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Abstract

In this paper we have reviewed some existing and proposed some new estimators for estimating the ridge parameter k. All in all 19 different estimators have been studied. The investigation has been carried out using Monte Carlo simulations. A large number of different models were investigated where the variance of the random error, the number of variables included in the model, the correlations among the explanatory variables, the sample size and the unknown coefficients vectors β have been varied. For each model we have performed 2000 replications and presented the results both in term of figures and tables. Based on the simulation study, we found that increasing the number of correlated variable, the variance of the random error and increasing the correlation between the independent variables have negative effect on the MSE. When the sample size increases the MSE decreases even when the correlation between the independent variables and the variance of the random error are large. In all situations, the proposed estimators have smaller MSE than the ordinary least squared and some other existing estimators.

Key words: Linear Model; LSE; MSE; Monte Carlo simulations; Multicollinearity; Ridge Regression;

AMS Subject classification: Primary 62J07, Secondary 62F10

1. Introduction

In most of the empirical works practitioners often concern about the specification of the models under consideration, especially with regards to problems associated with the residuals, with the aim of assessing white noise errors which in other word implies that the model is well specified. Model misspecification can be due to omission of one or several relevant variables, inclusion of un-necessary variables, wrong functional form, misspecified dynamics, autocorrelation, heteroscedasicity, etc. In the practical work, it is recommended that practitioners should conduct some diagnostic tests in order to assure the whiteness of the model under consideration, otherwise the estimated results can be inefficient, biased or inconsistent.

However, there are other problems that also might influent the results in wrong direction, e.g. multicollinearity. This problem happens in situations when the explanatory variables are highly inter-correlated. Then, it becomes difficult disentangle the separate effects of each of the explanatory variables on the explained variable. As a result, estimated parameters can be wrongly in-significant or have (unexpectedly) wrong signs. Note that multicollinearity is more a problem with the data than with the model itself, and hence this kind of problems can not be identified by residual analysis. As a result, a common deficiency in many applied studies is the absence of paying serious attention to this problem. Indeed, although model misspecification is an important area in the statistical modelling, multicollinearity is an important issue too.

The history of multicollinearity dates back at least to the paper by Frisch (1934) who introduced the concept to denote a situation where the variables dealt with are subject to two or more relations. One way to deal with this problem is called the ridge regression, first introduced by Horel and Kennard (1970 a,b). At this stage, the main interest lies in finding a value of the ridge parameter, say K, such that the reduction in the variance term of the slope parameter is greater than the increase in the squared bias of it. The authors proved that there is a nonzero value of such ridge parameter for which the Mean Squared Errors (MSE) for the slop parameter using the ridge regression is smaller than the variance of the Ordinary Least Square (OLS) estimator of the respective parameter. Many authors thereafter worked with this area of research and developed and proposed different estimates for the ridge regression parameter. To mention few, McDonald and Galarneau (1975), Lawless and Wang (1976), Saleh and Kibria (1996), Haq and Kibria (1996), Kibria (2003), Khalaf and Shukur (2005) and Alkhamisi, Khalaf and Shukur (2006). In Kibria (2003) and Alkhamisi, Khalaf and Shukur (2006), the authors used simulation techniques to study the properties of some new proposed estimators and compared their properties with some popular existing estimators. Under certain conditions, they found that the MSEs of some of the new proposed estimators are smaller than the corresponding MSE of the OLS estimator and other known existing estimators. Recently, Muniz and Kibria (2009) developed 5 new ridge parameters based on Kibria (2003) and Khalaf and Shukur (2005) in models with two explanatory variables. They found the new parameters outperform the previous ones in term of smaller MSEs.

In this paper we aim to extend the study by Muniz and Kibria (2009) by developing 9 more new ridge parameters and to increase the dimension of the models by including more explanatory variables. We also study models with 4 explanatory variables that are more realistic in empirical work than models with only 2 variables. Processing in this manner, it is possible to investigate the effect of the extra included variables on the MSEs.

The paper is organised as follows: In section 2 we present the model we analyse, and give the formal definition of the ridge regression parameters used in this study. In Section 3, the design of our Monte Carlo experiment together with the factors that can affect the small sample properties of these proposed parameters are introduced. In Section 4 we describe the results concerning the various parameters in term of MSE. The conclusions of the paper are presented in section 5.

2. Methodology

In this section we present the proposed ridge regression estimators. This includes a brief background on the methods suggested by Hoerl and Kennard (1970a), and that developed by Khalaf and Shukur (2005), Alkhamisi and Shukur (2008), Alkhamisi, Khalaf and Shukur (2006) and Muniz and Kibria (2009). Moreover, the new ridge parameter, (denoted by K_{AS}), together with the other five new versions

2.1 Notations and some preliminaries

The multiple linear regression model can be expressed as:

$$y = X\beta + e\,, \tag{2.1}$$

where y is an $n \times 1$ vector of responses, X is an $n \times p$ observed matrix of the regressors, β is a $p \times 1$ vector of unknown parameters, and e is an $n \times 1$ vector of errors.

The ordinary least square estimator (OLS) of the regression coefficients β is defined as

$$\hat{\beta} = \left(X'X\right)^{-1} X'y\,, \tag{2.2}$$

Suppose, there exists an orthogonal matrix D such that $D'CD = \Lambda$, where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)$ are the eigenvalues of the matrix $C = X'X$. The orthogonal (canonical form) version of the multiple regression model (2.1) is

$$\mathrm{Y} = \mathrm{X}^{*}\alpha + e$$

where $X^* = XD$ and $\alpha = D'\beta$. In case the matrix $X'X$ is ill-conditioned however (in the sense of there is a near-linear dependency among the columns of the matrix) the OLS of β has a large variance, and multicollinearity is said to be present. Ridge regression replaces $X'X$ with $X'X + kI$, ($k \rangle 0$). Then the generalized ridge regression estimators of α are given as follows:

$$\hat{\alpha}(k) = \left(X'^* X^* + kI_p\right)^{-1} X'^* Y \qquad (2.3)$$

where $k = diag(k_1, k_2, ..., k_p)$, $k_i \rangle 0$ and $\hat{\alpha} = \Lambda^{-1} X'^* Y$ is the ordinary least squares (OLS) estimates of α.

According to Hoerl and Kennard (1970) the value of k_i which minimizes the MSE $(\hat{\alpha}(k))$ is

$$k_i = \frac{\sigma^2}{\alpha_i^2}, \qquad (2.4)$$

where σ^2 represents the error variance of the multiple regression model, and α_i is the i^{th} element of α.

2.2 Proposed Estimators

In this section, we review some already available estimators and propose some new ridge parameters.

2.2.1 Estimators based on Hoerl and Kennard (1970)

Hocking, Speed and Lynn (1976) showed that for known optimal k_i, the generalized ridge regression estimator is superior to all other estimators within the class of biased estimators they considered. Nevertheless, the optimal value of k_i fully depends on the unknown σ^2 and α_i, and they must be estimated from the observed data. Hoerl and Kennard (1970), suggested to replace σ^2 and α_i^2 by their corresponding unbiased estimators in (2.4). That is,

$$\hat{k}_i = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \qquad (2.5)$$

where $\hat{\sigma}^2$ is the residual mean square estimate, which is unbiased estimator of σ^2 and $\hat{\alpha}_i$ is the i^{th} element of $\hat{\alpha}$, which is an unbiased estimator of α.

Hoerl and Kennard (1970) suggested k to be

$$k_{HK1} = \hat{k}_{HK} = \frac{\hat{\sigma}^2}{\hat{\alpha}_{\max}^2} \tag{2.6}$$

where $\hat{\alpha}_{\max}$ is the maximum element of $\hat{\alpha}$. Now, when σ^2 and α are known then $\hat{k}_{HK}$ will give smaller MSE than the OLS.

Hoerl et al. (1975), proposed a different estimator of k by taking the harmonic mean of $\hat{k}_i$. That is

$$k_{HK2} = \hat{k}_{HKB} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \alpha_i^2} = \frac{p\hat{\sigma}^2}{\hat{\alpha}'\hat{\alpha}} \tag{2.7}$$

2.2.2 Estimators based on Kibria (2003)

Kibria (2003) proposed some new estimators based on *generalized ridge regression* approach. They are as follows:

By using the geometric mean of $\hat{k}_i$, which produces the following estimator

$$k_{K1} = \hat{k}_{GM} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p} \hat{\alpha}_i^2\right)^{\frac{1}{p}}} \tag{2.8}$$

By using the median of $\hat{k}_i$, which produces the following estimator for $p \geq 3$

$$k_{K2} = \hat{k}_{MED} = Median\left\{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right\}, \quad i = 1,2,\ldots,p \tag{2.9}$$

2.2.3. Estimators based on Khalaf and Shukur (2005)

Khalaf and Shukur (2005) suggested a new method to estimate the ridge parameter k, as a modification of k_{HK1} as

$$k_{S1} = \hat{k}_{KS} = \frac{t_{\max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_{\max}\hat{\alpha}_{\max}^2} \tag{2.10}$$

where $t_{\max}$ is the maximum eigenvalue of $X'X$ matrix

Following Kibria (2003) and Khalaf and Shukur (2005), Alkhamisi Khalaf and Shukur (2006) proposed the following estimators for k :

$$k_{S2} = \hat{k}_{arith}^{KS} = \frac{1}{p}\sum_{i=1}^{p}\left(\frac{t_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_i\hat{\alpha}_i^2}\right), \qquad i = 1,2,..., p \qquad (2.11)$$

$$k_{S3} = \hat{k}_{\max}^{KS} = \max\left(\frac{t_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_i\hat{\alpha}_i^2}\right), \qquad i = 1,2,..., p \qquad (2.12)$$

$$k_{S4} = \hat{k}_{md}^{KS} = median\left(\frac{t_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_i\hat{\alpha}_i^2}\right). \qquad (2.13)$$

2.2.4 Some proposed new estimators

Following Kibria (2003), Khalaf and Shukur (2005), Alkhamisi Khalaf and Shukur (2006) and Alkhamisi and Shukur (2008), we proposed the following estimators. First, following Kibria (2003) and Khalaf and Shukur (2005), we propose the following estimator

$$k_{KM1} = \hat{k}_{gm}^{KS} = \left(\prod_{i=1}^{p} \frac{t_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_i\hat{\alpha}_i^2}\right)^{\frac{1}{p}} \qquad (2.14)$$

