9-30-2011

Inference about Reliability Parameter with Underlying Gamma and Exponential Distribution

Zeyi Wang
Florida International University, wzyznn@gmail.com

FLORIDA INTERNATIONAL UNIVERSITY

Miami, Florida

INFERENCE ABOUT RELIABILITY PARAMETER WITH UNDERLYING GAMMA AND EXPONENTIAL DISTRIBUTION

A thesis submitted in partial fulfillment of the

requirements for the degree of

MASTER OF SCIENCE

in

STATISTICS

by

Zeyi Wang

2011

To: Dean Kenneth Furton
College of Arts and Sciences

This thesis, written by Zeyi Wang, and entitled Inference about Reliability Parameter with underlying Gamma and Exponential Distribution, having been approved in respect to style and intellectual content, is referred to you for judgment.

We have read this thesis and recommend that it be approved.

Florence George

Kai Huang

Jie Mi, Major Professor

Date of Defense: September 30, 2011

The thesis of Zeyi Wang is approved.

Dean Kenneth Furton
College of Arts and Sciences

Dean Lakshmi N. Reddi
University Graduate School

Florida International University, 2011

ACKNOWLEDGMENTS

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. I would like to thank Dr. Florence George and Dr. Kai Huang for their time to serve on my committee. I really appreciate their suggestions and encouragement.

I am deeply indebted to my supervisor professor Dr. Jie Mi who helped me in all the time of research for and writing of this thesis. It is a pleasure to express my sincere gratitude for his patience, motivation, enthusiasm, and immense knowledge. His encouragement, guidance and support from the initial to the final level enabled me to complete the thesis.

I would like to give a special thanks to all other professors in the Department of Mathematics and Statistics who supported and encouraged me throughout my years in FIU.

Finally, I wish to thank my family and friends for their unflagging spiritual support and their absolute confidence in me.

ABSTRACT OF THE THESIS

INFERENCE ABOUT RELIABILITY PARAMETER WITH UNDERLYING GAMMA AND EXPONENTIAL DISTRIBUTION

by

Zeyi Wang

Florida International University, 2011

Miami, Florida

Professor Jie Mi, Major Professor

The statistical inference about the reliability parameter R involving independent gamma stress and exponential strength is considered. Assuming the shape parameter of gamma is a known arbitrary real number and the scale parameters of gamma and exponential are unknown, the UMVUE and MLE of R are obtained. A pivot is proposed. Some inference about R derived from this pivot is presented. It will be shown that the pivot can be used for testing hypothesis and constructing confidence interval. A procedure of constructing the confidence interval for R is derived. The performances of the UMVUE and MLE are compared numerically based on extensive Monte Carlo simulation. Simulation studies indicate that the performance of the two estimators is about the same. The MLE is preferred because of the simplicity of its computation.

Keywords: Stress-Strength, Exponential Distribution, Gamma Distribution, UMVUE, MLE, MSE, Pivot, Bias.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1. Introduction

Extensive research has been conducted on the stress-strength model. This model involves two independent random variables X and Y, and the parameter of interest is the probability $R \equiv P(X \geq Y)$. This function has attracted a great deal of attention in the literature because of its wide applications. Naturally, from the engineering point of view X and Y can represent the strength of a structure and the stress imposed on it. With this interpretation the probablility R is often called the reliability parameter which will be used in this article. In addition to its applications in engineering, stress-strength model is also applied in many other fields. For example, in a medical application let X represent the response for a control group and Y represent the response for a treatment group (Simonoff, Hochberg, and Reiser (1986); Hauck, Hyslop, and Anderson (2000)). In this case the probability $P(X \geq Y)$ measures the effect of the treatment. The probability $R \equiv P(X \geq Y)$ can also be used for bioequivalence assessment (Wellek (1993)). It is frequently used to assess the effectiveness of diagnostic markers in distinguishing between diseased and healthy individuals (Reiser (2000)). In biology, this probability is useful in estimating heritability of a generic trait (Schwarz and Wearen (1959)).

An extensive review of this model is presented in Kotz, Lumelsdii, and Pensky (2003). The most recent work on the topic was published by Saracoglu and Kaya (2007), Krishnamoorthy, Mukherjee, and Guo (2007), Baklizi (2008a, b), Eryilmaz (2008a, b, c; 2010), Kundu and Raqab (2009), Rezaei, Tahmasbi, and Mahmoodi (2010) and the references therein.

In the literature, various distributions of X and Y are considered which include exponential, normal, Weibull, etc. Krishnamoorthy et al. (2007) gave the test and interval estimation procedures derived from the generalized variable approach for two-parameter exponential case when both the location and scale parameters are unknown. Baklizi (2008a, b) derived estimators and confidence intervals by considering both one and two-parameter exponential cases with record values and lower record values. Constantine, Karson, and Tse (1986) considered the case when both X and Y have gamma distributions. It was assumed there that the two scale parameters are unknown but the two shape parameters are known integers. Under these assumptions they derived UMVUE and MLE of R and obtained exact interval estimation of R.

