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Constructible Sandwich Cut

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Cover Page Footnote

Dedicated to all present and future math students at FIU.

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Constructible Sandwich Cut

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In mathematical measure theory, the "Ham-Sandwich" theorem states that any n objects in an n-dimensional Euclidean space can be simultaneously divided in half with a single cut by an (n-1)-dimensional hyperplane. While it guarantees its existence, the theorem does not provide a way of finding this halving hyperplane, as it is only an existence result. In this paper, we look at the problem in dimension 2, more in the style of Euclid and the antique Greeks, that is from a constructible point of view, with straight edge and compass. For two arbitrary regions in the plane, there is certainly no hope for constructing the halving line by straight edge and compass. At the opposite end, for two circles the problem is very easy, as the halving line is simply the line which passes through the centers of both circles. It turns out that for the case of two equilateral triangles the problem is already interesting and challenging from a constructible point of view, and this is the main goal of our paper. We will investigate the "deltoid" of a triangle, that is, three hyperbolas which form an "envelope" of all halving lines of a given triangle, and how to construct halving lines with certain given properties.

Keywords: *halving line, deltoid, constructible, Ham-Sandwich Theorem, triangle, area*

1 Introduction

Imagine two aliens aboard a spacecraft exploring the vastness of the universe. After awakening from a centuries-long hyper-sleep they are in dire need of some breakfast. They look in the pantry and discover some pancake batter and syrup. Upon further inspection, however, they realize that there is only enough batter and syrup for one pancake. The pair decides to put the pancake on one plate and pour the syrup on another, agreeing to evenly split the meal. Unfortunately, the aliens only have enough power in their laser cutter for only one cut. Can they do it? They are not too concerned, as both are aware of the mathematical result known on Earth as the "Pancake Theorem." This theorem is a critically important theorem in measure theory that states that two objects in two dimensions can be cut into two regions such that each region has exactly half the area of the original object. One of the reasons this is so important is because it allows us to confirm that a precise halving line exists. Usually, most methods of cutting regions in half involve a process of repeated approximation. However, with this theorem, and later the results of this paper, it will be possible to find an exact halving line for not just one, but two regions simultaneously with just one cut.

1.1 The Pancake Theorem

Before we get into the theorem and proof, it is important to understand why this theorem is so crucial to this paper. This theorem proves that the discoveries in this paper are possible because it shows that regardless of where the regions are located in the plane, there is a line which bisects them. Furthermore this line (which becomes a hyperplane in higher dimensions) exists in all dimensions.

Theorem: Given two bounded regions $\mathscr{R}_1$ and $\mathscr{R}_2$ in a two dimensional plane $\mathbb{R}^2$, there exists a line which bisects both $\mathscr{R}_1$ and $\mathscr{R}_2$.

The following are definitions that we will use in this note and are necessary for a clear understanding of the above statement:

- A non-empty subset $\mathscr{R} \subseteq \mathbb{R}^2$ is said to be a *region* in the plane $\mathbb{R}^2$ if $\mathscr{R}$ is *open* and *connected.*
- A set $A \neq \emptyset$ is said to be *open* in $\mathbb{R}^2$if it has some "thickness" around each of its points; rigorously, for every point $p \in A$ there exists a small disk centered at p entirely contained in A.
- A non-empty set A is said to be *connected* in $\mathbb{R}^2$ it it is made from "one piece"; rigorously, there exist no two disjoint open sets in $\mathbb{R}^2$ having each non-empty intersection with A . For an open set A in $\mathbb{R}^2$, connectedness is equivalent to path-connectedness, that is for any two points $p_1, p_2 \in A$ there is a continuous path in A from p_1 to p_2.
- A set A is said to be *bounded* in $\mathbb{R}^2$ if it can be included in a large enough disk. Note that if $\mathscr{R}$ is a bounded region in $\mathbb{R}^2$, as in the theorem, automatically

  $$0 < \text{Area}(\mathscr{R}) < +\infty,$$

  where the first inequality comes from the fact that $\mathscr{R}$ is open, and the second from the assumption that $\mathscr{R}$ is bounded.

Finally, we will say that a line L *bisects* a bounded region $\mathscr{R}$ if it divides it into two regions of equal areas. We will call such line a *halving line* for the region $\mathscr{R}$.