Now, using equation (2.4) and square root transformations (Alkhamisi and Shukur (2008)), we propose the following estimators:

$$k_{KM2} = \max\left(\frac{1}{\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}}\right) \qquad (2.15)$$

$$k_{KM3} = \max\left(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}\right) \qquad (2.16)$$

$$k_{KM4} = \left(\prod_{i=1}^{p}\frac{1}{\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}}\right)^{\frac{1}{p}} \qquad (2.17)$$

$$k_{KM5} = \left(\prod_{i=1}^{p}\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}\right)^{\frac{1}{p}} \qquad (2.18)$$

$$k_{KM6} = median\left(\frac{1}{\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}}\right) \tag{2.19}$$

$$k_{KM7} = median\left(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}\right) \tag{2.20}$$

$$k_{KM8} = \max\left(\frac{1}{\sqrt{\frac{t_{\max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_{\max}\hat{\alpha}_i^2}}}\right) \tag{2.21}$$

$$k_{KM9} = \max\left(\sqrt{\frac{t_{\max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_{\max}\hat{\alpha}_i^2}}\right) \tag{2.22}$$

$$k_{KM10} = \left(\prod_{i=1}^{p} \frac{1}{\sqrt{\frac{t_{\max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_{\max}\hat{\alpha}_i^2}}}\right)^{\frac{1}{p}} \tag{2.23}$$

$$k_{KM11} = \left(\prod_{i=1}^{p} \sqrt{\frac{t_{\max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_{\max}\hat{\alpha}_i^2}}\right)^{\frac{1}{p}} \tag{2.24}$$

$$k_{KM12} = median\left(\frac{1}{\sqrt{\frac{t_{\max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_{\max}\hat{\alpha}_i^2}}}\right) \tag{2.25}$$

Note that the new proposed estimators: k_{S1} (in 2.10), k_{S2} (in 2.11), k_{KM3} (in 2.16), k_{KM7} (in 2.20), k_{KM8} (in 2.21), k_{KM9} (in 2.22), k_{KM10} (in 2.23), k_{KM11} (in 2.24) and finally k_{KM12} (in 2.25) were not investigated in Muniz and Kibria (2009).

3. The Monte Carlo Design

The aim of this paper is to compare the performance of our new proposed estimators the other estimators together with the OLS. Since a theoretical comparison is not possible, a simulation study has been conducted in this section. The design of a good simulation study is dependent on (i) what factors are expected to affect the properties of the estimators under investigation and (ii) what criteria are being used to judge the results. Since ridge estimators are supposed to have smaller MSE compared to OLS, the MSE will be used as criteria to measure the goodness of an estimator, while the first question will be treated shortly.

All in all about 20 different estimators have been discussed in this paper. From the preliminary simulation study we have, however, selected the following best 15 estimators: $k_{HK,}$ $k_{K1,}$ $k_{K2,}$ $k_{S1,}$ $k_{S2,}$ $k_{KM1,}$ $k_{KM2,}$ $k_{KM4,}$ $k_{KM5,}$ $k_{KM6,}$ $k_{KM8,}$ $k_{KM9,}$ $k_{KM10,}$ $k_{KM11,}$ and k_{KM12} . Among them the last 5 are newly proposed.

Since the degree of collinearity among the explanatory variable are of central importance, we followed Muniz and Kibria (2009) in generating the explanatory variable using the following device,

$$x_{ij} = \left(1-\gamma^2\right)^{(1/2)} z_{ij} + \gamma\ z_{ip}\,, \qquad i = 1,2,\ldots n \qquad j = 1,2,\ldots p \tag{3.1}$$

where γ^2 represents the correlation between the explanatory variables, and z_{ij} are independent standard pseudo-random numbers. The n observations for the dependent variable are then determined by:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i\,, \qquad i = 1,2,\ldots n \tag{3.2}$$

wher e_i are i.i.d. $\mathrm{N}\left(0,\sigma^2\right)$ pseudo-random numbers, and β_0 is taken to be zero without loss of generality.

Factors that vary in the Monte Carlo simulations

Since our primary interest lies in the performance of our proposed estimators according to the strength of the multcollinearity, we used different degrees of correlation between the variables and let (γ) = 0.7, 0.8 and 0.9. We also want to see the effect of the sample sizes on the performance of the estimators. Therefore, in this study, we considered $n = 10$, 20, 30, 40, 50 and 100 which will cover models with small, medium and large sample sizes. The number of the explanatory variables is also of great importance since the bad impact of the collinearity on the MSE might be stronger when more variables in the model are correlated. We hence generated models with and $p = 2$ and 4 explanatory variables. To see whether the magnitude error variance have a significant effect of the performances of the proposed estimators, we used different values of the error standard deviations $\sigma = 0.01$, 0.5, 1, 3, and 5. For each set of explanatory variables we considered the coefficient vector that corresponded to the largest eigen value of $X'X$ matrix subject to the constraint that $\beta'\beta = 1$. Newhouse and Oman (1971) stated that if the mean squared error (MSE) is a function of β, σ^2, and k, and if the explanatory variables are fixed, then the MSE is minimized when we choose this coefficient vector.

For given values of n, p, β, γ, and σ, the set of explanatory variables are generated. Then the experiment was repeated 2000 times and the average mean squared error was calculated for all 15 estimators.

4. Results Discussions

In this section we present the results of our Monte Carlo experiment concerning the MSEs of the different proposed estimators compare to the OLS. A conventional way to report the results of a Monte Carlo experiment is to tabulate the values of these MSEs under different conditions. When determining the manner of presentation, some account has to be taken to the results obtained. Our original intention was to start by presenting results for all the main effects in term of tables. However, since the results are too extensive, presenting the results in term of tables will make it difficult to follow the head line of the findings. We hence present our most important findings in form of figures which summaries most of the results in this with respect to the different factures under investigations. More exact results of the simulated MSEs for the 15 estimators are provided in the appendix (all results are not included in the tables however but are available up on request from the authors). Simulated MSEs for fixed n, p and γ and different values of σ are presented in Table A.1, for fixed n, p and σ and different values of γ are presented in Table A.2, for fixed p, γ and σ and different values of n are presented in Table A.3.

The performance of the estimators with respect to changes in σ. In Table A.1 we have provided the MSEs of the estimators as a function of the variance of the errors (σ). When the value of σ increases, the MSE of the estimators also increases. For all values of σ, the ridge regression estimators have smaller MSE than the OLS. However, the performance of the proposed estimators $k_{KM4}, k_{KM5}, k_{KM8}$, k_{KM10}, k_{KM12}, and k_{K1}, k_{K2} is better than the performance of the rest of the analyzed estimators. This behavior was almost constant for any sample size and number of variables considered. However, when the standard deviation is large, i.e. $(\sigma = 5)$, the new k_{KM8}, k_{KM12} outperform all the other estimators in term of producing less MSE.

For given $\gamma = 0.70$ and $n = 10$, the performance of estimators as a function of the standard deviation of the errors for $p = 2$ and $p = 4$ are provided in Figures 1 and 2 respectively. From these figures we observe that as the standard deviation increases, the MSE also increases. The same is true when shifting from 2 to 4 variables models especially for the OLS, k_{HK}, k_{S1}, k_{S2}, (see figure 2).

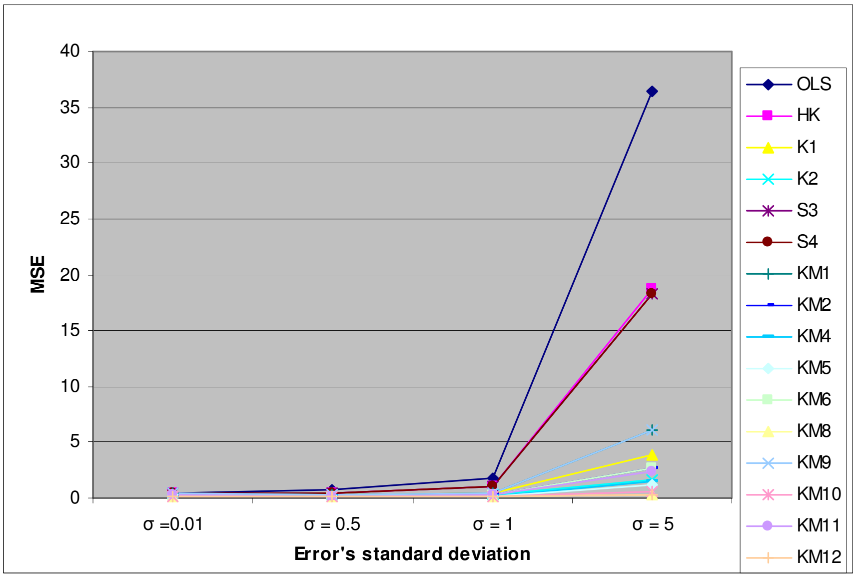

Figure 1 Performance of the estimators as a function of σ when $p = 2$

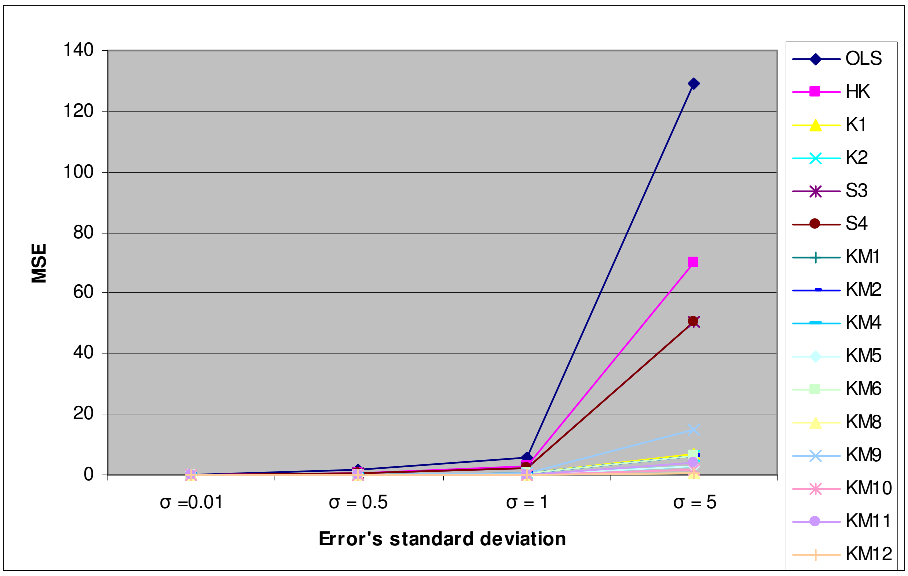

Figure 2 Performance of the estimators as a function of σ when $p = 4$

Performance as a function of γ

In Table A.2 we have provided the MSEs of the estimators as a function of the correlation between the explanatory variables. For smaller sigma ($\sigma = 0.01$) the change in the correlation between the explanatory variables had almost no effect on the MSEs. In all situations they remained almost the same for any sample size or number of parameters, and their MSEs are very small. When σ increases, the higher correlation between the independent variables resulted in an increase of the MSE of the *k*-estimators. In general, $k_{KM4}, k_{KM5}, k_{KM8}$, k_{KM10}, k_{KM12} and k_{K1}, k_{K2} performed better than others.

