E^S$ is always on the $45^o$ line. Thus given an asymmetric $S$ with a linear Pareto frontier, its outcome never coincides with any of the solutions above.

The *Equal Sacrifice solution* $ES$ is the other prominent solution concept that fails to satisfy Scale Invariance. Its outcome for $S$ is the point $(ES_1, ES_2)$ that is maximal in $S$ among all points obeying $b_1 - ES_1 = b_2 - ES_2$ (Chun, 1988). When the bargaining set is asymmetric, it never coincides with the solutions satisfying all three axioms.

One more well-known solution concept that fails one of the three axioms is the *Dictatorial solution*. There are two Dictatorial solutions, $D(1)$ and $D(2)$. Their outcomes for $S$ are defined by the equations $D^S(i) = b_i^S$. It fails to satisfy Symmetry, and therefore its outcome never coincides with any of the other solutions on any $S$ (Bigelow and Anbarci, 1993).

## 3. NASH'S DEMAND GAME AND ITS MODIFICATIONS

In *Nash's demand game* (Nash, 1953), players 1 and 2 simultaneously make demands $x_i$ with $x_i \in [0, b_i]$. If $x = (x_1, x_2) \in S$, player $i$ receives $x_i$. Otherwise, both players get 0.

The canonical form of this game is a prototype of a very general problem: how should the gains from cooperation be divided among the participants? As Binmore (1998) puts it: "Bargaining between two individuals is worthwhile when an agreement between them can create a surplus that would otherwise be unavailable.... The archetypal version of this problem is called *dividing the dollar*" (p. 21). "Many bargaining problems have this simple structure. For example, wage negotiations often reduce to a dispute over how the surplus created by the joint efforts of a firm and its workers should be divided" (p. 69).

This game, however, has a multiplicity of Nash equilibria since every point on the Pareto frontier is a Nash equilibrium outcome (as a matter of fact, $(0, 0)$ is also a Nash equilibrium outcome; in that equilibrium each player $i$ demands $b_i$). There have been several attempts to rectify this problem.

The use of such a mechanism can be motivated by thinking of an outside agent, an arbitrator, who tries to help the parties to resolve their disputes via a mechanism that will induce the parties to reach a desirable outcome on their own. In the industrial relations literature, for instance, providing an incentive for the parties to resolve their dispute voluntarily is considered an important virtue of an arbitration mechanism (for instance, see Bloom (1981), and the references therein). In addition, such a mechanism should not be too punitive, unlike the original Nash demand game.

It is evident that Nash's demand game punishes both players severely regardless of how close $(x_1, x_2)$ is to being in $S$. Nash (1953) was the first to address this problem. He proposed the following variation of his original demand game: Given $(x_1, x_2)$, denote player $i$'s payoff function by $L_i = x_i H(x_1, x_2)$ where $H(x_1, x_2) = 1$ for $(x_1, x_2) \in S$ and $H(x_1, x_2) = 0$ otherwise. In order to secure a unique Nash equilibrium, Nash (1953) suggested *smoothing* the payoff function $L$ by replacing the indicator function $H$ with a continuous approximation $h$ such that $h$ equals $H$ on $S$, but then drops off to zero in a continuous way. The smoothed payoff function for player $i$ is the expected utility $G_i(x_1, x_2) = x_i h(x_1, x_2)$. Nash (1953) proved that as $h$ approaches $H$, the Nash solution outcome becomes the unique Nash equilibrium outcome of this modified demand game. Remarkably, this holds regardless of the form of $h$.

Smoothing, however, has not been received too well by many other scholars since it lacked motivation. Luce and Raiffa (1957, p. 142) called it "a completely artificial mathematical escape from the troublesome nonuniqueness," and questioned its "relevance to the players." Schelling (1960, p. 283) stated that smoothing was "in no sense logically necessary" in such a prototypical bargaining setup. On the other hand, with smoothing, players are not necessarily assigned their disagreement payoffs when the demands are incompatible.

Howard (1992) studied a procedure whose subgame perfect equilibrium outcome coincides with that of the Nash solution. This procedure also assumes common knowledge of preference between players. Let $A$ be a finite set of alternatives. The players consider lotteries over the alternatives in $A$ as possible resolutions of the bargaining problem. Let $M$ be the set of all probability distributions over $A$. Given a distribution in $M$ define the choice (bargaining) set $S \subset \mathbf{R}_+^2$ using von Neumann-Morgenstern utility $u_i \colon M \to \mathbf{R}$ for $i = 1, 2$.