The proof of the Pancake Theorem is seen through intuition. It uses a theorem students usually see, without proof, in their first Calculus course two times:

- **The Intermediate Value Theorem:** For every function that is continuous on a closed interval, $f: [a, b] \to \mathbb{R}$ where a and b are real numbers and $a < b$, if c is any value between $f(a)$ and $f(b)$ there exists $x \in [a,b]$ such that $f(x) = c$. [Thomas]

The proof of the Ham-Sandwich Theorem, the generalization of the Pancake Theorem in n- dimensions, $n \geq 3$, requires, besides the Intermediate Value Theorem, the following topological result:

- **The Borsuk-Ulam Theorem:** For every continuous mapping, $f: \mathbf{S}^n \to \mathbb{R}^n$ where $\mathbf{S}^n$ is an n-dimensional sphere, there exists a point x on $\mathbf{S}^n$ such that $f(x) = f(-x)$. [Mendelson]

We demonstrate below the proof of the Pancake Theorem with one use of the Intermediate Value Theorem for the first step and the use of the 1-dimensional version of the Borsuk-Ulam Theorem for the second step. One can follow the full details of the proof with two applications of the IVT, with many pictures and even a video at this site [David]. The first step can be thought of as halving one pancake in a given direction.

Lemma 1: If $\mathscr{R}$ is a bounded region in $\mathbb{R}^2$ and L' is a given line, there exists a unique halving line L for the region $\mathscr{R}$, so that L is parallel to L'.

Proof of Lemma 1: Without loss of generality, we can choose a coordinate system in $\mathbb{R}^2$ so that the given line L' is the y-axis. Lines parallel to L' are then vertical lines $x = x_0$, and we need to show that there exists a vertical line that bisects $\mathscr{R}$. Again, without loss of generality, assume Area($\mathscr{R}$) = 1. For each real number x_0, define $f(x_0)$ = Area($\mathscr{R} \cap \{x < x_0\}$). That is, $f(x_0)$ is the area inside $\mathscr{R}$ and to the left of the line $x = x_0$. The function f is continuous and its values will go from 0 to 1 as x_0 goes from $-\infty$ to $+\infty$. Note that by its definition f is non-decreasing. Moreover, as the region $\mathscr{R}$ is bounded, there exists a unique interval $[x_1, x_2]$, so that $f(x_1) = 0$, $f(x_1) = 1$, and $0 < f(x_0) < 1$, for all $x_0 \in (x_1, x_2)$. Because of the openness assumption for $\mathscr{R}$, the function f is strictly increasing on the interval $[x_1, x_2]$. Using the Intermediate Value Theorem for f on $[x_1, x_2]$, it follows that there exists a value $\tilde{x}_0 \in (x_1, x_2)$ so that $f(\tilde{x}_0) = 1/2$. Moreover, because f is strictly increasing on the interval $[x_1,x_2]$, this value $\tilde{x}_0$ is unique. □

Conclusion of proof of the Pancake Theorem: Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be bounded regions in the plane. It is convenient to associate to every unit vector $\mathbf{n} = (a, b) \in S^1$ the family of parallel lines having $\mathbf{n}$ as normal vector. All the lines in this family have equations $ax + by = c$, for $c \in \mathbb{R}$. This association is convenient as for any line of equation $ax + by = c$ one can make a canonical choice for one of the two (unbounded) regions $\mathscr{P}_1$, $\mathscr{P}_2$ it divides the plane into. We will choose $\mathscr{P}_1$ to be the region that $\mathbf{n} = (a, b)$ points toward. Equivalently, this preferred region is described by

$$\mathscr{P}_1 = \{(x, y) | ax + by > c\}.$$

Now, given $\mathbf{n} = (a, b) \in S^1$, from Lemma 1 applied to the region $\mathscr{R}_1$, there exists a unique line with normal $\mathbf{n}$ that bisects $\mathscr{R}_1$. Let $\mathscr{P}_1$ denote the preferred region of the plane associated with this line as above. Then, we define now the function $f : S^1 \to \mathbb{R}$ by measuring how much area of the region $\mathscr{R}_2$ has been captured in $\mathscr{P}_1$,

$$f(\mathbf{n}) = \text{Area}(\mathscr{R}_2 \cap \mathscr{P}_1) .$$

This is a continuous function. Applying the 1-dimensional version of the Borsuk-Ulam theorem, there exists a point $\tilde{\mathbf{n}} \in S^1$ so that $f(\tilde{\mathbf{n}}) = f(-\tilde{\mathbf{n}})$. Note that the lines associated to $\tilde{\mathbf{n}}$ and $-\tilde{\mathbf{n}}$ are the same, but $f(-\tilde{\mathbf{n}})$ represents the area of the region $\mathscr{R}_2$ that has been captured in $\mathscr{P}_2$. Thus, this line will bisect both regions $\mathscr{R}_2$ and $\mathscr{R}_2$. □