For given $\sigma = 1$ and $p = 4$ the performance of estimators as a function of the correlation between the explanatory variables for $n = 20$ and $n = 50$ are provided in Figures 3 and 4 respectively. From these figures we observed that as correlation increases, the MSE also increases. The MSE decreases however when the number of observations increases from 20 to 50. All of the ridge estimators have smaller MSE compared with OLS and they are very close to each other.

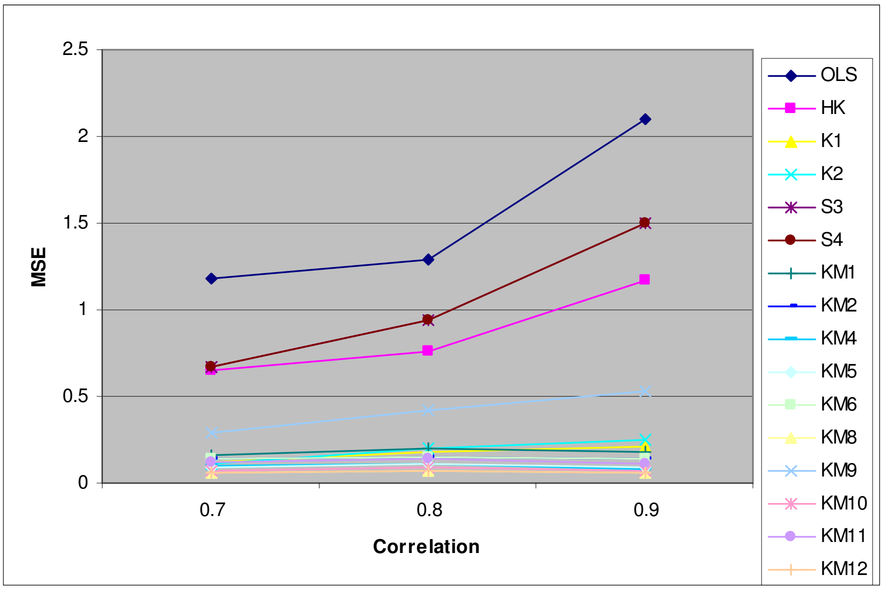

Figure 3 Performance of the estimators as a function of γ when n=20

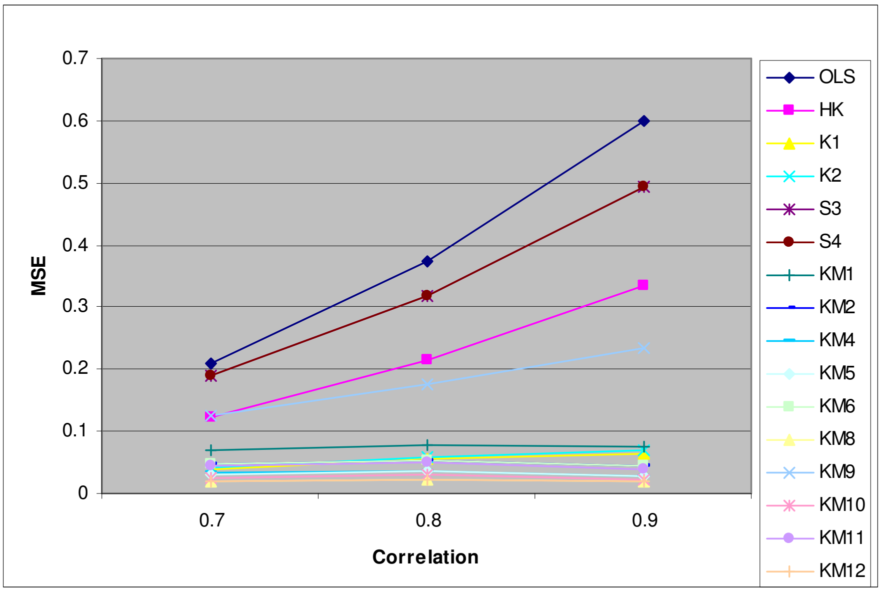

Figure 4 Performance of the estimators as a function of γ when n=50

Performance as a function of n

In Table A.3 we have provided the MSEs of the estimators as a function of the sample size. We observed that, in general, when the sample size increases the MSE decreases, or remained the same. Even for the large values of γ and σ, if we increase the sample size the MSE of estimators decrease. Again in this situation, as n and p increased the performance of $k_{KM4}, k_{KM5}, k_{KM8}$, k_{KM10}, k_{KM12}, and k_{K1}, k_{K2} is better than the rest of the *k*-estimators.

For given $\gamma = 0.90$ and $p = 2$, the performance of the estimators as a function of the sample size for $\sigma = 0.5$ and $\sigma = 5$ are provided in Figures 5 and 6 respectively. From these figures, we observed that as the sample size increases, the MSE decreases. Except for a few situations, this pattern was constant for all of the estimators. Note the huge increase in the MSE when shifting from $\sigma = 0.5$ to $\sigma = 5$.

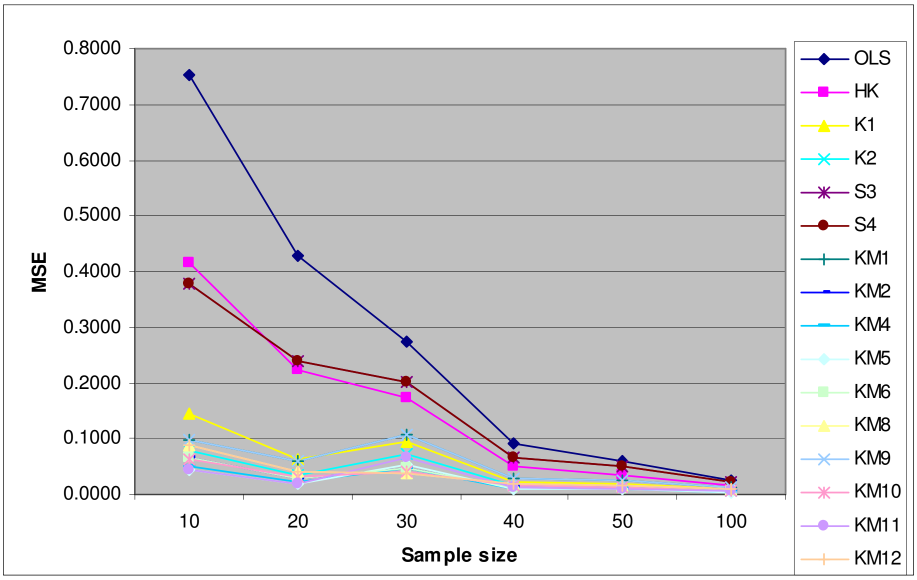

Figure 5 Performance of the estimators as a function of n when $\sigma = 0.5$

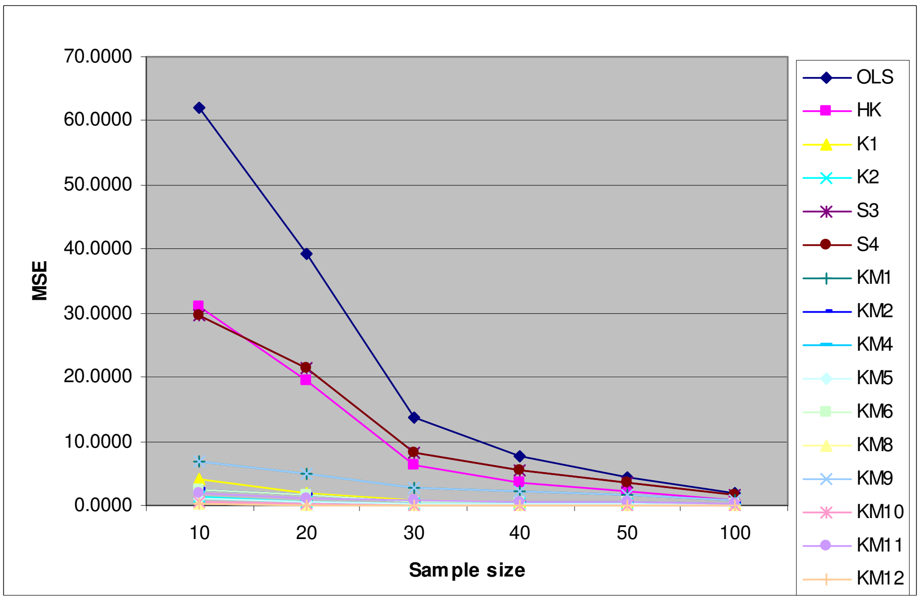

Figure 6 Performance of the estimators as a function of n when $\sigma = 5$

5. Concluding Remarks

In this paper we have reviewed and proposed some new estimators for estimating the ridge parameter k. The new proposed estimators are defined based on the work of Kibria (2003), Khalaf and Shukur (2005) and Alkhamisi Khalaf and Shukur (2006). The performance of the estimators depends on the variance of the random error (σ), the correlations among the explanatory variables (γ), the sample size (n) and the unknown coefficients vectors β. Based on the simulation study, some conclusions might be drawn. However, these conclusions might be restricted to the set of experimental conditions which are investigated. We used the MSE criteria to measure the goodness of the estimators. The increase of number of correlated variable, σ and the increase of the correlation between the independent variables have a negative effect in the MSE, in the sense that it also increases. When the sample size increases the MSE decreases, even when the correlation between the independent variables and σ are large. In all situations, the proposed estimators have smaller MSE than the ordinary least squared estimators. Five of them, $k_{KM4}, k_{KM5}, k_{KM8}$, k_{KM10}, k_{KM12}, and the k_{K1}, k_{K2} performed better than the rest in the sense of smaller MSE. Finally, it appears that the proposed estimators k_{KM4}, k_{KM5}, k_{KM8}, k_{KM10}, k_{KM12} are useful and may be recommended to the practitioners. The k_{KM8}, k_{KM12} estimators are particularly also recommended when working with model with large residual variances since they outperform all the others in such cases.