In this article we will study the same model while X has exponential distribution and Y has gamma distribution. It is assumed that the shape parameter of the gamma distribution is known, however it can be any positive real number and is not restricted to be an integer. The thesis is organized as follows. The MLE and UMVUE of R are derived in Section 2. A pivotal quantity is presented in Section 3. Some inference about R derived from this pivot will also be shown in Section 3. It has been discovered that the pivot can be used for testing hypothesis, and the rejection region are obtained accordingly. A procedure of constructing the confidence intervals for R is derived. In section 4, I present results of numerical studies based on Monte Carlo simulations. It is observed numerically that even though the MLE of R is biased, it is superior to the UMVUE for its MSE is about the same as that of the UMVUE, and is actually smaller most the time, and furthermore its computation is much easier.

2. Point Estimation of Reliability Parameter

Let the random strength X have exponential distribution with probability density function

$$f_X(x) = \lambda e^{-\lambda x}, x > 0$$

and the random stress Y have gamma distribution with probability density function

$$f_Y(y) = \frac{\tau^\gamma}{\Gamma(\gamma)} y^{\gamma-1} e^{-\tau y}, y > 0$$

where the shape parameter γ is known but the scale parameters λ and τ are unknown.

Our quantity of interest is the reliability parameter R defined as

$$R = P(X > Y).$$

Suppose that $\mathbf{X} = \{X_1, ..., X_m\}$ is a simple random sample from the strength population and $\mathbf{Y} = \{Y_1, ..., Y_n\}$ is a simple random sample from the stress population, where m and n do not have to be the same.

In this section we will first derive the uniformly minimum variance unbiased estimator (UMVUE) of R. Then we will derive the maximum likelihood estimator (MLE) of R.

Theorem 1. The UMVUE of R based on sample $\{X_1, ..., X_m\}$ and $\{Y_1, ..., Y_n\}$ is given by

$$\tilde{R} = \begin{cases} \int\limits_0^1 (m-1)(1-s)^{m-2}\Big(\int\limits_0^{us/v} \frac{1}{B\left(\gamma,(n-1)\gamma\right)} \cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds, & \\ \qquad \text{if } u = \sum\limits_{i=1}^m x_i \leq \sum\limits_{j=1}^n y_j = v & \\ \int\limits_0^{v/u} (m-1)(1-s)^{m-2}\Big(\int\limits_0^{us/v} \frac{1}{B\left(\gamma,(n-1)\gamma\right)} t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds & \\ \quad + (1-\frac{v}{u})^{m-1}, \qquad \text{if } u = \sum\limits_{i=1}^m x_i > \sum\limits_{j=1}^n y_j = v. & \end{cases}$$

Proof. Obviously the indicator function

$$I(X_1 > Y_1) = \begin{cases} 1, & \text{if } X_1 > Y_1, \\ 0, & \text{if } X_1 \leq Y_1. \end{cases}$$

is an unbiased estimator of R. Moreover, (U, V) is the sufficient and complete statistic, where $U = \sum\limits_{i=1}^m X_i$ and $V = \sum\limits_{j=1}^n Y_j$. Hence, according to the Blackwell-Rao and Lehmann-Scheffe Theorem $E\big(I(X_1 > Y_1)\big|U,V\big) = P(X_1 > Y_1\big|U,V)$ is the unique UMVUE of R.

In order to find $P(X_1 > Y_1\big|U,V)$ we need to derive the conditional distribution of X_1 given $U = u$ and the conditional distribution of Y_1 given $V = v$. By using the routine way of multivariate change of variables it can be shown that the conditional probability density function (pdf) of X_1 given $U = u$ is

$$f_{X_1|U}(x|u) = \frac{m-1}{u}(1-\frac{x}{u})^{m-2}, \ 0 < x < u < \infty;$$

and the conditional pdf of Y_1 given $V = v$ is

$$f_{Y_1|V}(y|v) \quad = \quad \frac{\Gamma(n\gamma)}{\Gamma(\gamma)\Gamma\big((n-1)\gamma\big)} \cdot \frac{y^{\gamma-1}}{v^{\gamma}}(1-\frac{y}{v})^{(n-1)\gamma-1}$$

$$= \quad \frac{1}{B\big(\gamma,(n-1)\gamma\big)} \cdot \frac{y^{\gamma-1}}{v^{\gamma}}(1-\frac{y}{v})^{(n-1)\gamma-1}, \ 0<y<v<\infty.$$

Here $B(a,b)$ is the beta function, i.e.,

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

.

Thus, the desired UMVUE $\tilde{R}$ is given by

$$\begin{aligned}
\tilde{R} &= P(X_1 > Y_1 \big| U=u, V=v) \\
&= \int_0^\infty \int_0^\infty I(x>y) f_{X_1|U}(x|u) f_{Y_1|V}(y|v) dx dy \\
&= \int_0^\infty \int_0^\infty \frac{I(y<v)}{B\big(\gamma,(n-1)\gamma\big)} \cdot \frac{y^{\gamma-1}}{v^{\gamma}}(1-\frac{y}{v})^{(n-1)\gamma-1} \cdot \frac{(m-1)I(x<u)I(y<x)}{u} \\
&\quad \cdot (1-\frac{x}{u})^{m-2} dx dy \\
&= \int_0^\infty \int_0^\infty \frac{(m-1)I(x<u)}{u}(1-\frac{x}{u})^{m-2} \cdot \frac{I\big(y<min\{x,\ v\}\big)}{B\big(\gamma,(n-1)\gamma\big)} \cdot \frac{y^{\gamma-1}}{v^{\gamma}}(1-\frac{y}{v})^{(n-1)\gamma-1} \\
&\quad dx dy \\
&= \int_0^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Bigg(\int_0^{min\{x,\ v\}} \frac{1}{B\big(\gamma,(n-1)\gamma\big)} \cdot \frac{y^{\gamma-1}}{v^{\gamma}}(1-\frac{y}{v})^{(n-1)\gamma-1} dy\Bigg) dx \\
&= \int_0^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Bigg(\int_0^{min\{x,\ v\}} \frac{1}{B\big(\gamma,(n-1)\gamma\big)} \cdot (\frac{y}{v})^{\gamma-1}(1-\frac{y}{v})^{(n-1)\gamma-1} d\frac{y}{v}\Bigg) \\
&\quad dx. \qquad (1)
\end{aligned}$$