Let us now come back to the situation described in the introduction with our aliens and their cutting dilemma. Even though the theorem guarantees the existence of the line, for arbitrary regions $\mathscr{R}_1$ and $\mathscr{R}_2$finding this line precisely is likely impossible even for advanced alien civilizations. We investigate below particular cases when the halving line may be obtained constructively, with a straight-edge and compass.

2 Constructible Objects

In this section, we will briefly overview some basic notions about constructible numbers and constructible objects from additional given data. The reader could refer to [Moise] and [Martin] for more details.

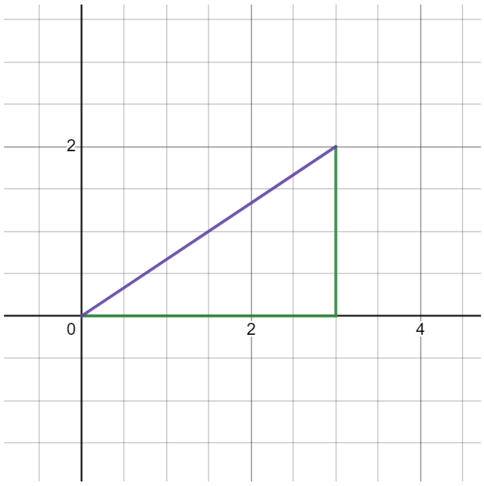
2.1 Constructible Numbers

Given a line segment of unit length, a real number r is said to be *constructible* if one can construct with straight-edge and compass a segment of length r in a finite number of steps. It is known from Galois theory, that the set of constructible numbers are precisely those numbers r which can be expressed in finite steps from integers using only the operations of addition, sub- traction, multiplication, division, and square-roots (but no roots of higher order). For example, $\frac{5+\sqrt{13}}{3}$ is a constructible number as one can follow these steps for its construction:

To construct the number 5 is trivial as all that is needed is to conjoin 5 segments of unit length. The construction of $\sqrt{13}$ can be done by first constructing a segment of length 3, then at one of the segment's endpoints construct a line perpendicular to the segment which has length 2. Finally, one can connect the final two endpoints creating a triangle, by the Pythagorean Theorem the resulting segment will have length $\sqrt{13}$. After this one can copy and conjoin the two finalized segments to construct a segment of length $5 + \sqrt{13}$. In order to construct one third of this value we will need to construct a line segment beginning at one of the endpoints of our segment and extending as long as needed. Then mark 3 successive, equally spaced points P_1, P_2, P_3. Connect P_3 to the end of our original segment and lines parallel to P_3 that pass through P_2 and P_1. These lines will divide $5 + \sqrt{13}$ into 3 equal pieces, completing the construction.

Figure 1

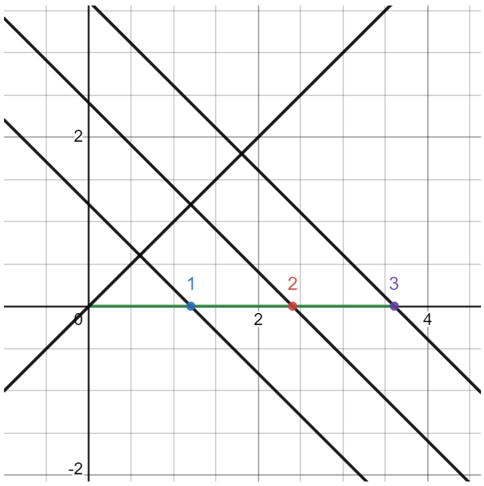
The triangle whose hypotenuse is exactly $\sqrt{13}$



The set of constructible numbers form a field, let us denote it by $\mathbb{F}$, with $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{R}$, both inclusions being strict. For example, it is well-known that is $\sqrt[3]{2}$ not constructible. Likewise, $\sqrt[5]{6}$, π are not constructible numbers.