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APPENDIX A

Table A.1 Simulated MSE for fixed n, p, and γ and different values of σ

$n = 10, \ p = 2, \ \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.427	0.426	0.426	0.426	0.426	0.426	0.389	0.125	0.129	0.272	0.125	0.125	0.389	0.129	0.272	0.125
0.5	0.765	0.516	0.336	0.269	0.517	0.517	0.304	0.172	0.171	0.189	0.172	0.136	0.304	0.149	0.222	0.136
1	1.799	1.051	0.438	0.297	1.042	1.042	0.479	0.270	0.223	0.215	0.270	0.143	0.479	0.163	0.291	0.143
5	36.39	18.75	3.898	1.701	18.36	18.36	6.025	2.733	1.446	1.204	2.733	0.317	6.025	0.547	2.373	0.317

$n = 10, \ p = 2, \ \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.073	0.073	0.073	0.073	0.073	0.073	0.057	0.121	0.094	0.025	0.121	0.121	0.057	0.095	0.025	0.120
0.5	0.438	0.244	0.117	0.084	0.243	0.243	0.087	0.065	0.053	0.047	0.065	0.084	0.087	0.063	0.048	0.084
1	1.608	0.866	0.254	0.133	0.840	0.840	0.271	0.121	0.084	0.082	0.121	0.089	0.271	0.075	0.115	0.089
5	38.59	20.17	4.069	1.577	20.19	20.19	6.265	2.712	1.384	1.138	2.712	0.271	6.265	0.477	3.325	0.271

$n = 10, \ p = 2, \ \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.073	0.073	0.073	0.073	0.073	0.073	0.052	0.120	0.094	0.020	0.120	0.052	0.094	0.020	0.121	0.052.
0.5	0.618	0.329	0.125	0.076	0.313	0.313	0.087	0.063	0.051	0.043	0.063	0.085	0.087	0.063	0.042	0.085
1	2.445	1.267	0.296	0.137	1.191	1.191	0.301	0.133	0.195	0.081	0.133	0.094	0.301	0.074	0.109	0.090
5	61.65	31.81	4.630	1.542	30.49	30.49	7.311	2.890	1.224	1.222	2.890	0.256	7.311	0.438	2.203	0.256

$n = 20, \ p = 2, \ \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.035	0.032	0.032	0.032	0.032	0.032	0.030	0.032	0.032	0.020	0.032	0.032	0.030	0.032	0.020	0.032
0.5	0.090	0.061	0.045	0.041	0.072	0.072	0.043	0.029	0.025	0.025	0.029	0.038	0.043	0.029	0.028	0.038
1	0.264	0.152	0.080	0.064	0.209	0.209	0.119	0.059	0.047	0.043	0.059	0.040	0.119	0.035	0.068	0.040
5	5.529	2.823	1.010	0.645	4.299	4.299	2.393	1.198	0.853	0.524	0.198	0.116	2.393	0.246	1.293	0.116

$n = 20, \ p = 2, \ \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.189	0.189	0.189	0.189	0.189	0.189	0.174	0.055	0.057	0.123	0.055	0.055	0.174	0.057	0.123	0.057
0.5	0.334	0.225	0.149	0.20	0.255	0.255	0.156	0.077	0.077	0.086	0.077	0.05	0.156	0.065	0.107	0.059
1	0.776	0.444	0.182	0.127	0.535	0.535	0.265	0.117	0.099	0.098	0.117	0.061	0.265	0.071	0.146	0.061
5	13.77	6.66	1.312	0.619	8.96	8.96	3.500	1.265	0.703	0.521	1.265	0.113	3.500	0.217	1.280	0.113

$n = 20, \ p = 2, \ \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.032	0.032	0.032	0.032	0.032	0.032	0.025	0.032	0.032	0.011	0.032	0.032	0.025	0.032	0.011	0.032
0.5	0.190	0.104	0.049	0.035	0.121	0.121	0.047	0.028	0.023	0.020	0.028	0.040	0.047	0.029	0.022	0.040
1	0.666	0.340	0.094	0.056	0.419	0.419	0.155	0.050	0.037	0.037	0.050	0.042	0.155	0.034	0.058	0.042
5	15.76	7.692	1.441	0.599	9.601	9.601	3.440	1.303	0.635	0.457	1.303	0.093	3.440	0.176	1.141	0.093

$n = 50, \ p = 2, \ \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.013	0.012	0.012	0.012	0.012	0.012	0.011	0.012	0.013	0.008	0.012	0.012	0.011	0.011	0.008	0.012
0.5	0.027	0.020	0.016	0.015	0.025	0.025	0.018	0.011	0.010	0.010	0.011	0.016	0.018	0.012	0.016	0.016
1	0.076	0.046	0.027	0.023	0.071	0.071	0.049	0.023	0.020	0.017	0.023	0.016	0.049	0.014	0.030	0.016
5	1.561	0.783	0.323	0.231	1.444	1.444	0.994	0.434	0.329	0.204	0.434	0.029	0.994	0.075	0.578	0.029

$n = 50, \ p = 2, \ \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.073	0.071	0.071	0.071	0.071	0.071	0.068	0.020	0.022	0.054	0.020	0.020	0.068	0.022	0.054	0.020
0.5	0.096	0.070	0.053	0.047	0.089	0.089	0.067	0.033	0.032	0.035	0.033	0.022	0.067	0.025	0.048	0.022
1	0.172	0.107	0.061	0.049	0.156	0.156	0.106	0.050	0.043	0.039	0.050	0.022	0.106	0.026	0.067	0.022
5	2.381	1.184	0.388	0.250	2.099	2.099	1.246	0.467	0.322	0.217	0.467	0.035	1.246	0.081	0.616	0.035

$n = 50, \quad p = 2, \quad \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.013	0.012	0.012	0.012	0.012	0.012	0.009	0.020	0.015	0.004	0.012	0.012	0.009	0.015	0.004	0.012
0.5	0.072	0.039	0.017	0.012	0.054	0.054	0.023	0.010	0.008	0.007	0.010	0.016	0.023	0.012	0.009	0.016
1	0.248	0.123	0.033	0.019	0.189	0.189	0.079	0.020	0.013	0.013	0.020	0.016	0.079	0.012	0.025	0.016
5	6.132	2.978	0.464	0.199	4.572	4.572	1.858	0.452	0.225	0.169	0.452	0.026	1.858	0.052	0.539	0.026

$n = 100, \quad p = 2, \quad \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.010	0.006	0.006	0.006	0.006	0.006	0.005	0.009	0.008	0.004	0.009	0.009	0.005	0.008	0.004	0.009
0.5	0.014	0.010	0.008	0.007	0.013	0.013	0.010	0.005	0.005	0.005	0.005	0.008	0.010	0.006	0.006	0.008
1	0.039	0.023	0.013	0.011	0.038	0.038	0.028	0.011	0.009	0.008	0.011	0.008	0.028	0.007	0.017	0.008
5	0.812	0.412	0.163	0.113	0.779	0.779	0.572	0.221	0.165	0.102	0.221	0.012	0.572	0.031	0.326	0.012

$n = 100, \quad p = 2, \quad \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.010	0.006	0.006	0.006	0.006	0.006	0.005	0.009	0.007	0.003	0.009	0.009	0.005	0.007	0.003	0.009
0.5	0.016	0.010	0.008	0.007	0.015	0.015	0.010	0.005	0.004	0.004	0.005	0.008	0.010	0.006	0.006	0.008
1	0.048	0.027	0.013	0.011	0.045	0.045	0.031	0.010	0.008	0.008	0.010	0.008	0.031	0.006	0.016	0.008
5	1.026	0.518	0.178	0.114	0.968	0.968	0.653	0.212	0.150	0.096	0.212	0.011	0.653	0.028	0.033	0.011

$n = 100, \quad p = 2, \quad \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.036	0.034	0.034	0.034	0.034	0.034	0.032	0.010	0.010	0.023	0.010	0.010	0.032	0.010	0.023	0.010
0.5	0.058	0.039	0.026	0.021	0.053	0.053	0.036	0.014	0.014	0.016	0.014	0.010	0.036	0.011	0.023	0.010
1	0.118	0.067	0.029	0.021	0.106	0.106	0.064	0.021	0.018	0.017	0.021	0.010	0.064	0.012	0.032	0.010
5	2.115	1.005	0.222	0.118	1.861	1.861	0.960	0.226	0.132	0.099	0.226	0.013	0.960	0.029	0.013	0.013

$n = 10, \quad p = 4, \quad \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.184	0.183	0.183	0.183	0.183	0.183	0.071	0.159	0.103	0.063	0.159	0.159	0.133	0.103	0.063	0.159
0.5	1.482	0.847	0.190	0.171	0.664	0.664	0.133	0.140	0.104	0.105	0.140	0.124	0.234	0.103	0.111	0.124
1	5.564	3.088	0.392	0.359	2.278	2.278	0.329	0.327	0.185	0.188	0.327	0.139	0.701	0.151	0.233	0.139
5	129.0	70.10	6.667	6.191	50.63	50.63	6.225	6.286	2.659	2.616	6.286	0.568	14.72	1.587	3.912	0.568

$n = 10, \quad p = 4, \quad \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.170	0.169	0.169	0.169	0.169	0.169	0.192	0.169	0.190	0.193	0.169	0.169	0.193	0.190	0.193	0.169
0.5	12.04	6.365	0.341	0.289	3.308	3.308	0.264	0.217	0.213	0.223	0.217	0.183	0.409	0.205	0.234	0.183
1	47.61	24.49	0.661	0.495	11.43	11.43	0.485	0.384	0.514	0.283	0.306	0.384	0.196	0.247	0.359	0.196
5	1190.	613.5	10.40	6.984	292.0	292.0	7.455	5.690	2.463	2.868	5.690	0.582	21.58	1.507	4.237	0.582

$n = 10, \quad p = 4, \quad \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.188	0.177	0.179	0.179	0.179	0.179	0.209	0.164	0.160	0.184	0.164	0.164	0.243	0.159	0.184	0.164
0.5	23.17	11.50	0.405	0.321	5.521	5.521	0.270	0.190	0.182	0.197	0.190	0.164	0.480	0.173	0.215	0.164
1	94.17	48.44	0.865	0.627	22.26	22.26	0.517	0.406	0.241	0.275	0.406	0.173	1.278	0.205	0.336	0.173
5	2313	1174	13.87	9.616	553.3	553.3	8.052	6.160	2.021	2.583	6.160	0.452	26.49	1.187	3.965	0.452

$n = 20, \quad p = 4, \quad \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.063	0.063	0.063	0.063	0.063	0.063	0.060	0.060	0.047	0.054	0.060	0.060	0.062	0.047	0.054	0.060
0.5	0.359	0.223	0.082	0.079	0.221	0.221	0.081	0.067	0.059	0.060	0.067	0.052	0.117	0.053	0.068	0.052
1	1.185	0.646	0.121	0.114	0.673	0.673	0.160	0.145	0.097	0.093	0.145	0.057	0.288	0.075	0.121	0.057
5	28.52	14.95	1.624	1.495	15.52	15.52	2.851	2.599	1.390	1.208	2.599	0.209	5.958	0.785	1.921	0.209

$n = 20, \quad p = 4, \quad \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.064	0.064	0.064	0.064	0.064	0.064	0.064	0.064	0.063	0.064	0.064	0.063	0.063	0.063	0.063	0.063
0.5	0.367	0.241	0.111	0.110	0.280	0.280	0.102	0.087	0.080	0.082	0.087	0.067	0.154	0.076	0.088	0.067
1	1.295	0.756	0.182	0.196	0.936	0.936	0.198	0.154	0.109	0.110	0.154	0.069	0.419	0.090	0.139	0.069
5	30.56	16.70	2.512	3.022	21.69	21.69	3.336	2.627	1.147	1.041	2.627	0.169	8.985	0.588	1.850	0.169