In the following let us consider two cases:

Case 1: $u \le v$.

In this case $x \le u$ implies $x \le u \le v$ and $min(x,\ v) = x$. Let $t = y/v$ and

$s = x/u$, then from (1)

$$\begin{aligned}
\tilde{R} &= \int_0^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^x \frac{1}{B\big(\gamma,(n-1)\gamma\big)}(\frac{y}{v})^{\gamma-1}(1-\frac{y}{v})^{(n-1)\gamma-1}d\frac{y}{v}\Big)dx \\
&= \int_0^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^{x/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)dx \\
&= \int_0^1 (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds. \qquad (2)
\end{aligned}$$

Case 2: $\quad u > v$.

We express the integral in (1) as the sum of two other integrals as follows

$$\tilde{R} = \int_0^v + \int_v^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^{min\{x,\ v\}} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot(\frac{y}{v})^{\gamma-1}(1-\frac{y}{v})^{(n-1)\gamma-1}d\frac{y}{v}\Big)dx$$

Note that on the interval $(0,\ v)$ we have $x \leq v$ and so $min\{x,\ v\} = x$, and on the interval $(v,\ u)$ it holds that $x \geq v$ and consequently $min\{x,\ v\} = v$. From these observations we see that

$$\begin{aligned}
\tilde{R} &= \int_0^v \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^x \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot(\frac{y}{v})^{\gamma-1}(1-\frac{y}{v})^{(n-1)\gamma-1}d\frac{y}{v}\Big)dx \\
&\quad + \int_v^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^v \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot(\frac{y}{v})^{\gamma-1}(1-\frac{y}{v})^{(n-1)\gamma-1}d\frac{y}{v}\Big)dx \\
&= \int_0^v \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^{x/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)dx \\
&\quad + \int_v^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}\Big(\int_0^1 \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{v/u} (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds \\
&\quad + \int_v^u \frac{m-1}{u}(1-\frac{x}{u})^{m-2}dx \\
&= \int_0^{v/u} (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds \\
&\quad + \int_{v/u}^1 (m-1)(1-s)^{m-2}ds \\
&= \int_0^{v/u} (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)}\cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds \\
&\quad +(1-\frac{v}{u})^{m-1}. \qquad (3)
\end{aligned}
$$

Finally, from (2) and (3) we obtain the desired result. ■

The expression of $\tilde{R}$ given in Theorem 1 is complicated because the known shape parameter γ can be any positive real number. However, in the case when γ is an integer, then simple expression for $\tilde{R}$ can be obtained as follows.

Theorem 2. Suppose that the shape parameter γ is a known integer. Then it follows that

$$
\tilde{R} = \begin{cases}
\frac{m-1}{B\left(\gamma,(n-1)\gamma\right)} \sum_{j=0}^{(n-1)\gamma-1} \binom{(n-1)\gamma-1}{j} \frac{(-1)^{(n-1)\gamma-1-j} B\left(n\gamma-j,\ m-1\right)}{n\gamma-j-1} & \\
\cdot(\frac{u}{v})^{n\gamma-j-1}, & if\ u=\sum_{i=1}^m x_i \le \sum_{j=1}^n y_j = v \\
\frac{m-1}{B\left(\gamma,(n-1)\gamma\right)} \sum_{i=0}^{m-2}\sum_{j=0}^{(n-1)\gamma-1} \binom{m-2}{i}\binom{(n-1)\gamma-1}{j} \frac{(-1)^{(n-1)\gamma-j+m-i-3}}{(n\gamma-j-1)(n\gamma-j+m-i-2)} & \\
\cdot(\frac{v}{u})^{m-i-1} + (1-\frac{v}{u})^{m-1}, & if\ u=\sum_{i=1}^m x_i > \sum_{j=1}^n y_j = v
\end{cases}
$$