Figure 2

The completed construction of $\frac{5+\sqrt{13}}{3}$ *(shown in green)*



2.2 Constructible Objects Given Additional Data

In our case, and in other situations, for constructions with straight-edge and compass we may be given not only a segment of unit length, but also some additional data. In this case, the set of numbers that can be constructed, given the additional data, may be larger than $\mathbb{F}$. As a first example, suppose that in the plane $\mathbb{R}^2$ we are given on a coordinate system the points $P_1(1, 0)$ and $P_2(a, b)$, where let's say $a = \sqrt[3]{2}$, $b = \pi$. Then, as part of the given data, we have segments of lengths $\sqrt[3]{2}$ and π, thus we'll be able to construct points with coordinates more than just F. For example, with the given data in this case, a number of the form $2 + \sqrt{2}\pi - 3\sqrt[3]{2}$ can be constructed with straight edge and compass.

The following example is very relevant for our problem. Suppose we are given the two foci of a certain hyperbola in the plane, so, implicitly, we also know the focal axis (the line going through the foci). Suppose we are also given the two points where the hyperbola intersects the focal axis. If a coordinate system is chosen so that the x-axis is the focal axis and the origin is the midpoint between the foci, the hyperbola will have standard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,. \tag{1}$$

The given data amounts to knowing a and $c = \sqrt{a^2 + b^2}$. Of course, knowing a and c, then one can construct a segment of length b as this only involves the given data, usual arithmetic operations and square-root. Note also that, point by point, we construct the all points on the hyperbola from the given data. Indeed, for each point on the x-axis with $x \geq a$, or $x \leq -a$, we consider a segment of length x as given, and we will be able to construct a segment of length $y = \frac{b}{a}\sqrt{x^2 - a^2}$ as this only involves the given data (and x) and constructible operations.

Proposition 1: Assume a hyperbola in a standard form (1) is given, for example, by knowing the points of coordinates $(a, 0)$ and $(0, b)$.

a. Given a point $P_0(x_0, y_0)$ anywhere in the plane, then one can construct with straight-edge and compass from the given data the point $P_1(x_1, y_1)$ so that P_1 is on the hyperbola [1] and the line P_0P_1 is tangent to the hyperbola.

b. Given a slope $m \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$ then one can construct with straight-edge and compass from the given data the point $P_1(x_1, y_1)$ on the hyperbola [1], so that the tangent line to the hyperbola at P_1 has slope m.

Proof: For the first part of the proposition, we can begin by expressing the slope of the line in two different ways. One way would be using a point which is on the hyperbola and using the given point, and the other would be using implicit differentiation. This yields a second degree system which automatically proves constructibility because all second degree systems have solutions which involve rational numbers and square roots. □

For the second part of the proposition, it is even simpler as all we would need to do is take the derivative of the hyperbolic function and set it equal to the slope we were given. Once again we will have a second-degree equation that will only yield values in the constructible field. □

3 Two Regions with Central Point Symmetry

Given a point P in the plane $\mathbb{R}^2$, denote by s_P the symmetry with respect to the point P, that is, s_P: $\mathbb{R}^2 \to \mathbb{R}^2$, is the rotation map by 180° around the point P.

Definition: A set $\mathscr{R} \subset \mathbb{R}^2$ is said to have a central point symmetry with respect to a point C

if for any point A in $\mathscr{R}$, $s_C(A)$ also belongs to $\mathscr{R}$.

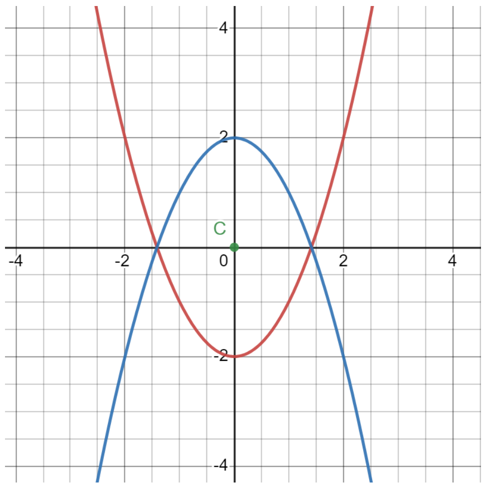
Example 1: A circular disk has central point symmetry with respect to its center, this is also true for regions bounded by any regular polygon with an even number of sides. A parallelogram also has central point symmetry with respect to the point at the intersection of its diagonals.

By contrast, note that regular polygons with odd number of sides (in particular, equilateral triangles) do not have a central point symmetry (although they still have other symmetries).