$n = 20, \quad p = 4, \quad \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.01	0.0630	0.0620	0.062	0.062	0.062	0.062	0.049	0.061	0.057	0.022	0.061	0.061	0.054	0.057	0.052	0.061
0.5	0.579	0.344	0.112	0.113	0.419	0.419	0.085	0.070	0.063	0.064	0.070	0.059	0.171	0.060	0.070	0.059
1	2.099	1.170	0.210	0.248	1.500	1.500	0.179	0.136	0.083	0.087	0.136	0.061	0.530	0.070	0.112	0.061
5	51.09	27.72	3.112	35.62	35.62	3.155	2.742	8.514	0.835	0.820	2.742	0.129	11.745	0.416	1.478	0.129

$n = 50, \quad p = 4, \quad \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM10	KM11	KM12
0.01	0.023	0.022	0.022	0.022	0.022	0.022	0.018	0.021	0.016	0.016	0.021	0.021	0.022	0.016	0.016	0.021
0.5	0.071	0.049	0.026	0.027	0.066	0.066	0.030	0.023	0.020	0.020	0.023	0.019	0.047	0.018	0.0241	0.019
1	0.210	0.124	0.040	0.043	0.191	0.191	0.069	0.047	0.034	0.031	0.047	0.019	0.126	0.024	0.046	0.019
5	4.868	2.686	0.489	0.552	4.413	4.413	1.386	0.969	0.517	0.408	0.969	0.042	2.817	0.225	0.810	0.042

$n = 50, \quad p = 4, \quad \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM10	KM11	KM12
0.01	0.021	0.020	0.020	0.020	0.020	0.020	0.020	0.021	0.021	0.021	0.021	0.021	0.020	0.021	0.021	0.020
0.5	0.112	0.073	0.035	0.035	0.098	0.098	0.037	0.029	0.028	0.028	0.029	0.023	0.061	0.026	0.031	0.023
1	0.374	0.214	0.056	0.059	0.319	0.319	0.078	0.053	0.037	0.037	0.053	0.023	0.176	0.030	0.050	0.023
5	9.112	4.978	0.698	0.911	7.735	7.735	1.481	0.924	0.394	0.356	0.924	0.040	4.065	0.176	0.730	0.040

$n = 50, \quad p = 4, \quad \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM10	KM11	KM12
0.01	0.023	0.022	0.022	0.022	0.022	0.022	0.015	0.021	0.017	0.015	0.021	0.021	0.019	0.017	0.015	0.021
0.5	0.166	0.099	0.032	0.033	0.138	0.138	0.030	0.021	0.019	0.019	0.021	0.020	0.071	0.018	0.022	0.020
1	0.600	0.335	0.063	0.071	0.494	0.494	0.074	0.045	0.027	0.028	0.045	0.020	0.235	0.022	0.040	0.020
5	14.84	8.068	0.999	1.504	12.04	12.04	1.516	1.023	0.314	0.301	1.023	0.035	5.570	0.139	0.635	0.035

$n = 100, \quad p = 4, \quad \gamma = 0.7$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM10	KM11	KM12
0.01	0.010	0.010	0.010	0.010	0.010	0.010	0.008	0.010	0.008	0.008	0.010	0.010	0.010	0.008	0.008	0.010
0.5	0.030	0.021	0.012	0.012	0.029	0.029	0.015	0.011	0.010	0.010	0.011	0.009	0.023	0.009	0.012	0.009
1	0.089	0.053	0.019	0.020	0.085	0.085	0.037	0.024	0.018	0.016	0.024	0.010	0.063	0.012	0.024	0.010
5	1.966	1.067	0.213	0.245	1.879	1.879	0.727	0.440	0.253	0.199	0.440	0.016	1.365	0.103	0.423	0.016

$n = 100, \quad p = 4, \quad \gamma = 0.8$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM10	KM11	KM12
0.01	0.010	0.010	0.010	0.010	0.010	0.010	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010
0.5	0.040	0.027	0.015	0.015	0.038	0.038	0.018	0.013	0.012	0.012	0.013	0.010	0.028	0.011	0.014	0.010
1	0.131	0.077	0.023	0.025	0.123	0.123	0.041	0.026	0.018	0.017	0.026	0.010	0.083	0.013	0.025	0.010
5	3.065	1.679	0.278	0.330	2.869	2.869	0.816	0.458	0.219	0.180	0.458	0.015	1.889	0.084	0.409	0.015

$n = 100, \quad p = 4, \quad \gamma = 0.9$

σ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM10	KM11	KM12
0.01	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010
0.5	0.069	0.043	0.018	0.018	0.063	0.063	0.019	0.013	0.012	0.012	0.013	0.010	0.040	0.011	0.014	0.010
1	0.236	0.132	0.029	0.035	0.213	0.213	0.045	0.026	0.016	0.016	0.026	0.010	0.125	0.013	0.024	0.010
5	5.757	3.139	0.420	0.599	5.195	5.195	0.913	0.432	0.165	0.158	0.432	0.015	2.967	0.068	0.369	0.015

Table A.2 Simulated MSE for fixed n, p, and σ and different values of γ

$n = 10,\ \ p = 2,\ \ \sigma = 0.01$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.427	0.426	0.426	0.426	0.426	0.426	0.389	0.125	0.129	0.272	0.125	0.125	0.389	0.129	0.272	0.125
0.80	0.073	0.073	0.073	0.073	0.073	0.073	0.057	0.121	0.094	0.025	0.121	0.121	0.057	0.095	0.025	0.120
0.90	0.073	0.073	0.073	0.073	0.073	0.073	0.052	0.120	0.094	0.020	0.120	0.052	0.094	0.020	0.121	0.052.

$n = 10,\ \ p = 2,\ \ \sigma = 1$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	1.799	1.051	0.438	0.297	1.042	1.042	0.479	0.270	0.223	0.215	0.270	0.143	0.479	0.163	0.291	0.143
0.80	1.608	0.866	0.254	0.133	0.840	0.840	0.271	0.121	0.084	0.082	0.121	0.089	0.271	0.075	0.115	0.089
0.90	2.445	1.267	0.296	0.137	1.191	1.191	0.301	0.133	0.195	0.081	0.133	0.094	0.301	0.074	0.109	0.090

$n = 10,\ \ p = 2,\ \ \sigma = 5$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	36.39	18.75	3.898	1.701	18.36	18.36	6.025	2.733	1.446	1.204	2.733	0.317	6.025	0.547	2.373	0.317
0.80	38.59	20.17	4.069	1.577	20.19	20.19	6.265	2.712	1.384	1.138	2.712	0.271	6.265	0.477	3.325	0.271
0.90	61.65	31.81	4.630	1.542	30.49	30.49	7.311	2.890	1.224	1.222	2.890	0.256	7.311	0.438	2.203	0.256

$n = 10,\ \ p = 4,\ \ \sigma = 0.01$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.184	0.183	0.183	0.183	0.183	0.183	0.071	0.159	0.103	0.063	0.159	0.159	0.133	0.103	0.063	0.159
0.80	0.170	0.169	0.169	0.169	0.169	0.169	0.192	0.169	0.190	0.193	0.169	0.169	0.193	0.190	0.193	0.169
0.90	0.188	0.177	0.179	0.179	0.179	0.179	0.209	0.164	0.160	0.184	0.164	0.164	0.243	0.159	0.184	0.164

$n = 10,\ \ p = 4,\ \ \sigma = 1$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	5.564	3.088	0.392	0.359	2.278	2.278	0.329	0.327	0.185	0.188	0.327	0.139	0.701	0.151	0.233	0.139
0.80	47.61	24.49	0.661	0.495	11.43	11.43	0.485	0.384	0.514	0.283	0.306	0.384	0.196	0.247	0.359	0.196
0.90	94.17	48.44	0.865	0.627	22.26	22.26	0.517	0.406	0.241	0.275	0.406	0.173	1.278	0.205	0.336	0.173

$n = 10,\ \ p = 4,\ \ \sigma = 5$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	129	70.10	6.667	6.191	50.63	50.63	6.225	6.286	2.659	2.616	6.286	0.568	14.72	1.587	3.912	0.568
0.80	1190	613.5	10.40	6.984	292	292	7.455	5.690	2.463	2.868	5.690	0.582	21.58	1.507	4.237	0.582
0.90	2313.	1174	13.87	9.616	553.3	553.3	8.052	6.160	2.021	2.583	6.160	0.452	26.49	1.187	3.965	0.452

$n = 20,\ \ p = 2,\ \ \sigma = 0.01$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.035	0.032	0.032	0.032	0.032	0.032	0.030	0.032	0.032	0.020	0.032	0.032	0.030	0.032	0.020	0.032
0.80	0.189	0.189	0.189	0.189	0.189	0.189	0.174	0.055	0.057	0.123	0.055	0.055	0.174	0.057	0.123	0.057
0.90	0.032	0.032	0.032	0.032	0.032	0.032	0.025	0.032	0.032	0.011	0.032	0.032	0.025	0.032	0.011	0.032

$n = 20,\ \ p = 2,\ \ \sigma = 1$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.264	0.152	0.080	0.064	0.209	0.209	0.119	0.059	0.047	0.043	0.059	0.040	0.119	0.035	0.068	0.040
0.80	0.776	0.444	0.182	0.127	0.535	0.535	0.265	0.117	0.099	0.098	0.117	0.061	0.265	0.071	0.146	0.061
0.90	0.666	0.340	0.094	0.056	0.419	0.419	0.155	0.050	0.037	0.037	0.050	0.042	0.155	0.034	0.058	0.042