Proof. For the case of $u = \sum_{i=1}^{m} X_i \leq \sum_{j=1}^{n} Y_j = v$, according to Theorem 1 the UMVUE $\tilde{R}$ is given as

$$\begin{aligned}
\tilde{R} &= \int_0^1 (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)} \cdot t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)ds \\
&= \int_0^1 (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B\big(\gamma,(n-1)\gamma\big)} \cdot t^{\gamma-1}\sum_{j=0}^{(n-1)\gamma-1}\binom{(n-1)\gamma-1}{j} \\
&\quad (-t)^{(n-1)\gamma-1-j}dt\Big)ds \\
&= \int_0^1 \frac{(m-1)(1-s)^{m-2}}{B\big(\gamma,(n-1)\gamma\big)}\sum_{j=0}^{(n-1)\gamma-1}\binom{(n-1)\gamma-1}{j}(-1)^{(n-1)\gamma-1-j} \\
&\quad \cdot\Big(\int_0^{us/v} t^{n\gamma-j-2}dt\Big)ds \\
&= \int_0^1 \frac{(m-1)(1-s)^{m-2}}{B\big(\gamma,(n-1)\gamma\big)}\sum_{j=0}^{(n-1)\gamma-1}(-1)^{(n-1)\gamma-1-j}\binom{(n-1)\gamma-1}{j}\frac{1}{n\gamma-j-1} \\
&\quad \cdot(\frac{us}{v})^{n\gamma-j-1}ds \\
&= \frac{m-1}{B\big(\gamma,(n-1)\gamma\big)}\sum_{j=0}^{(n-1)\gamma-1}\binom{(n-1)\gamma-1}{j}\frac{(-1)^{(n-1)\gamma-1-j}}{n\gamma-j-1}(\frac{u}{v})^{n\gamma-j-1} \\
&\quad \cdot\int_0^1 s^{n\gamma-j-1}(1-s)^{m-2}ds \\
&= \frac{m-1}{B\big(\gamma,(n-1)\gamma\big)}\sum_{j=0}^{(n-1)\gamma-1}\binom{(n-1)\gamma-1}{j}\frac{(-1)^{(n-1)\gamma-1-j}B\big(n\gamma-j,\ m-1\big)}{n\gamma-j-1} \\
&\quad \cdot(\frac{u}{v})^{n\gamma-j-1}
\end{aligned}$$

On the other hand, if $u > v$, then

$$\begin{aligned}
\tilde{R} &= \int_0^{v/u} (m-1)(1-s)^{m-2}\Big(\int_0^{us/v} \frac{1}{B(\gamma,(n-1)\gamma)} t^{\gamma-1}(1-t)^{(n-1)\gamma-1}dt\Big)dt \\
&\quad +(1-\frac{v}{u})^{m-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{m-1}{B\big(\gamma,(n-1)\gamma\big)} \sum_{j=0}^{(n-1)\gamma-1} \binom{(n-1)\gamma-1}{j} \frac{(-1)^{(n-1)\gamma-j-1}}{n\gamma-j-1} (\frac{u}{v})^{n\gamma-j-1} \\
&\quad \cdot \int_0^{v/u} s^{n\gamma-j-1}(1-s)^{m-2}ds + (1-\frac{v}{u})^{m-1} \\
&= \frac{m-1}{B\big(\gamma,(n-1)\gamma\big)} \sum_{j=0}^{(n-1)\gamma-1} \binom{(n-1)\gamma-1}{j} \frac{(-1)^{(n-1)\gamma-j-1}}{n\gamma-j-1} (\frac{u}{v})^{n\gamma-j-1} \\
&\quad \cdot \sum_{i=0}^{m-2} \binom{m-2}{i} (-1)^{m-2-i} \int_0^{v/u} s^{n\gamma-j+m-i-3}ds + (1-\frac{v}{u})^{m-1} \\
&= \frac{m-1}{B\big(\gamma,(n-1)\gamma\big)} \sum_{j=0}^{(n-1)\gamma-1} \binom{(n-1)\gamma-1}{j} \frac{(-1)^{(n-1)\gamma-j-1}}{n\gamma-j-1} (\frac{u}{v})^{n\gamma-j-1} \\
&\quad \cdot \sum_{i=0}^{m-2} \binom{m-2}{i} \frac{(-1)^{m-2-i}}{n\gamma-j+m-i-2} (\frac{v}{u})^{n\gamma-j+m-i-2} + (1-\frac{v}{u})^{m-1} \\
&= \frac{m-1}{B\big(\gamma,(n-1)\gamma\big)} \sum_{i=0}^{m-2} \sum_{j=0}^{(n-1)\gamma-1} \binom{m-2}{i} \binom{(n-1)\gamma-1}{j} \\
&\quad \cdot \frac{(-1)^{(n-1)\gamma-j+m-i-3}}{(n\gamma-j-1)(n\gamma-j+m-i-2)} (\frac{v}{u})^{m-i-1} + (1-\frac{v}{u})^{m-1}
\end{aligned}
$$

Therefore, the desired result follows. ■

To derive the MLE $\hat{R}$ of R we need to derive the closed-form formula of R.

Theorem 3. The reliability parameter R is given as

$$R = (\frac{\tau}{\lambda+\tau})^{\gamma}.$$

Proof. The reliability parameter R is given by $R = P(X > Y)$. By conditioning on Y we have

$$R = \int_0^{\infty} P(X > y) f_Y(y) dy$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda y} \cdot \frac{\tau^\gamma}{\Gamma(\gamma)} y^{\gamma-1} e^{-\tau y} dy \\
&= \tau^\gamma \int_0^\infty \frac{1}{\Gamma(\gamma)} y^{\gamma-1} e^{-(\lambda+\tau)y} dy \\
&= \frac{\tau^\gamma}{(\lambda+\tau)^\gamma} \int_0^\infty \frac{(\lambda+\tau)^\gamma}{\Gamma(\gamma)} y^{\gamma-1} e^{-(\lambda+\tau)y} dy \\
&= (\frac{\tau}{\lambda+\tau})^\gamma
\end{aligned}
$$

So, the result follows. ■

With the help of Theorem 3 the MLE of R is readily available.