There are three phases in Howard's procedure:

> ***Phase 1:*** Players 1 and 2 propose $(x_1, x_2^*)$ and $(x_1^*, x_2)$ in $S$, respectively. If $(x_1, x_2) \in S$, then each player $i$ receives his demand $x_i$. Otherwise, we go to Phase 2.

***Phase 2:*** Player 1 announces some $r \in [0, 1]$. Player 2 has three options: (1) He may accept $(x_1, x_2^*)$; (2) he may 'counter by announcing some $t \in (r, 1]$, then Player 1 may choose between $t(x_1, x_2^*)$ and $(x_1^*, x_2)$; or (3) he may 'challenge', then Player 1 may either accept $(x_1^*, x_2)$ or counter by announcing some $r' \in (r, 1]$ so that Player 2 may choose either $r'(x_1^*, x_2)$ or $(x_1, x_2^*)$.

***Phase 3:*** If no agreement is reached in Phase 2, the players get the disagreement point $(0, 0)$.

We propose a new *Simultaneous Procedure* which retains the first phase of Howard's procedure, but replaces the remainder by a simultaneous move scheme that rewards more generous proposals:

***Phase 1:*** Players 1 and 2 propose $(x_1, x_2^*)$ and $(x_1^*, x_2)$ in $S$, respectively. If $(x_1, x_2) \in S$, the players get $(x_1, x_2)$. Otherwise we proceed to Phase 2.

***Phase 2:*** Each Player $i$ announces some $t_i \in [0, 1]$. After these announcements, each Player $i$ may or may not choose $t_j(x_i, x_j^*)$.

   (1) If Player $i$ chooses $t_j(x_i, x_j^*)$ and Player $j$ does not choose $t_i(x_i^*, x_j)$, then $t_j(x_i, x_j^*)$ is the outcome.

   (2) If both players choose $t_j(x_i, x_j^*)$ and $t_i(x_i^*, x_j)$ simultaneously, a coin toss determines the outcome.

   (3) If neither $t_j(x_i, x_j^*)$ nor $t_i(x_i^*, x_j)$ is chosen, but $t_i = t_j$, then a coin toss determines the outcome.

   (4) If neither $t_j(x_i, x_j^*)$ nor $t_i(x_i^*, x_j)$ is chosen, but $t_i > t_j$, then $t_i(x_i, x_j^*)$ becomes the outcome.

If $t_i > t_j$, then Player $i$ is more generous in that he is willing to accept a bigger proportion of his opponent's initial proposal than the proportion of his initial proposal that his opponent is willing to accept. That is, at the last part of Phase 2, if there is no agreement, but Player $i$ is more generous, then he secures the $t_i$ proportion of his initial proposal. Thus, if his generosity is not appreciated by his opponent, that generosity is applied to his own initial proposal. In a sense, each player has to be prepared to taste his own medicine, the medicine that he has prescribed for the other player.

**Theorem 1.** *The iterated strict dominance equilibrium outcome of the Simultaneous Procedure is the Nash solution outcome.*

*Proof.* We know that the highest possible product of the form $x_1 x_2^*$ is the Nash product. Suppose Player 1 proposes the Nash product $(x_1, x_2^*)$. If Player 2 proposes $(x_1^*, x_2)$ with $x_2 < x_2^*$, then $(x_1, x_2) \in S$ and becomes the outcome. This is worse for 2 than the Nash outcome. Thus Player 2 will not respond to the Nash outcome $(x_1, x_2^*)$ with $x_2 < x_2^*$. Moreover, if $x_1^* > x_1$, $x_2 < x_2^*$, so Player 2's response will obey $x_1^* \leq x_1$, $x_2 \geq x_2^*$. Of course, if Player 2 also proposes the Nash outcome, the procedure ends after Phase 1.