$n = 20,\ \ p = 2,\ \ \sigma = 5$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	5.529	2.823	1.010	0.645	4.299	4.299	2.393	1.198	0.853	0.524	0.198	0.116	2.393	0.246	1.293	0.116
0.80	13.77	6.66	1.312	0.619	8.96	8.96	3.500	1.265	0.703	0.521	1.265	0.113	3.500	0.217	1.280	0.113
0.90	15.76	7.692	1.441	0.599	9.601	9.601	3.440	1.303	0.635	0.457	1.303	0.093	3.440	0.176	1.141	0.093

$$n = 20, \quad p = 4, \quad \sigma = 0.01$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.063	0.063	0.063	0.063	0.063	0.063	0.060	0.060	0.047	0.054	0.060	0.060	0.062	0.047	0.054	0.060
0.80	0.064	0.064	0.064	0.064	0.064	0.064	0.064	0.064	0.063	0.064	0.064	0.063	0.063	0.063	0.063	0.063
0.90	0.063	0.062	0.062	0.062	0.062	0.062	0.049	0.061	0.057	0.022	0.061	0.061	0.054	0.057	0.052	0.061

$$n = 20, \quad p = 4, \quad \sigma = 1$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	1.185	0.646	0.121	0.114	0.673	0.673	0.160	0.145	0.097	0.093	0.145	0.057	0.288	0.075	0.121	0.057
0.80	1.295	0.756	0.182	0.196	0.936	0.936	0.198	0.154	0.109	0.110	0.154	0.069	0.419	0.090	0.139	0.069
0.90	2.099	1.170	0.210	0.248	1.500	1.500	0.179	0.136	0.083	0.087	0.136	0.061	0.530	0.070	0.112	0.061

$$n = 20, \quad p = 4, \quad \sigma = 5$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	28.52	14.95	1.624	1.495	15.52	15.52	2.851	2.599	1.390	1.208	2.599	0.209	5.958	0.785	1.921	0.209
0.80	30.56	16.70	2.512	3.022	21.69	21.69	3.336	2.627	1.147	1.041	2.627	0.169	8.985	0.588	1.850	0.169
0.90	51.09	27.72	3.112	35.62	35.62	3.155	2.742	8.514	0.835	0.820	2.742	0.129	11.745	0.416	1.478	0.129

$$n = 50, \quad p = 2, \quad \sigma = 0.01$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.013	0.012	0.012	0.012	0.012	0.012	0.011	0.012	0.013	0.008	0.012	0.012	0.011	0.011	0.008	0.012
0.80	0.073	0.071	0.071	0.071	0.071	0.071	0.068	0.020	0.022	0.054	0.020	0.020	0.068	0.022	0.054	0.020
0.90	0.013	0.012	0.012	0.012	0.012	0.012	0.009	0.020	0.015	0.004	0.012	0.012	0.009	0.015	0.004	0.012

$$n = 50, \quad p = 2, \quad \sigma = 1$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.076	0.046	0.027	0.023	0.071	0.071	0.049	0.023	0.020	0.017	0.023	0.016	0.049	0.014	0.030	0.016
0.80	0.172	0.107	0.061	0.049	0.156	0.156	0.106	0.050	0.043	0.039	0.050	0.022	0.106	0.026	0.067	0.022
0.90	0.248	0.123	0.033	0.019	0.189	0.189	0.079	0.020	0.013	0.013	0.020	0.016	0.079	0.012	0.025	0.016

$$n = 50, \quad p = 2, \quad \sigma = 5$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	1.561	0.783	0.323	0.231	1.444	1.444	0.994	0.434	0.329	0.204	0.434	0.029	0.994	0.075	0.578	0.029
0.80	2.381	1.184	0.388	0.250	2.099	2.099	1.246	0.467	0.322	0.217	0.467	0.035	1.246	0.081	0.616	0.035
0.90	6.132	2.978	0.464	0.199	4.572	4.572	1.858	0.452	0.225	0.169	0.452	0.026	1.858	0.052	0.539	0.026

$$n = 50, \quad p = 4, \quad \sigma = 0.01$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.023	0.022	0.022	0.022	0.022	0.022	0.018	0.021	0.016	0.016	0.021	0.021	0.022	0.016	0.016	0.021
0.80	0.021	0.020	0.020	0.020	0.020	0.020	0.020	0.021	0.021	0.021	0.021	0.021	0.020	0.021	0.021	0.020
0.90	0.023	0.022	0.022	0.022	0.022	0.022	0.015	0.021	0.017	0.015	0.021	0.021	0.019	0.017	0.015	0.021

$$n = 50, \quad p = 4, \quad \sigma = 1$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.210	0.124	0.040	0.043	0.191	0.191	0.069	0.047	0.034	0.031	0.047	0.019	0.126	0.024	0.046	0.019
0.80	0.374	0.214	0.056	0.059	0.319	0.319	0.078	0.053	0.037	0.037	0.053	0.023	0.176	0.030	0.050	0.023
0.90	0.600	0.335	0.063	0.071	0.494	0.494	0.074	0.045	0.027	0.028	0.045	0.020	0.235	0.022	0.040	0.020

$$n = 50, \quad p = 4, \quad \sigma = 5$$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	4.868	2.686	0.489	0.552	4.413	4.413	1.386	0.969	0.517	0.408	0.969	0.042	2.817	0.225	0.810	0.042
0.80	9.112	4.978	0.698	0.911	7.735	7.735	1.481	0.924	0.394	0.356	0.924	0.040	4.065	0.176	0.730	0.040
0.90	14.84	8.068	0.999	1.504	12.04	12.04	1.516	1.023	0.314	0.301	1.023	0.035	5.570	0.139	0.635	0.035

$n = 100, \quad p = 2, \quad \sigma = 0.01$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.010	0.006	0.006	0.006	0.006	0.006	0.005	0.009	0.008	0.004	0.009	0.009	0.005	0.008	0.004	0.009
0.80	0.010	0.006	0.006	0.006	0.006	0.006	0.005	0.009	0.007	0.003	0.009	0.009	0.005	0.007	0.003	0.009
0.90	0.036	0.034	0.034	0.034	0.034	0.034	0.032	0.010	0.010	0.023	0.010	0.010	0.032	0.010	0.023	0.010

$n = 100, \quad p = 2, \quad \sigma = 1$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.039	0.023	0.013	0.011	0.038	0.038	0.028	0.011	0.009	0.008	0.011	0.008	0.028	0.007	0.017	0.008
0.80	0.048	0.027	0.013	0.011	0.045	0.045	0.031	0.010	0.008	0.008	0.010	0.008	0.031	0.006	0.016	0.008
0.90	0.118	0.067	0.029	0.021	0.106	0.106	0.064	0.021	0.018	0.017	0.021	0.010	0.064	0.012	0.032	0.010

$n = 100, \quad p = 2, \quad \sigma = 5$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.812	0.412	0.163	0.113	0.779	0.779	0.572	0.221	0.165	0.102	0.221	0.012	0.572	0.031	0.326	0.012
0.80	1.026	0.518	0.178	0.114	0.968	0.968	0.653	0.212	0.150	0.096	0.212	0.011	0.653	0.028	0.033	0.011
0.90	2.115	1.005	0.222	0.118	1.861	1.861	0.960	0.226	0.132	0.099	0.226	0.013	0.960	0.029	0.013	0.013

$n = 100, \quad p = 4, \quad \sigma = 0.01$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.010	0.010	0.010	0.010	0.010	0.010	0.008	0.010	0.008	0.008	0.010	0.010	0.010	0.008	0.008	0.010
0.80	0.010	0.010	0.010	0.010	0.010	0.010	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010
0.90	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010

$n = 100, \quad p = 4, \quad \sigma = 1$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	0.089	0.053	0.019	0.020	0.085	0.085	0.037	0.024	0.018	0.016	0.024	0.010	0.063	0.012	0.024	0.010
0.80	0.131	0.077	0.023	0.025	0.123	0.123	0.041	0.026	0.018	0.017	0.026	0.010	0.083	0.013	0.025	0.010
0.90	0.236	0.132	0.029	0.035	0.213	0.213	0.045	0.026	0.016	0.016	0.026	0.010	0.125	0.013	0.024	0.010

$n = 100, \quad p = 4, \quad \sigma = 5$

γ	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
0.70	1.966	1.067	0.213	0.245	1.879	1.879	0.727	0.440	0.253	0.199	0.440	0.016	1.365	0.103	0.423	0.016
0.80	3.065	1.679	0.278	0.330	2.869	2.869	0.816	0.458	0.219	0.180	0.458	0.015	1.889	0.084	0.409	0.015
0.90	5.757	3.139	0.420	0.599	5.195	5.195	0.913	0.432	0.165	0.158	0.432	0.015	2.967	0.068	0.369	0.015

Table A.3 Simulated MSE for fixed p, γ and σ and different values of n

$p=2,\ \gamma=0.7,\ \sigma=0.01$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.427	0.426	0.426	0.426	0.426	0.426	0.389	0.125	0.129	0.272	0.125	0.125	0.389	0.129	0.272	0.125
20	0.035	0.032	0.032	0.032	0.032	0.032	0.030	0.032	0.032	0.020	0.032	0.032	0.030	0.032	0.020	0.032
30	0.122	0.121	0.121	0.121	0.121	0.121	0.117	0.035	0.037	0.093	0.035	0.035	0.117	0.037	0.093	0.035
40	0.089	0.089	0.089	0.089	0.089	0.089	0.086	0.026	0.028	0.071	0.026	0.026	0.086	0.028	0.071	0.026
50	0.013	0.012	0.012	0.012	0.012	0.012	0.011	0.012	0.013	0.008	0.012	0.012	0.011	0.011	0.008	0.012
100	0.010	0.006	0.006	0.006	0.006	0.006	0.005	0.009	0.008	0.004	0.009	0.009	0.005	0.008	0.004	0.009

$p=2,\ \gamma=0.7,\ \sigma=0.5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.765	0.516	0.336	0.269	0.517	0.517	0.304	0.172	0.171	0.189	0.172	0.136	0.304	0.149	0.222	0.136
20	0.090	0.061	0.045	0.041	0.072	0.072	0.043	0.029	0.025	0.025	0.029	0.038	0.043	0.029	0.028	0.038
30	0.161	0.118	0.089	0.078	0.145	0.145	0.107	0.056	0.055	0.060	0.056	0.038	0.107	0.044	0.079	0.038
40	0.114	0.086	0.067	0.059	0.107	0.107	0.083	0.043	0.042	0.046	0.043	0.028	0.083	0.032	0.063	0.028
50	0.027	0.020	0.016	0.015	0.025	0.025	0.018	0.011	0.010	0.010	0.011	0.016	0.018	0.012	0.016	0.016
100	0.014	0.010	0.008	0.007	0.013	0.013	0.010	0.005	0.005	0.005	0.005	0.008	0.010	0.006	0.006	0.008

$p=2,\ \gamma=0.7,\ \sigma=1$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	1.799	1.051	0.438	0.297	1.042	1.042	0.479	0.270	0.223	0.215	0.270	0.143	0.479	0.163	0.291	0.143
20	0.264	0.152	0.080	0.064	0.209	0.209	0.119	0.059	0.047	0.043	0.059	0.040	0.119	0.035	0.068	0.040
30	0.279	0.171	0.098	0.080	0.240	0.240	0.158	0.083	0.073	0.066	0.083	0.039	0.158	0.047	0.104	0.039
40	0.194	0.124	0.073	0.059	0.177	0.177	0.126	0.066	0.058	0.051	0.066	0.029	0.126	0.035	0.084	0.029
50	0.076	0.046	0.027	0.023	0.071	0.071	0.049	0.023	0.020	0.017	0.023	0.016	0.049	0.014	0.030	0.016
100	0.039	0.023	0.013	0.011	0.038	0.038	0.028	0.011	0.009	0.008	0.011	0.008	0.028	0.007	0.017	0.008