Theorem 4. The MLE $\hat{R}$ of R is given as

$$
\hat{R} = (\frac{\gamma\bar{X}}{\bar{Y}+\gamma\bar{X}})^\gamma. \tag{4}
$$

Proof. It is easy to see that the MLEs of λ and τ are

$$
\hat{\lambda} = \frac{1}{\bar{X}} \quad \text{and} \quad \hat{\tau} = \frac{\gamma}{\bar{Y}}
$$

respectively where

$$
\bar{X} = \frac{\sum_{i=1}^{m} X_i}{m} \quad \text{and} \quad \bar{Y} = \frac{\sum_{j=1}^{n} Y_j}{n}.
$$

Because of the invariance property of MLE we immediately obtain

$$
\hat{R} = (\frac{\hat{\tau}}{\hat{\lambda}+\hat{\tau}})^\gamma = \Big(\frac{\gamma/\bar{Y}}{(1/\bar{X})+(\gamma/\bar{Y})}\Big)^\gamma = (\frac{\gamma\bar{X}}{\bar{Y}+\gamma\bar{X}})^\gamma.
$$

Thus, the desired result follows. ■

3. Inference about R Based on Pivot

We will keep the same notation used in the previous section. For example, the MLE of λ is $\hat{\lambda} = 1/\bar{X}$ and the MLE of τ is $\hat{\tau} = \gamma/\bar{Y}$.

First let us search for a pivot.

Theorem 5. Define

$$Q = \frac{\lambda}{\tau} \cdot \frac{\gamma \bar{X}}{\bar{Y}} = \frac{1}{\rho} \frac{\gamma \bar{X}}{\bar{Y}}$$

where $\rho = \tau/\lambda$. Then Q is a pivotal quantity and has F-distribution with numerator degrees of freedom $2m$ and denominator degrees of freedom $2n\gamma$, i.e., $Q \sim F(2m,\ 2n\gamma)$.

Proof. Note that $2\lambda X_i \sim \chi^2(2)$, where $\chi^2(\nu)$ denote Chi-Square distribution with ν degrees of freedom, since X_i follows exponential ditribution with mean $1/\lambda$. Thus

$$2\lambda \sum_{i=1}^{m} X_i \sim \chi^2(2m).$$

Similarly, we see that

$$2\tau \sum_{j=1}^{n} Y_j \sim \chi^2(2n\gamma).$$

Therefore,

$$Q = \frac{1}{\rho} \frac{\gamma \bar{X}}{\bar{Y}} = \frac{\gamma\lambda \sum_{i=1}^{m} X_i/m}{\tau \sum_{j=1}^{n} Y_j/n} = \frac{2\lambda \sum_{i=1}^{m} X_i/2m}{2\tau \sum_{j=1}^{n} Y_j/2n\gamma} \stackrel{\mathrm{d}}{=} F(2m,\ 2n\gamma),$$

because of the independence of $\bar{X}$ and $\bar{Y}$. ■

On the basis of Theorem 5 the reliability parameter R can be expressed in terms of F-distribution.

Theorem 6. The reliability parameter R is given by $R = R(\rho) = 1 - F(\gamma/\rho;\ 2,\ 2\gamma)$ and strictly increase in ρ.

Proof. By the definition of the reliability parameter R it follows that

$$\begin{aligned} R &= P(X > Y) \\ &= P(\frac{X}{Y} > 1) \\ &= P(\frac{2\lambda X}{2\tau Y} > \frac{\lambda}{\tau}) \\ &= P(\frac{2\lambda X/2}{2\tau Y/2\gamma} > \frac{\gamma}{\rho}) \\ &= P(\xi > \frac{\gamma}{\rho}) \\ &= 1 - F(\frac{\gamma}{\rho}; 2, 2\gamma) \qquad (5) \end{aligned}$$

where random variable $\xi \sim F(2, 2\gamma)$ and $F(\ \cdot\ ; 2, 2\gamma)$ is its cumulative distribution function (CDF).

From Eq.(5) it is clear that R strictly increases in $\rho > 0$. ■

For any $R_0 \in (0, 1)$ it is easy to see that there exists a unique $\rho_0 > 0$ such that $R(\rho_0) = R_0$. Actually ρ_0 is determined by the equation

$$F(\gamma/\rho_0; 2, 2\gamma) = 1 - R_0 \qquad (6)$$

according to Theorem 6.

Now we can use Q for testing hypothesis about R.

Theorem 7. (Two-Sided Test) Consider testing hypothesis

$$H_0 : R = R_0 \qquad \text{vs.} \qquad H_a : R \neq R_0.$$

Then statistic $Q_0 = \gamma\bar{X}/\rho_0\bar{Y}$ can be used as test statistic with rejection region

$$R.R. = \{(\mathbf{X}, \mathbf{Y}) : Q_0 \leq F_{1-\alpha/2}(2m, 2n\gamma) \text{ or } \mathrm{Q_0 \geq F_{\alpha/2}(2m, 2n\gamma)}\}$$

where $\mathbf{X} = (X_1, ..., X_m)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$.