Example 2: Let $\mathscr{L}$ be a line in the plane. Obviously, $\mathscr{L}$ divides the plane in two disjoint regions. Let $\mathscr{R}_1$ be a subset of one of the two regions and assume that the closure of $\mathscr{R}_1$ inter- sected with L is a segment AB on $\mathscr{L}$. Let C be the midpoint of the segment AB. Denote by $\mathscr{R}_2$ the image of $\mathscr{R}_1$ under s_c, the symmetry with respect to the point C. Let also $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2$. By construction, it is clear that R has central point symmetry with respect to the point C.

Figure 3

A visual of example 2, in this case $\mathscr{R}$ is the region bounded by the parabolas $f(x) = x^2 - 2$ and $f(x) = -x^2 + 2$ and C is the point of central symmetry



Example 3: The annular region (the ring) $\mathscr{R} = \{(r, \theta) | 1 < r < 2\}$ has central point symmetry with respect to the origin of the coordinate plane. This example shows that it is possible for the center of symmetry to be *outside* the region. The construction in example 2 can be adjusted to obtain many other such examples (but all of them will have a "hole" inside and the center is in this hole, so they will not be simply-connected).

Lemma 2: Let $\mathscr{R}$ be a bounded region with central point symmetry with respect to a point C. Then a line L is a halving line for $\mathscr{R}$ if and only if L passes through C.

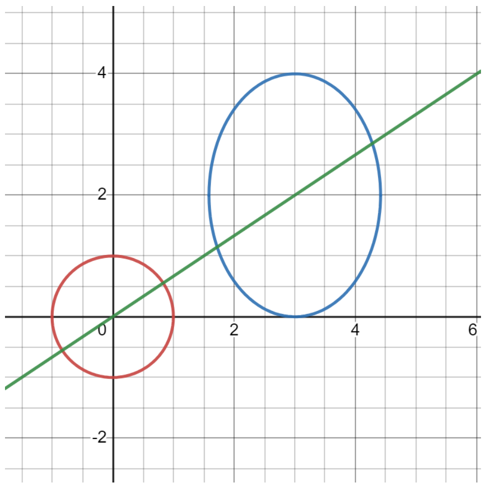
Proof: Assume first that L is a line that passes through the center of symmetry C. Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be the two regions obtained from $\mathscr{R}$ on each of the sides of L. Because $\mathscr{R}$ is assumed to have central point symmetry with respect to C and L goes through C, $s_C(\mathscr{R}_1) = \mathscr{R}_2$. In particular, Area$(\mathscr{R}_1)$ = Area$(\mathscr{R}_2)$. As a side-note, observe that since $\mathscr{R}$ is connected and has central point symmetry with respect to the point C, any line through C must intersect $\mathscr{R}$.

For the converse direction, assume by contradiction that L is a halving line for $\mathscr{R}$ that does not go through its center of symmetry C. Then, the image of L through the symmetry with respect to C, $L' = s_C(L)$ is also a halving line for $\mathscr{R}$, with L' parallel to L, but $L' \neq L$. However, this contradicts the uniqueness statement in **Lemma 1**. □

We have now shown that for any region with central point symmetry any halving line must pass through the center of the region, this implies that for any two regions with central point symmetry the line which halves both regions is the line which passes through both centers. Thus, assuming one knows the coordinates of the centers of both regions, one uses a straight edge to construct the line.

Figure 4

A visual of the solved case created by using one circle and one ellipse



4 One Triangle and a Figure with Central Point Symmetry

4.1 Deltoid of a Triangle

Definition: The deltoid of a triangle is formed by three hyperbolas [Blue]. Each of the three hyperbolas intersects the other two and the midpoint of one of the median lines of the triangle. Each of the three hyperbolas is centered at each vertex of the triangle. Each of the hyperbolas demonstrates asymptotic behavior with respect to the two sides which share the vertex constituting the hyperbola's center. All lines which bisect the given triangle must be tangent to one of the three hyperbolas, and must pass through the region they collectively form [Todd].

Finding the Hyperbolas: Since we know that each of the hyperbolas which form the deltoid behave asymptotically with respect to two of the sides of the triangle and that each of the hyperbolas will intersect two of the midpoints of two of the median lines of the given triangle, this makes finding a formula for the hyperbolas relatively simple. Given the lines $L_1 : Ax + By + C = 0$ and $L_2 : Dx + Ey + F = 0$ which constitute the asymptotes of a given hyperbola, the formula for the family of hyperbolas is $(Ax + By + C)(Dx + Ey + F) = k$ [amd]. The constant k can be easily solved for when plugging in one of the medians midpoints that the hyperbola passes through.