$p=2,\ \gamma=0.7,\ \sigma=5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	36.39	18.75	3.898	1.701	18.36	18.36	6.025	2.733	1.446	1.204	2.733	0.317	6.025	0.547	2.373	0.317
20	5.529	2.823	1.010	0.645	4.299	4.299	2.393	1.198	0.853	0.524	0.198	0.116	2.393	0.246	1.293	0.116
30	4.417	2.277	0.757	0.465	3.654	3.654	2.041	0.795	0.553	0.402	0.795	0.072	2.041	0.160	1.022	0.072
40	2.685	1.378	0.534	0.358	2.373	2.373	1.470	0.587	0.426	0.295	0.587	0.049	1.470	0.114	0.786	0.049
50	1.561	0.783	0.323	0.231	1.444	1.444	0.994	0.434	0.329	0.204	0.434	0.029	0.994	0.075	0.578	0.029
100	0.812	0.412	0.163	0.113	0.779	0.779	0.572	0.221	0.165	0.102	0.221	0.012	0.572	0.031	0.326	0.012

$p=2,\ \gamma=0.8,\ \sigma=0.01$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.073	0.073	0.073	0.073	0.073	0.073	0.057	0.121	0.094	0.025	0.121	0.121	0.057	0.095	0.025	0.120
20	0.035	0.032	0.032	0.032	0.032	0.032	0.030	0.032	0.032	0.020	0.032	0.032	0.030	0.032	0.020	0.032
30	0.021	0.020	0.020	0.020	0.020	0.020	0.018	0.034	0.027	0.009	0.034	0.034	0.034	0.027	0.009	0.034
40	0.089	0.089	0.089	0.089	0.089	0.089	0.086	0.026	0.028	0.071	0.026	0.026	0.086	0.028	0.071	0.026
50	0.073	0.071	0.071	0.071	0.071	0.071	0.068	0.020	0.022	0.054	0.020	0.020	0.068	0.022	0.054	0.020
100	0.010	0.006	0.006	0.006	0.006	0.006	0.005	0.009	0.007	0.003	0.009	0.009	0.005	0.007	0.003	0.009

$p=2,\ \gamma=0.8,\ \sigma=0.5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.438	0.244	0.117	0.084	0.243	0.243	0.087	0.065	0.053	0.047	0.065	0.084	0.087	0.063	0.048	0.084
20	0.334	0.225	0.149	0.20	0.255	0.255	0.156	0.077	0.077	0.086	0.077	0.05	0.156	0.065	0.107	0.059
30	0.084	0.051	0.031	0.025	0.065	0.065	0.032	0.017	0.014	0.013	0.017	0.026	0.032	0.019	0.016	0.026
40	0.045	0.029	0.020	0.018	0.039	0.039	0.023	0.013	0.011	0.011	0.013	0.019	0.023	0.014	0.013	0.019
50	0.096	0.070	0.053	0.047	0.089	0.089	0.067	0.033	0.032	0.035	0.033	0.022	0.067	0.025	0.048	0.022
100	0.016	0.010	0.008	0.007	0.015	0.015	0.010	0.005	0.004	0.004	0.005	0.008	0.010	0.006	0.006	0.008

$p=2,\ \gamma=0.8,\ \sigma=1$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	1.608	0.866	0.254	0.133	0.840	0.840	0.271	0.121	0.084	0.082	0.121	0.089	0.271	0.075	0.115	0.089
20	0.776	0.444	0.182	0.127	0.535	0.535	0.265	0.117	0.099	0.098	0.117	0.061	0.265	0.071	0.146	0.061
30	0.273	0.143	0.053	0.037	0.207	0.207	0.099	0.038	0.027	0.025	0.038	0.027	0.099	0.043	0.027	0.099
40	0.140	0.076	0.036	0.027	0.120	0.120	0.068	0.028	0.021	0.019	0.028	0.020	0.068	0.016	0.034	0.020
50	0.172	0.107	0.061	0.049	0.156	0.156	0.106	0.050	0.043	0.039	0.050	0.022	0.106	0.026	0.067	0.022
100	0.048	0.027	0.013	0.011	0.045	0.045	0.031	0.010	0.008	0.008	0.010	0.008	0.031	0.006	0.016	0.008

$p=2,\ \gamma=0.8,\ \sigma=5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	38.59	20.17	4.069	1.577	20.19	20.19	6.265	2.712	1.384	1.138	2.712	0.271	6.265	0.477	3.325	0.271
20	13.77	6.66	1.312	0.619	8.96	8.96	3.500	1.265	0.703	0.521	1.265	0.113	3.500	0.217	1.280	0.113
30	6.236	3.051	0.788	0.430	4.653	4.653	2.168	0.762	0.469	0.329	0.762	0.058	2.168	0.124	0.898	0.058
40	3.133	1.549	0.480	0.294	2.660	2.660	1.490	0.559	0.371	0.252	0.559	0.038	1.490	0.08	0.711	0.038
50	2.381	1.184	0.388	0.250	2.099	2.099	1.246	0.467	0.322	0.217	0.467	0.035	1.246	0.081	0.616	0.035
100	1.026	0.518	0.178	0.114	0.968	0.968	0.653	0.212	0.150	0.096	0.212	0.011	0.653	0.028	0.033	0.011

$p=2,\ \gamma=0.9,\ \sigma=0.01$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.073	0.073	0.073	0.073	0.073	0.073	0.052	0.120	0.094	0.020	0.120	0.052	0.094	0.020	0.121	0.052.
20	0.032	0.032	0.032	0.032	0.032	0.032	0.025	0.032	0.032	0.011	0.032	0.032	0.025	0.032	0.011	0.032
30	0.021	0.020	0.020	0.020	0.020	0.020	0.016	0.034	0.027	0.007	0.034	0.034	0.016	0.027	0.007	0.034
40	0.090	0.089	0.089	0.089	0.089	0.089	0.079	0.026	0.027	0.053	0.026	0.026	0.079	0.027	0.054	0.026
50	0.013	0.012	0.012	0.012	0.012	0.012	0.009	0.020	0.015	0.004	0.012	0.012	0.009	0.015	0.004	0.012
100	0.036	0.034	0.034	0.034	0.034	0.034	0.032	0.010	0.010	0.023	0.010	0.010	0.032	0.010	0.023	0.010

$p=2,\ \gamma=0.9,\ \sigma=0.5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.618	0.329	0.125	0.076	0.313	0.313	0.087	0.063	0.051	0.043	0.063	0.085	0.087	0.063	0.042	0.085
20	0.190	0.104	0.049	0.035	0.121	0.121	0.047	0.028	0.023	0.020	0.028	0.040	0.047	0.029	0.022	0.040
30	0.123	0.066	0.030	0.022	0.087	0.087	0.035	0.017	0.014	0.013	0.017	0.027	0.035	0.019	0.015	0.027
40	0.195	0.125	0.070	0.053	0.151	0.151	0.084	0.035	0.035	0.039	0.035	0.027	0.084	0.029	0.050	0.027
50	0.072	0.039	0.017	0.012	0.054	0.054	0.023	0.010	0.008	0.007	0.010	0.016	0.023	0.012	0.009	0.016
100	0.058	0.039	0.026	0.021	0.053	0.053	0.036	0.014	0.014	0.016	0.014	0.010	0.036	0.011	0.023	0.010

$p=2,\ \gamma=0.9,\ \sigma=1$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	2.445	1.267	0.296	0.137	1.191	1.191	0.301	0.133	0.195	0.081	0.133	0.094	0.301	0.074	0.109	0.090
20	0.666	0.340	0.094	0.056	0.419	0.419	0.155	0.050	0.037	0.037	0.050	0.042	0.155	0.034	0.058	0.042
30	0.449	0.230	0.061	0.034	0.304	0.304	0.116	0.034	0.023	0.023	0.034	0.027	0.116	0.022	0.039	0.027
40	0.499	0.275	0.093	0.057	0.358	0.358	0.160	0.052	0.044	0.045	0.052	0.027	0.160	0.031	0.071	0.027
50	0.248	0.123	0.033	0.019	0.189	0.189	0.079	0.020	0.013	0.013	0.020	0.016	0.079	0.012	0.025	0.016
100	0.118	0.067	0.029	0.021	0.106	0.106	0.064	0.021	0.018	0.017	0.021	0.010	0.064	0.012	0.032	0.010

$p=2,\ \gamma=0.9,\ \sigma=5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	61.65	31.81	4.630	1.542	30.49	30.49	7.311	2.890	1.224	1.222	2.890	0.256	7.311	0.438	2.203	0.256
20	15.76	7.692	1.441	0.599	9.601	9.601	3.440	1.303	0.635	0.457	1.303	0.093	3.440	0.176	1.141	0.093
30	10.65	5.177	0.686	0.382	7.235	7.235	2.718	0.754	0.385	0.301	0.754	0.052	2.718	0.101	0.826	0.052
40	9.842	4.804	0.761	0.301	6.910	6.910	2.565	0.538	0.288	0.256	0.538	0.042	2.565	0.085	0.698	0.042
50	6.132	2.978	0.464	0.199	4.572	4.572	1.858	0.452	0.225	0.169	0.452	0.026	1.858	0.052	0.539	0.026
100	2.115	1.005	0.222	0.118	1.861	1.861	0.960	0.226	0.132	0.099	0.226	0.013	0.960	0.029	0.013	0.013

$p=4,\ \gamma=0.7,\ \sigma=0.01$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.184	0.183	0.183	0.183	0.183	0.183	0.071	0.159	0.103	0.063	0.159	0.159	0.133	0.103	0.063	0.159
20	0.063	0.063	0.063	0.063	0.063	0.063	0.060	0.060	0.047	0.054	0.060	0.060	0.062	0.047	0.054	0.060
30	0.379	0.378	0.378	0.378	0.378	0.378	0.031	0.037	0.033	0.030	0.037	0.037	0.036	0.033	0.030	0.037
40	0.025	0.025	0.025	0.025	0.025	0.025	0.015	0.025	0.016	0.012	0.025	0.025	0.022	0.016	0.012	0.025
50	0.023	0.022	0.022	0.022	0.022	0.022	0.018	0.021	0.016	0.016	0.021	0.021	0.022	0.016	0.016	0.021
100	0.010	0.010	0.010	0.010	0.010	0.010	0.008	0.010	0.008	0.008	0.010	0.010	0.010	0.008	0.008	0.010

$p=4,\ \gamma=0.7,\ \sigma=0.5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	1.482	0.847	0.190	0.171	0.664	0.664	0.133	0.140	0.104	0.105	0.140	0.124	0.234	0.103	0.111	0.124
20	0.359	0.223	0.082	0.079	0.221	0.221	0.081	0.067	0.059	0.060	0.067	0.052	0.117	0.053	0.068	0.052
30	0.143	0.097	0.051	0.051	0.123	0.123	0.052	0.043	0.038	0.039	0.043	0.035	0.084	0.036	0.043	0.035
40	0.089	0.057	0.027	0.027	0.078	0.078	0.028	0.022	0.018	0.018	0.022	0.021	0.049	0.016	0.021	0.021
50	0.071	0.049	0.026	0.027	0.066	0.066	0.030	0.023	0.020	0.020	0.023	0.019	0.047	0.018	0.024	0.019
100	0.030	0.021	0.012	0.012	0.029	0.029	0.015	0.011	0.010	0.010	0.011	0.009	0.023	0.009	0.012	0.009