Proof. Note that the null hypothesis $R = R_0$ is equivalent to $H_0' : \rho = \rho_0$. From Theorem 5, it follows that $Q_0 \sim F(2m, 2n\gamma)$. Therefore, the size of this test is α. ■

Following the similar argument, we can obtain the two results below.

Theorem 8. The pivot Q_0 can be used for testing hypothesis

$$H_0 : R \leq R_0 \qquad \text{vs.} \qquad H_a : R > R_0.$$

The associated rejection region is

$$R.R. = \{(\mathbf{X}, \mathbf{Y}) : Q_0 \geq F_\alpha(2m, 2n\gamma)\}.$$

Proof. It is sufficient to show that the size of this test is α, i.e.,

$$\sup_{R \leq R_0} P_R(\text{rejecting } \mathrm{H_0}) = \alpha. \tag{7}$$

Then null hypothesis $R \leq R_0$ is equivalent to $H_0' : \rho \leq \rho_0$ by Theorem 6, where ρ_0 is determined by Eq.(6). Hence, it suffices to show that

$$\sup_{\rho \leq \rho_0} P_\rho(\text{rejecting H}_0') = \alpha. \tag{8}$$

We have

$$\begin{aligned} P_\rho(\text{rejecting H}_0') &= P_\rho\big(\frac{\gamma \bar{X}}{\rho_0 \bar{Y}} \geq F_\alpha(2m, 2n\gamma)\big) \\ &= P_\rho\big(\frac{\gamma \bar{X}}{\rho \bar{Y}} \geq \frac{\rho_0}{\rho} F_\alpha(2m, 2n\gamma)\big) \\ &= 1 - F\big(\frac{\rho_0}{\rho} F_\alpha(2m, 2n\gamma); 2m, 2n\gamma\big). \end{aligned}$$

Thus

$$\begin{aligned} &\sup_{\rho \leq \rho_0} P_\rho(\text{rejecting H}_0') \\ =&\sup_{\rho \leq \rho_0} \{1 - F\big(\frac{\rho_0}{\rho} F_\alpha(2m, 2n\gamma); 2m, 2n\gamma\big)\} \\ =&1 - F\big(F_\alpha(2m, 2n\gamma); 2m, 2n\gamma\big) \\ =&1 - (1 - \alpha) \\ =&\alpha. \end{aligned}$$

This ends the proof. ■

Theorem 9. Consider testing hypothesis

$$H_0 : R \geq R_0 \qquad \text{vs.} \qquad H_a : R < R_0.$$

The test that uses test statistic Q_0 along with rejection region

$$R.R. = \{(\mathbf{X}, \mathbf{Y}) : Q_0 \leq F_{1-\alpha}(2m, 2n\gamma)\}$$

has size α.

Proof. It is the same as Theorem 8. ■

At the end of this section we derive a procedure for constructing confidence interval of R can be obtained as

$$\Bigg(1 - F\big(\frac{\bar{Y}}{\bar{X}} F_{\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big), 1 - F\big(\frac{\bar{Y}}{\bar{X}} F_{1-\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big)\Bigg).$$

Proof. According to Eq.(5) the reliability parameter R is given as

$$R = 1 - F\big(\frac{\gamma}{\rho}; 2, 2\gamma\big).$$

It then follows from Theorem 5 that

$$\begin{aligned}
&1-\alpha \\
=&P\big(F_{1-\alpha/2}(2m, 2n\gamma) < Q < F_{\alpha/2}(2m, 2n\gamma)\big) \\
=&P\big(F_{1-\alpha/2}(2m, 2n\gamma) < \frac{1}{\rho}\frac{\gamma\bar{X}}{\bar{Y}} < F_{\alpha/2}(2m, 2n\gamma)\big) \\
=&P\big(\frac{\bar{Y}}{\bar{X}} F_{1-\alpha/2}(2m, 2n\gamma) < \frac{\gamma}{\rho} < \frac{\bar{Y}}{\bar{X}} F_{\alpha/2}(2m, 2n\gamma)\big) \\
=&P\Bigg(F\big(\frac{\bar{Y}}{\bar{X}} F_{1-\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big) < F\big(\frac{\gamma}{\rho}; 2, 2\gamma\big) < F\big(\frac{\bar{Y}}{\bar{X}} F_{\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big)\Bigg) \\
=&P\Bigg(1 - F\big(\frac{\bar{Y}}{\bar{X}} F_{\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big) < 1 - F\big(\frac{\gamma}{\rho}; 2, 2\gamma\big) < \\
&\quad 1 - F\big(\frac{\bar{Y}}{\bar{X}} F_{1-\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big)\Bigg) \\
=&P\Bigg(1 - F\big(\frac{\bar{Y}}{\bar{X}} F_{\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big) < R < 1 - F\big(\frac{\bar{Y}}{\bar{X}} F_{1-\alpha/2}(2m, 2n\gamma); 2, 2\gamma\big)\Bigg).
\end{aligned}$$

and consequently the desired result. ■

4. Numerical study

In this section we will conduct simulation studies in which the shape parameter of the gamma distribution is $\gamma = 1.25$ and $\gamma = 4.5$ in Example 1 and Example 2 respectively. Three cases of $R = 0.3$, 0.6 and 0.9 are considered. Using $N = 10000$ replications, in Tables 1-3 we list the average of $\tilde{R}$, $\widehat{MSE}(\tilde{R})$ which is the estimates of MSE of $\tilde{R}$, and the counterparts of the MLE $\hat{R}$ corresponding to various sample sizes $m = n = 5, 6, \cdots, 40$, for the case of $\gamma = 1.25$. Tables 4-6 show the results for the case of $\gamma = 4.5$. In these tables $e_{\hat{R},\tilde{R}} = \widehat{MSE}(\hat{R})/\widehat{MSE}(\tilde{R})$ is the relative efficiency.