Suppose that the proposals by the two players do not coincide and that $x_1 x_2^* > x_1^* x_2$ (we consider the case $x_1 x_2^* > x_1^* x_2$ later). Since the proposals do not coincide, $1 > x_2^*/x_2 > x_1^*/x_1 \geq 0$. In order to secure $t_1(x_1, x_2^*)$ as the outcome, Player 1 must announce some $t_1$ in Phase 2 such that $t_1$ is slightly below $x_2^*/x_2$ but greater than $x_1^*/x_1$. To see that, note that if $t_1 < x_1^*/x_1$, then Player 2 making an announcement of $t_2 = 1$ at Phase 2 would reject choosing $t_1(x_1^*, x_2)$ and would prefer having $t_2(x_1, x_2^*) = (x_1, x_2^*)$ instead. In addition, if $t_1$ is not slightly below $x_2^*/x_2$, then Player 2 might choose $t_1(x_1^*, x_2)$ which is worse than $t_1(x_1, x_2^*)$ for Player 1.

Likewise, when $x_1 x_2^* < x_1^* x_2$, in order to secure $t_2(x_1^*, x_2)$, Player 2 must announce some $t_2$ in Phase 2 such that $t_2$ is slightly below $x_1^*/x_1$ but greater than $x_2^*/x_2$.

When $x_1 x_2^* = x_1^* x_2$, then observe that each Player $i$ has to announce a $t_i = x_1^*/x_1 = x_2^*/x_2$. Then in Phase 1, each Player $i$ will propose a point in $S$ with a higher Nash product than $x_1 x_2^* = x_1^* x_2$. This bidding will stop when $x_1 x_2^* = x_1^* x_2$ is indeed the Nash product. $\square$

Note that the player with a more generous proposal (in the sense that it complies more with his opponent's demand for himself) gets rewarded and the one with less generous demand gets relatively punished.

**Example 1.** Consider the convex hull of $\{(0,0), (4,0), (0,2)\}$. The Nash solution outcome is $(2, 1)$. Thus the Nash product is 2. Suppose Player 1 proposes $(2.5, 0.75)$ (with a Nash product of 1.875) and Player 2 proposes $(1, 1.5)$ (with a Nash product of 1.5). Note that $x_1^*/x_1 =$

$1/2.5 = 0.4$ and $x_2^*/x_2 = 0.75/1.5 = 0.5$. Then Player 1 can announce some $t_1 = 0.5 - \epsilon$ where $\epsilon$ is a small and positive with $\epsilon < 0.1$. Then $t_1 = 0.5 - \epsilon > 0.4 = x_1^*/x_1$. In that case, Player 1 will not choose any $t_2(x_1, x_2^*)$ since his $t_1$ will be greater than Player 2's $t_2$ and Player 1 prefers $t_1(x_1, x_2^*) = (0.5 - \epsilon)(2.5, 0.75) = (1.25, 0.375) - \epsilon(2.5, 0.75)$ to any such $t_2(x_1, x_2^*) \leq 0.4(2.5, 0.75) = (1, 0.3)$. In order to have a higher payoff than 0.375, Player 2 has to make a proposal with a higher Nash product than $(2.5, 0.75)$ back in Phase 1, and so on.

## 4. PROBABILISTIC DEMAND GAMES

Rubinstein et al. (1992) simplified Howard's procedure by introducing a probability of terminating the game.

**Phase 1 :** Player 1 proposes $(x_1, x_2^*) \in S$.

**Phase 2 :** Player 2 can either accept Player 1's proposal or proposes $(x_1^*, x_2) \in S$ and announces some $p \in [0, 1]$.

**Phase 3 :** Nature makes a choice: With probability $(1 - p)$ the game terminates with payoffs $d = (0, 0)$ and with probability $p$ the game continues.

**Phase 4 :** Player 1 chooses between Player 2's proposal $(x_1^*, x_2)$ and a lottery that yields 1's original proposal $(x_1, x_2^*)$ with probability $p$ and $d = (0, 0)$ with probability $(1 - p)$.

This procedure also yields a subgame perfect equilibrium outcome which coincides with that of the Nash solution. The presence of Nature's choice in that procedure is very appealing. It points out the possibility that a relationship may end after a failure to reach an agreement with some probability. But the fact that this probability is determined by one of the players may not be deemed very realistic.

Instead, we consider a very simple but simultaneous version of the Rubinstein et al. (1992) procedure where the probability with which the relationship may end after a failure to reach agreement will be exogenous and may take any value in $[0, 1]$.