$p = 4, \ \gamma = 0.7, \ \sigma = 1$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	5.564	3.088	0.392	0.359	2.278	2.278	0.329	0.327	0.185	0.188	0.327	0.139	0.701	0.151	0.233	0.139
20	1.185	0.646	0.121	0.114	0.673	0.673	0.160	0.145	0.097	0.093	0.145	0.057	0.288	0.075	0.121	0.057
30	0.455	0.266	0.083	0.087	0.383	0.383	0.119	0.091	0.063	0.060	0.091	0.037	0.225	0.048	0.083	0.037
40	0.291	0.167	0.047	0.050	0.253	0.253	0.077	0.055	0.036	0.033	0.055	0.022	0.150	0.025	0.050	0.022
50	0.210	0.124	0.040	0.043	0.191	0.191	0.069	0.047	0.034	0.031	0.047	0.019	0.126	0.024	0.046	0.019
100	0.089	0.053	0.019	0.020	0.085	0.085	0.037	0.024	0.018	0.016	0.024	0.010	0.063	0.012	0.024	0.010

$p = 4, \ \gamma = 0.7, \ \sigma = 5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	129.0	70.10	6.667	6.191	50.63	50.63	6.225	6.286	2.659	2.616	6.286	0.568	14.72	1.587	3.912	0.568
20	28.52	14.95	1.624	1.495	15.52	15.52	2.851	2.599	1.390	1.208	2.599	0.209	5.958	0.785	1.921	0.209
30	10.56	5.809	1.060	1.178	8.725	8.725	2.214	1.668	0.849	0.709	1.668	0.096	4.821	0.410	1.304	0.096
40	6.643	3.564	0.639	0.723	5.760	5.760	1.634	1.154	0.629	0.518	1.154	0.056	3.380	0.292	0.978	0.056
50	4.868	2.686	0.489	0.552	4.413	4.413	1.386	0.969	0.517	0.408	0.969	0.042	2.817	0.225	0.810	0.042
100	1.966	1.067	0.213	0.245	1.879	1.879	0.727	0.440	0.253	0.199	0.440	0.016	1.365	0.103	0.423	0.016

$p = 4, \ \gamma = 0.8, \ \sigma = 0.01$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.170	0.169	0.169	0.169	0.169	0.169	0.192	0.169	0.190	0.193	0.169	0.169	0.193	0.190	0.193	0.169
20	0.064	0.064	0.064	0.064	0.064	0.064	0.064	0.064	0.063	0.064	0.064	0.063	0.063	0.063	0.063	0.063
30	0.040	0.039	0.039	0.039	0.039	0.039	0.036	0.038	0.037	0.038	0.038	0.038	0.038	0.038	0.037	0.038
40	0.026	0.026	0.026	0.026	0.026	0.026	0.016	0.026	0.017	0.013	0.026	0.026	0.023	0.017	0.013	0.026
50	0.021	0.020	0.020	0.020	0.020	0.020	0.020	0.021	0.021	0.021	0.021	0.021	0.020	0.021	0.021	0.020
100	0.010	0.010	0.010	0.010	0.010	0.010	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010

$p = 4, \ \gamma = 0.8, \ \sigma = 0.5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	12.04	6.365	0.341	0.289	3.308	3.308	0.264	0.217	0.213	0.223	0.217	0.183	0.409	0.205	0.234	0.183
20	0.367	0.241	0.111	0.110	0.280	0.280	0.102	0.087	0.080	0.082	0.087	0.067	0.154	0.076	0.088	0.067
30	0.184	0.121	0.059	0.059	0.154	0.154	0.059	0.047	0.044	0.044	0.047	0.038	0.094	0.041	0.049	0.038
40	0.113	0.071	0.029	0.029	0.097	0.097	0.030	0.023	0.018	0.019	0.023	0.024	0.058	0.017	0.022	0.022
50	0.112	0.073	0.035	0.035	0.098	0.098	0.037	0.029	0.028	0.028	0.029	0.023	0.061	0.026	0.031	0.023
100	0.040	0.027	0.015	0.015	0.038	0.038	0.018	0.013	0.012	0.012	0.013	0.010	0.028	0.011	0.014	0.010

$p = 4, \ \gamma = 0.8, \ \sigma = 1$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	47.61	24.49	0.661	0.495	11.43	11.43	0.485	0.384	0.514	0.283	0.306	0.384	0.196	0.247	0.359	0.196
20	1.295	0.756	0.182	0.196	0.936	0.936	0.198	0.154	0.109	0.110	0.154	0.069	0.419	0.090	0.139	0.069
30	0.653	0.385	0.101	0.109	0.532	0.532	0.129	0.093	0.064	0.063	0.093	0.040	0.282	0.050	0.085	0.040
40	0.378	0.215	0.055	0.059	0.322	0.322	0.083	0.057	0.035	0.033	0.057	0.023	0.182	0.024	0.050	0.023
50	0.374	0.214	0.056	0.059	0.319	0.319	0.078	0.053	0.037	0.037	0.053	0.023	0.176	0.030	0.050	0.023
100	0.131	0.077	0.023	0.025	0.123	0.123	0.041	0.026	0.018	0.017	0.026	0.010	0.083	0.013	0.025	0.010

$p = 4, \ \gamma = 0.8, \ \sigma = 5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	1190	613	10.40	6.984	292.0	292.0	7.455	5.690	2.463	2.868	5.690	0.582	21.58	1.507	4.237	0.582
20	30.56	16.70	2.512	3.022	21.69	21.69	3.336	2.627	1.147	1.041	2.627	0.169	8.985	0.588	1.850	0.169
30	14.29	7.724	1.260	1.633	11.44	11.44	2.190	1.643	0.703	0.606	1.643	0.083	5.743	0.328	1.159	0.083
40	8.662	4.724	0.760	0.890	7.315	7.315	1.701	1.283	0.579	0.473	1.283	0.056	4.022	0.257	0.933	0.056
50	9.112	4.978	0.698	0.911	7.735	7.735	1.481	0.924	0.394	0.356	0.924	0.040	4.065	0.176	0.730	0.040
100	3.065	1.679	0.278	0.330	2.869	2.869	0.816	0.458	0.219	0.180	0.458	0.015	1.889	0.084	0.409	0.015

$p = 4, \ \gamma = 0.9, \ \sigma = 0.01$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.038	0.038	0.038	0.038	0.038	0.038	0.032	0.037	0.033	0.032	0.037	0.037	0.035	0.033	0.032	0.037
20	0.063	0.062	0.062	0.062	0.062	0.062	0.049	0.061	0.057	0.022	0.061	0.061	0.054	0.057	0.052	0.061
30	0.038	0.038	0.038	0.038	0.038	0.038	0.032	0.037	0.033	0.032	0.037	0.037	0.035	0.033	0.032	0.037
40	0.028	0.028	0.028	0.028	0.028	0.028	0.024	0.027	0.024	0.023	0.027	0.027	0.026	0.024	0.024	0.027
50	0.023	0.022	0.022	0.022	0.022	0.022	0.015	0.021	0.017	0.015	0.021	0.021	0.019	0.017	0.015	0.021
100	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010

$p = 4,\ \ \gamma = 0.9,\ \ \sigma = 0.5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	0.308	0.184	0.065	0.066	0.247	0.247	0.053	0.040	0.036	0.037	0.040	0.035	0.118	0.035	0.041	0.035
20	0.579	0.344	0.112	0.113	0.419	0.419	0.085	0.070	0.063	0.064	0.070	0.059	0.171	0.060	0.070	0.059
30	0.308	0.184	0.065	0.066	0.247	0.247	0.053	0.040	0.036	0.037	0.040	0.035	0.118	0.035	0.041	0.035
40	0.171	0.105	0.043	0.044	0.144	0.144	0.042	0.030	0.027	0.028	0.030	0.026	0.080	0.025	0.032	0.026
50	0.166	0.099	0.032	0.033	0.138	0.138	0.030	0.021	0.019	0.019	0.021	0.020	0.071	0.018	0.022	0.020
100	0.069	0.043	0.018	0.018	0.063	0.063	0.019	0.013	0.012	0.012	0.013	0.010	0.040	0.011	0.014	0.010

$p = 4,\ \ \gamma = 0.9,\ \ \sigma = 1$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	1.149	0.651	0.123	0.147	0.900	0.900	0.121	0.091	0.051	0.051	0.091	0.036	0.383	0.041	0.070	0.036
20	2.099	1.170	0.210	0.248	1.500	1.500	0.179	0.136	0.083	0.087	0.136	0.061	0.530	0.070	0.112	0.061
30	1.149	0.651	0.123	0.147	0.900	0.900	0.121	0.091	0.051	0.051	0.091	0.036	0.383	0.041	0.070	0.036
40	0.600	0.336	0.073	0.084	0.490	0.490	0.092	0.067	0.039	0.039	0.067	0.026	0.241	0.030	0.055	0.026
50	0.600	0.335	0.063	0.071	0.494	0.494	0.074	0.045	0.027	0.028	0.045	0.020	0.235	0.022	0.040	0.020
100	0.236	0.132	0.029	0.035	0.213	0.213	0.045	0.026	0.016	0.016	0.026	0.010	0.125	0.013	0.024	0.010

$p = 4,\ \ \gamma = 0.9,\ \ \sigma = 5$

n	OLS	HK	K1	K2	S3	S4	KM1	KM2	KM4	KM5	KM6	KM8	KM9	KM 10	KM 11	KM 12
10	27.93	15.14	1.876	2.760	21.46	21.46	2.269	1.690	0.509	0.507	1.690	0.068	8.729	0.246	0.987	0.068
20	51.09	27.72	3.112	35.62	35.62	3.155	2.742	8.514	0.835	0.820	2.742	0.129	11.74	0.416	1.478	0.129
30	27.93	15.14	1.876	2.760	21.46	21.46	2.269	1.690	0.509	0.507	1.690	0.068	8.729	0.246	0.987	0.068
40	9.842	4.804	0.761	0.301	6.910	6.910	2.565	0.538	0.288	0.256	0.538	0.042	2.565	0.085	0.698	0.042
50	14.84	8.068	0.999	1.504	12.04	12.04	1.516	1.023	0.314	0.301	1.023	0.035	5.570	0.139	0.635	0.035
100	5.757	3.139	0.420	0.599	5.195	5.195	0.913	0.432	0.165	0.158	0.432	0.015	2.967	0.068	0.369	0.015