Example 1.

In this example, shape parameter of gamma distribution $\gamma = 1.25$ is used. The scale parameters $\lambda = 2$ and $\tau = 1.235,\ 3.962,\ 22.742$ so that the reliability parameter $R = P(X > Y)$ equals to 0.3, 0.6, 0.9 accordingly based on Theorem 6. From the table 1-3, it is obvious that the average of the 10000 values of the UMVUE $\tilde{R}$ is very close to the true value of R, which is certainly consistent with the unbiasedness property of $\tilde{R}$, the estimate MSE of $\tilde{R}$ is only 0.02 even the sample sizes are only m=n=5 and it decreases to 0.0025 as the sample sizes increase to m=n=40. As far as the MLE $\hat{R}$, similar pattern is also observed. In addition, the bias of $\hat{R}$ is positive when $R = 0.3, 0.6$ and is negative when $R = 0.9$. Tables and Figures can be found in the Appendix.

Figure 1 is the plot of the average of the observed $N = 10000$ values of $\tilde{R}$ and the average of the observed $N = 10000$ values of $\hat{R}$ versus the true value of $R \in (0, 1)$.

Because of the unbiasedness property of $\tilde{R}$, the average of $\tilde{R}$ is almost a straight line from $(0,0)$ to $(1,1)$. The average of $\hat{R}$ has only very slight deviation from the true R.

For the case of $m = n = 5$ the solid line curve in Figure 2 shows the behavior of the difference between the average of the observed $N = 10000$ values of $\tilde{R}$ and the true value of $R \in (0,1)$. The difference is of the order of 10^{-3}. The dotted curve in Figure 2 shows the similar picture for the MLE $\hat{R}$ with error of order of 10^{-2}. And it is notable that the MLE $\hat{R}$ overestimates when R is small and underestimates when R is large.

For the case of $\gamma = 1.25$ Figure 3 shows the behavior of the Mean Squared Error of $\tilde{R}$ for sample sizes $m = n = 5, 10, 20, 30, 40$. It is obvious that the MSE of $\tilde{R}$ decrease while the sample sizes increase. And the decreasing speed is also decelerated with the increasing sample sizes. The behavior of the MSE of $\hat{R}$ follow the same pattern as shown in Figure 4.

Example 2.

In this second example, similar computation is done but with shape parameter of gamma distribution $\gamma = 4.5$ and the scale parameter $\lambda = 2$ and $\tau = 6.520,\ 16.638,\ 84.425$ according to $R = 0.3,\ 0.6,\ 0.9$ following Theorem 6. The estimates of R and MSE in the cases $R = 0.3,\ 0.6$ and 0.9 are listed in the Table 4-6. For $m = n = 5$ the average of the observed $N = 10000$ values of $\tilde{R}$ and the average of the observed $N = 10000$ values of $\hat{R}$ versus the true value of $R \in (0,1)$ is given in Figure 5. The difference of R and the average of $N = 10000$ simulated values are plotted in Figure 6. Figure 7 and Figure 8 show the behavior of the MSE of $\tilde{R}$ and $\hat{R}$ respectively.

From the above two examples, it is noticed that even though $\hat{R}$ is a biased estimator of R its MSE is about the same as that of $\tilde{R}$ and actually is smaller than the MSE of $\tilde{R}$. Considering this and the fact that the computation of $\tilde{R}$ based on Theorem 1, is more complicated than that of $\hat{R}$ based on Theorem 4, the MLE $\hat{R}$ is recommended for estimating the reliability parameter R.

REFERENCES

Baklizi, A., 2008a. Estimation of Pr(X < Y) using record values in the one and two parameter exponential distributions. Comm. Statist. - Theory Methods 37, 692-698.

Baklizi, A., 2008b. Likelihood and Bayesian estimation of Pr(X < Y) using lower record values from the generalized exponential distribution. Comput. Statist. Data Anal. 52, 3468-3473.

Church, J. D., Harris, B., 1970. The Estimation of Reliability from Stress-Strength Relationships. Technometrics 12, 49-54.

Eryilmaz, S., 2008a. A new perspective to stress-strength models. Ann. Inst. Statist. Math. DOI: 10.1017/s10463-008-0211-3.

Eryilmaz, S., 2008b. Consecutive k-out-of-n: G system in stress-strength setup. Comm. Statist. - Simul. Comput. 37, 579-589.

Eryilmaz, S., 2008c. Multivariate stress-strength reliability model and its evaluation for coherent structures. J.Multivar. Anal. 99, 1878-1887.

Eryilmaz, S., 2010. On system reliability in stress-strength setup. Statist. Probab. Lett. 80, 834-839.

Hauck, W.W., Hyslop, T., Anderson, S., 2000. Generalized treatment elects for clinical trials. Statist. Med. 19, 887-899.