Define the *Probabilistic Simultaneous Procedure* as follows:

**Phase 1 :** Players 1 and 2 respectively propose $(x_1, x_2^*)$ and $(x_1^*, x_2)$ in $S$. If $(x_1, x_2) \in S$, then each Player $i$ receives his demand

$x_i$. Otherwise, the game continues to Phase 2 with probability $p \in [0,1]$ and terminates paying $d = (0,0)$ with probability $(1-p)$.

**Phase 2:** One of the initial proposals $(x_1, x_2^*)$ and $(x_1^*, x_2)$ is selected at random with probability $\frac{1}{2}$.

Apart from the original Nash demand game (which ends up with a continuum of Nash equilibria), this is one of the simplest demand games that has been studied in the literature—if not the simplest. An interesting question is whether this procedure provides any non-cooperative foundations for the Nash or any other bargaining solution concept.

It is easy to see that, when $p = 0$, our scheme reduces to the original Nash demand game. As $p$ tends to 1, it in a sense resembles the conventional arbitration with chilling effect where parties make extreme demands and the arbitrator splits the difference. The interesting cases will certainly lie between these two extremes where $p$ is neither close to 0 nor to 1.

One question is whether any bargaining solution outcomes would be admitted in the Nash equilibrium set of our scheme given all or most $p$ levels. We can readily infer that the $p = 0$ case would admit all known solution outcomes (including the Dictatorial solution outcomes). But when we consider the $p = 1$ case, a simple setup with a linear Pareto frontier will lead to the midpoint becoming the only Nash equilibrium outcome. From that one can infer that only bargaining solutions satisfying Symmetry, Weak Pareto Optimality, and Scale Invariance would be initial candidates, precluding the outcomes from other well-known solution concepts, including the Egalitarian, Equal Sacrifice, and Dictatorial solutions, from being in every Nash equilibrium outcome set.

In addition, the equilibrium outcome obtained in the simple setup with a linear Pareto frontier has a drawback too. The demands are far from the equilibrium outcome payoffs. Then we pose the following question: are there any Nash equilibria for some levels of $p$, where the demands coincide with some of the resulting Nash equilibrium outcome payoffs such that the demand vector also coincides with one of the solution outcomes? In addition, we also ask whether any other

solution outcomes would be admitted in the Nash equilibrium set with the above property (where the demand vector coincides with the equilibrium payoffs) at some levels of $p$?

If there are several such solution concepts, then the crucial question becomes which of these solution concepts' outcome will be the Nash equilibrium outcome with that property at the highest possible $p$ level (i.e., with the least amount of possible punishment via the probabilistic imposition of the disagreement outcome). Thus one can imagine a procedure similar to the Dutch auction (or another procedure similar to the English auction) where one starts with $p = 1$ (or $p = 0$) and keeps decreasing (increasing) $p$ until one solution outcome becomes the unique equilibrium outcome (until only one solution outcome remains as the equilibrium outcome) where the demand vector coincides with the equilibrium payoff vector.

Another way of approaching this scheme would be to consider the following setup. Suppose one segment in a society strictly adheres to one of the solution concepts as the norm (see Binmore, 1998, for instance) and always demands the payoff that is prescribed by that solution concept for him in any bargaining situation. Suppose another segment in that society does not adhere to that norm. Consider bargaining situations between members of different segments. Then one can find the lowest probability of assigning the disagreement payoff to induce a member from the second segment as well to demand the payoff that is prescribed by that solution concept for him. That would be the norm that can be sustained most easily.

**Theorem 2.** *In the Probabilistic Simultaneous Procedure, there is a maximum probability $p^*$ for which there is a Nash equilibrium that both players agree on (i.e., the game does not go to Phase 2). The Kalai-Smorodinsky solution outcome is a Nash equilibrium at $p^*$ and at every lower value of $p$. Moreover, if the probability of continuing to Phase 2 is $p^*$, and $(x_1, x_2)$ is a Nash equilibrium that both players agree on, then $(x_1, x_2) = (KS_1, KS_2)$. Finally, $p^* \geq 2/3$.*