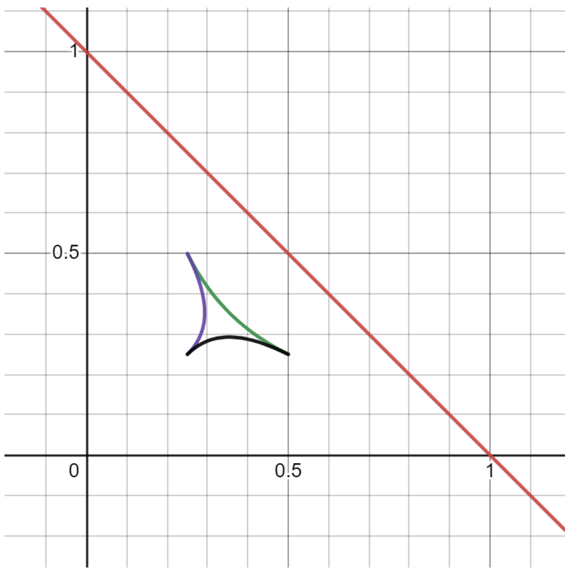
Example: In the case of the triangle formed by the line $y = 1 - x$ and the coordinate axis, the three hyperbolas would be centered at (0, 0), (1, 0), and (0, 1). The hyperbola centered at (0, 0) will have asymptotic behavior with respect to the x-axis and the y-axis so the resulting hyperbola would be in the form $xy = k$. Since we know that the hyperbola will intersect two of the midpoints of the median lines of the triangle, we can plug in the midpoints of all three medians to solve for k. We will end up with two like equations which will show the correct value of k. In this case the hyperbola is $xy = 1/8$.

4.2 Constructing a Halving Line For a Triangular Region Given a Point

Lemma 3: Given a triangular region T and a point P which is outside of T, one can find a line through P that halves the triangle T.

Figure 5

Triangle mentioned in example shown with all three hyperbolas forming the deltoid

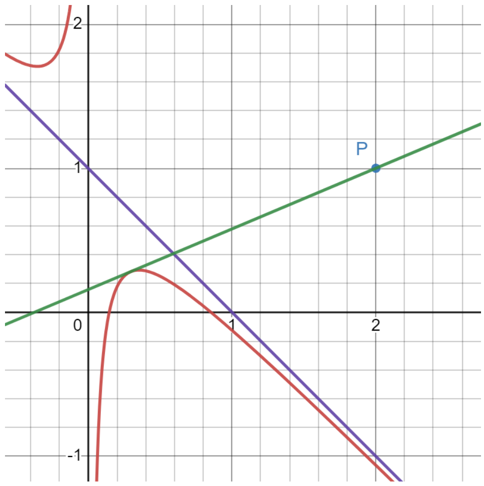


Proof: Given a triangular region $\mathscr{T}$, we can used the method described above in order to derive the formulas for the three hyperbolas of the deltoid of $\mathscr{T}$. Once one has all three deltoids, one can use **Proposition 1a** [Tutoring], to find the tangent lines of all three hyperbolas passing through the given point. One of these lines will be tangent at a point which the hyperbolas form the deltoid of the triangle. This line will automatically bisect $\mathscr{T}$, by the definition of the deltoid, and pass through P.

Remark on Figure 6: The point at which the line is tangent to the deltoid has an x value of exactly $\frac{-1+\sqrt{33}}{16}$. The explicit form of the equation for the hyperbola shown is $y = \frac{-1}{8x} - x + 1$ hence, when the x value is plugged in you still get a value with a combination of square roots and rational numbers. This is significant because it highlights that the coordinates of the point at which the halving line is tangent to the deltoid is constructible, meaning that there exists a way where, one can construct a halving line for the given triangle which passes through the given point.

Figure 6

An example of Lemma 2 where the triangle is formed by the line $y = 1x$ and its intersection with the x-axis and y-axis, and P is the point $(2, 1)$



Since we can now construct a line which bisects a triangle and passes through a point which is outside the triangle, given a triangle, T, and a region, R, with central point symmetry we can simply set C, the center of R, to be the point used in our construction, this will create a line which bisects both R and T.