*Proof.* As before, let $b_i = \max\{x_i : (x_1, x_2) \in S\}$. We start by supposing that Player $j$ has offered $x_i^*$ to Player $i$. Player $i$ can either accept,

or choose $(x_1, x_2) \in S$ with $x_i \neq x_i^*$. In the latter case we go to Phase 2, and Player $i$ obtains a payoff of $(p/2)x_i + (p/2)x_i^*$. Player $i$ will prefer to go to Phase 2 if

$$\frac{p}{2}x_i + \frac{p}{2}x_i^* > x_i^*, \text{ that is, if } \frac{p}{2}x_i > \left(1 - \frac{p}{2}\right)x_i^*.$$

If Player $i$ prefers to go to Phase 2, his best response is to pick his ideal point, which pays him $b_i$. Thus Player $i$ prefers to go to Phase 2 whenever $(p/2)b_i > (1 - (p/2))x_i^*$. We simplify this to $pb_i > (2 - p)x_i^*$

Let $(KS_1, KS_2)$ be the Kalai-Smorodinsky outcome. We know that $KS_1/b_1 = KS_2/b_2$. For $(KS_1, KS_2)$ to be a Nash equilibrium where both players choose the same point, we must have $pb_i \leq (2 - p)KS_i$. Define $p^*$ by $p^*b_i = (2 - p^*)KS_i)$. If $(KS_1, KS_2) = (b_1, b_2)$, $p^* = 1$. Otherwise, $p^* < 1$. Moreover, if $p \leq p^*$, $pb_i \leq p^*pb_i = (2 - p)KS_i \leq (2 - p^*)KS_i$. It follows that $(KS_1, KS_2)$ is a Nash equilibrium for all $p \leq p^*$.

Now suppose $p^* < 1$. If $(x_1, x_2)$ is a Nash equilibrium for $p$ with both players choosing the same point, then

$$b_i \leq \left(\frac{2 - p}{p}\right)x_i < \left(\frac{2 - p^*}{p^*}\right)x_i, \text{ so } x_i > KS_i.$$

But $(KS_1, KS_2)$ is weakly Pareto optimal, so this is impossible. There are no such Nash equilibria. It follows that if there is a Nash equilibrium that both players agree on, then $p \leq p^*$. But then the Kalai-Smorodinsky outcome is also a Nash equilibrium.

Now suppose $(x_1, x_2)$ is a Nash equilibrium that both players agree on with a continuation probability of $p^*$. Then $x_i \geq p^*b_i/(2 - p^*) = KS_i$. We cannot have both $x_i > KS_i$ as the Kalai-Smorodinsky solution is weakly Pareto optimal. Choose $i$ so $x_i > KS_i$ and let $j$ denote the other player. Then we may take a convex combination of $(x_1, x_2)$ and $j$'s ideal point that leads to a strong Pareto improvement over $(KS_1, KS_2)$. This is impossible, so $(KS_1, KS_2)$ is the only Nash equilibrium that both players agree on when $p^*$ is the continuation probability.

The Kalai-Smorodinsky outcome gives each player at least half of the ideal point. The minimum value of $p^*$ occurs when each gets exactly half of the ideal value, as occurs when $S$ is the convex hull of

$\{(0,0), (b_1, 0), (0, b_2)\}$. Then $b_i = p^* b_i / (2 - p^*)$, which implies $p^* = 2/3$ is the minimum value.    $\square$

A natural question is whether the various bargaining solution outcomes appear in a predictable order when $p$ decreases. The next theorem addresses one such case.

**Theorem 3.** *Suppose the Egalitarian solution outcome is a Phase 1 Nash equilibrium of the Probabilistic Simultaneous Procedure at probability $p$. Then the Equal Sacrifice outcome is also a Phase 1 Nash equilibrium at probability $p$.*

*Proof.* Let $S$ be the bargaining set and $b = (b_1, b_2)$ the ideal point corresponding to $S$. Because $S$ is comprehensive, we know $(b_1, 0), (0, b_2) \in S$. Let $(z, z) \in S$ be the Egalitarian outcome. Without loss of generality, we may assume $b_1 \geq b_2$. In fact, if $b_1 = b_2$, we must have $b_1 = b_2 = z$, in which case $(z, z)$ is also the Equal Sacrifice outcome.