5 Two Congruent Triangles, the Second Obtained From the First by a Certain Euclidean Motion

5.1 Second Triangle is Obtained by a Translation

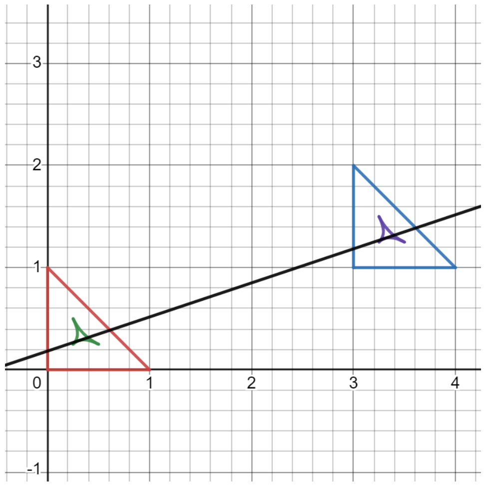
Lemma 4: Given a triangular region $\mathscr{T}$ and a slope $m \in \mathbb{R}$, one can construct a halving line L for $\mathscr{T}$ such that L has slope m.

Proof: Assuming we are given a triangular region $\mathscr{T}$ in the plane, a segment of unit length, a segment representing the translation of the triangles with slope m, and the segments representing the horizontal and vertical components of the translation. Then one can find the three hyperbolas which form the deltoid of $\mathscr{T}$. Following this one can apply **Proposition 1b**, yielding 3 lines tangent to each of the three hyperbolas. One of these lines will be tangent to a hyperbola at a point where it forms the deltoid, and by the definition of the deltoid that line will half $\mathscr{T}$.

Since **Lemma 4** allows us to find a line with an arbitrary slope which can bisect a given triangle, **Lemma 4** also allows us to solve the case of halving two congruent triangles where the second triangle is obtained by translating the first. If the triangle is translated a units parallel to the x-axis and b units parallel to the y-axis then $m = \frac{a}{b}$ (Note: m can also be found by applying the slope formula on two of the corresponding vertices of the triangles). After m is found simply apply **Lemma 4** to find the having line for one of the triangles and it will automatically half the other triangle as the cut would be the same on both triangles.

Figure 7

An example of the solved case of translated triangles where $m = \frac{1}{3}$



5.2 Second Triangle is obtained by a Reflection

Lemma 5: Given a triangular region $\mathscr{T}$ and a line L' , one can construct a halving line L for $\mathscr{T}$, so that L is perpendicular to L' .

Remark: The proof is simply the same as the proof of **Lemma 4** which is the change that the slope m is equivalent to the opposite reciprocal of the slope of L'.

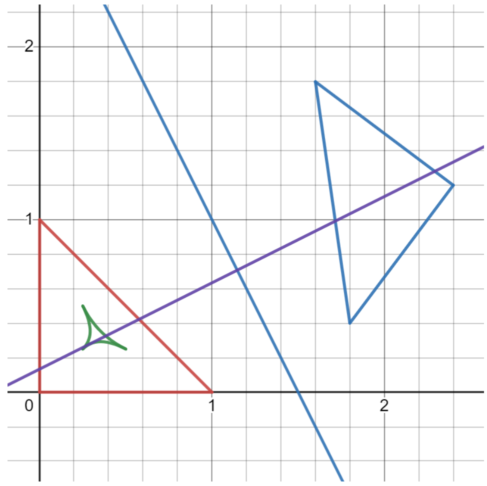
Since **Lemma 5** allows us to find a line with a perpendicular slope to a given line that can bisect a given triangle, **Lemma 5** also allows us to solve the case of halving two congruent triangles where the second triangle is obtained by reflecting the first along a given line. If the triangle is reflected along a line $y = ax + b$, then $m = -\frac{1}{a}$. After m is found, apply **Lemma 4** to find the having line for one of the triangles and it will automatically half the other triangle as the cut would be the same on both triangles.

Further Open Questions

Now that we have shown the solution to the problem with respect to triangles which have either been translated or reflected once, there are still many more cases to be explored. The logical continuation of this paper would be to attempt to solve this problem in the for any two given triangles in the plane. This is far more consequential and includes cases not mentioned here such as the case where the second triangle is obtained by a rotation, or by a composition of two or more transformations.

Figure 8

An example of the solved case where the line of reflection has a slope of −2 meaning that $m = \frac{1}{2}$



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