For the remainder of the proof we consider the non-trivial case $b_1 > b_2$. Let $T$ be the convex hull of $\{(0,0), (b_1, 0), (z, z), (0, b_2)\}$. Let $x = (x_1, x_2)$ be the Equal Sacrifice outcome for $T$. The Equal Sacrifice outcome for $S$ is to the northeast of $x$, and so will be an equilibrium at any value of $p$ where $x$ is an equilibrium. Thus it is enough to show the result for $T$.

The line through $(z, z)$ and $(b_1, 0)$ is $y = z(b_1 - x)/(b_1 - z)$. The Equal Sacrifice outcome obeys $x_2 = zb_2/b_1$. Equal Sacrifice then yields $x_1 = b_1 - b_2 + x_2 = b_1 + b_2(z - b_1)/b_1$. Now $x_1 \leq b_1$, so $(b_1 - b_2)x_1 \leq (b_1 - b_2)b_1 = (x_1 - x_2)b_1$. It follows that $x_2 b_1 \leq x_1 b_2$.

Since the Egalitarian outcome is an equilibrium, $z \geq b_i p / (2 - p)$ for $i = 1, 2$. As $b_1 > b_2$, we can sum this up as $z \geq b_1 p / (2 - p)$. But then $x_2 = zb_2/b_1 \geq b_2 p / (2 - p)$. We use the fact that $x_1 \geq x_2(b_1/b_2)$ to see that $x_1 \geq b_1 p / (2 - p)$. This establishes that Equal Sacrifice outcome $x$ is an equilibrium at probability $p$.    $\square$

What about the other cases? Can we rank other solution concepts using the probability level they appear at? No!

The following examples show that the no other unambiguous rankings are possible for the other solutions. Denote the solution outcomes

as follows: Nash (N), Kalai-Smorodinsky (KS), Egalitarian (E), Equal Sacrifice (ES), Equal Area (EA), and Average Payoff (AP). Sometimes all of these will coincide, as happens when $S$ is the convex hull of $\{(0,0),(2,0),(0,2)\}$. Note that $p^* = 2/3$ here.

**Example 2.** Let $S$ be the convex hull of $\{(0,0),(0,2),(4,0)\}$. Then the Nash, Kalai-Smorodinsky, Equal Area, and Average Payoff solution outcomes coincide at $(2,1)$. Egalitarian is $(4/3,4/3)$ and Equal Sacrifice is $(8/3,2/3)$. Here $p^* = 2/3$, which supports the Kalai-Smorodinsky, Nash, Equal Area, and Average Payoff solution outcomes. The Equal Sacrifice and Egalitarian solution outcomes do not appear as equilibria until $p \le 1/2$. Of course, if we had taken $(2,0)$ as the right corner of $S$, all of these solution outcomes would coincide at $(1,1)$.

**Example 3.** We modify the previous example slightly by truncating at $x_1 = 3$. In other words, we take $S$ as the convex hull of $\{(0,0),(0,2),(3,1/2),(3,0)\}$. The Kalai-Smorodinsky solution outcome is now $(12/7,8/7)$ with $p^* = 8/11$. Then Nash solution outcome remains at $(2,1)$ with $p = 2/3$. The Equal Area solution outcome is now $(15/8,17/16)$ with $p = 34/49$. The Egalitarian solution outcome remains at $(4/3,4/3)$, but now has $p = 8/13$. Equal Sacrifice joins the Nash solution outcome at $(2,1)$ with $p = 2/3$. The Average Payoff solution outcome is $(24/13,14/13)$ with $p = 14/23$.

**Example 4.** We now truncate at $x_1 = 2$. In other words, we take $S$ as the convex hull of $\{(0,0),(0,2),(2,1),(2,0)\}$. The Kalai-Smorodinsky, Egalitarian, and Equal Sacrifice solution outcomes are now $(4/3,4/3)$ with $p^* = 4/5$. Then Nash solution outcome remains at $(2,1)$ with $p = 2/3$. The Equal Area solution outcome is now $(3/2,5/4)$ with $p = 2/5$. The Average Payoff solution outcome is $(16/11,14/11)$ with $p = 4/19$.

**Example 5.** Let $S$ be the convex hull of $\{(0,0),(3,0),(3,1),(2,2),(0,3)\}$. The point $(2,2)$ is the Kalai-Smorodinsky, Nash, Equal Sacrifice, and

Egalitarian solution outcome. However, the Equal Area solution outcome is $(17/8, 15/8)$. This requires $p \leq 10/13$. The Average Payoff solution outcome is $(40/19, 36/19)$, which is supported by $p \leq 24/31$.

We can sum up the rankings by probability of the various solutions.

| Example 2 | Example 3 | Example 4 | Example 5 |
|---|---|---|---|
| KS, N, AP, EA | KS | KS, ES, E | KS, N, ES, E |
| ES, E | EA | AP | AP |
| | ES, N | EA | EA |
| | E | N | |
| | AP | | |

As we know, the Kalai-Smorodinsky solution outcome is always at the top. We see that each of the other solutions sometimes ties it. These also examples show that any of the solutions other than Kalai-Smorodinsky can be last in the rankings, although when Equal Sacrifice is last, it must tie with Egalitarian. Example 3 shows that Egalitarian can rank strictly below Equal Sacrifice. In sum, the examples show that no other general rankings of these solutions are possible beyond the requirements of Theorems 2 and 3.

## 5. Concluding Remarks

We have introduced two variations on the Nash demand game that involve simultaneous moves. The first is a simultaneous game related to Howard's procedure. As in most Nash demand games, the Nash solution outcome is the equilibrium outcome. The second has a range of possible solutions that depends on an exogenous breakdown probability. Of these, the Kalai-Smorodinsky outcome is most robust equilibrium outcome. The breakdown probabilities at which other solutions occur allow us to rank the other outcomes. We show there is no possible general ranking among a variety of standard solution concepts, save that the Equal Sacrifice solution is more robust than the Egalitarian solution.

Whether there are other simple Nash demand games that yield some of the other well-known bargaining solutions remains an open question.

## References

Anbarci, N., 1993. Noncooperative foundations of the area monotonic solution. Quart. J. Econ. 108, 245–258.

Anbarci, N., 1995. Reference functions and balanced concessions in bargaining. Can. J. Econ. 28, 675–682.

Anbarci, N., Bigelow, J., 1994. The area monotonic solution to the cooperative bargaining problem. Math. Social Sci. 28, 133–142.

Bigelow, J., Anbarci, N., 1993. Non-dictatorial, pareto-monotonic, cooperative bargaining: An impossibility theorem. European Journal of Political Economy 9, 551–558.

Binmore, K. G., 1998. Game Theory and the Social Contract II: Just Playing. MIT Press, Cambridge, MA.

Bloom, D., 1981. Is arbitration really compatible with bargaining? Ind. Relat. 20, 233–244.

Chun, Y., 1988. The equal-loss principle for bargaining problems. Econ. Letters 26, 103–106.

Howard, J. V., 1992. A social choice rule and its implementation in perfect equilibrium. J. Econ. Theory 56, 142–159.

Kalai, E., 1977. Proportional solutions to bargaining situations: Interpersonal utility comparisons. Econometrica 45 (7), 1623–1630.

Kalai, E., Smorodinsky, M., 1975. Other solutions to Nash's bargaining problem. Econometrica 43 (3), 513–518.

Luce, R. D., Raiffa, H., 1957. Games and Decisions. Wiley, New York.

Moulin, H., 1984. Implementing the kalai-smorodinsky bargaining solution. J. Econ. Theory 33, 32–45.

Nash, J. F., 1950. The bargaining problem. Econometrica 28, 155–162.

Nash, J. F., 1953. Two-person cooperative games. Econometrica 21, 128–140.

Roth, A. E., 1977. Independence of irrelevant alternatives, and solutions to Nash's bargaining problem. J. Econ. Theory 16 (2), 247–251.

Rubinstein, A., Safra, Z., Thomson, W., 1992. On the interpretation of the Nash bargaining solution and its extension to non-expected utility preferences. Econometrica 60, 1171–1186.

Schelling, T. C., 1960. The Strategy of Conflict. Harvard University Press, Cambridge, MA.

Deakin University, School of Accounting, Economics and Finance, 221 Burwood Highway, Burwood, VIC 3125, Australia. Phone: +61 3 9244 6530

*E-mail address*: nejat.anbarci@deakin.edu.au

Department of Economics, Florida International University, Miami, FL 33199

*E-mail address*: boydj@fiu.edu