Krishnamoorthy, K., Mukherjee, S., Guo, H., 2007. Inference on reliability in two-parameter exponential stress-strength model. Metrika 65, 261-273.

Kundu, D., Raqab, M.Z., 2009. Estimation of R = P(Y < X) for three-parameter Weibull distribution. Statist.Probab.Lett. 79, 1839-1846.

Kotz, S., Lumelsdii, Y., Pensky, M., 2003. The Stress-Strength Model and its Generalization. World Scientific, Singapore.

O'Brien, P.C., 1988. Comparing two samples: Extensions of the t, and log-rank tests. J. Amer. Statist. Assoc. 83, 52-61.

Reiser, B., Guttman, I., 1986. Statistical Inference for Pr(Y < X): The Normal Case. Technometrics 28, 253-257.

Rezaei, S., Tahmasbi, R., Mahmoodi, M., 2010. Estimation of P(Y < X) for generalized Pareto distribution. J. Statist. Plann. Inference 140, 480-494.

Reiser, B., 2000. Measuring the effectiveness of diagnostic markers in the presence of measurement error through the use of ROC curves. Statist. Med. 19, 2115-2129.

Saracoglu, B., Kaya, M.F., 2007. Maximum likelihood estimation and confidence intervals of system reliability for Gomperts distribution in stress-strength models. Selcuk J.Appl.Math. 8, 25-36.

Simonoff, J.S., Hochberg, Y., Reiser, B., 1986. Alternative estimation procedures for Pr(X < Y) in categorized data. Biometrics 42, 895-907.

Schwarz, L., Wearden, S., 1959. A distribution-free asymptotic method of estimating, testing and setting confidence limits to heritability. Biometrics 15, 227-235.

Wellek, S., 1993. Basing the analysis of comparative bioavailability trials on an individualized statistical definition of equivalence. Biometrical J. 35, 47-55.

APPENDIX

Table 1: $\gamma = 1.25,\ N = 10000,\ R = 0.3$

$R = 0.3$					
n=m	UMVUE		MLE		$e_{\hat{R},\tilde{R}}$
	$\bar{\tilde{R}}$	$\widehat{MSE}(\tilde{R})$	$\bar{\hat{R}}$	$\widehat{MSE}(\hat{R})$	
5	0.299369975	0.021279000	0.310376827	0.01829164	0.859609965
6	0.299523479	0.017760656	0.308708997	0.015683022	0.883020411
7	0.299292717	0.014429986	0.307318211	0.012968356	0.898708837
8	0.300009584	0.012872568	0.306994261	0.011736233	0.911724289
9	0.299753452	0.011397359	0.305995936	0.010497408	0.921038646
10	0.300468633	0.010144365	0.306096257	0.009427346	0.929317856
15	0.297289393	0.006532654	0.301154861	0.006203776	0.949656208
20	0.299774005	0.004927652	0.302650196	0.004749709	0.963888827
30	0.300264421	0.003291566	0.302169468	0.003214554	0.976603089
40	0.300526101	0.002447211	0.301964775	0.002404573	0.982577042

Table 2: $\gamma = 1.25,\ N = 10000,\ R = 0.6$

$R = 0.6$					
n=m	UMVUE		MLE		$e_{\hat{R},\tilde{R}}$
	$\bar{\tilde{R}}$	$\widehat{MSE}(\tilde{R})$	$\bar{\hat{R}}$	$\widehat{MSE}(\hat{R})$	
5	0.599437387	0.025788302	0.584773681	0.022158098	0.859230578
6	0.598951392	0.021275287	0.586602364	0.018790494	0.883207535
7	0.599970666	0.017585521	0.589170604	0.015815761	0.899362678
8	0.600670353	0.015258057	0.591108310	0.013906348	0.911410146
9	0.601236847	0.013549913	0.592677321	0.012478542	0.920931571
10	0.598439557	0.012276553	0.590800967	0.011435631	0.931501813
15	0.601083104	0.007743077	0.595824107	0.007374462	0.952394235
20	0.601455459	0.005807437	0.597482208	0.005594458	0.963326552
30	0.600660345	0.003859331	0.597988242	0.003765890	0.975788279
40	0.598942225	0.002933121	0.596957569	0.002886043	0.983949489

Table 3: $\gamma = 1.25,\ N = 10000,\ R = 0.9$

$R = 0.9$					
n=m	UMVUE		MLE		$e_{\hat{R},\tilde{R}}$
	$\bar{\tilde{R}}$	$\widehat{MSE}(\tilde{R})$	$\bar{\hat{R}}$	$\widehat{MSE}(\hat{R})$	
5	0.899956395	0.003996736	0.883237769	0.004917556	1.230393001
6	0.899237509	0.003331532	0.885480325	0.004000368	1.200759341
7	0.900913634	0.002561014	0.889323033	0.003006483	1.173942495
8	0.899956223	0.002219019	0.889836889	0.002584197	1.164567713
9	0.900111754	0.001893262	0.891174761	0.002175035	1.148829488
10	0.899542968	0.001704741	0.891516085	0.001946139	1.141603929
15	0.900531192	0.001072896	0.895301095	0.001167840	1.088493108
20	0.900265017	0.000796210	0.896371635	0.000850527	1.068218699
30	0.900390881	0.000503877	0.897815471	0.000526223	1.044348352
40	0.900002028	0.000389978	0.898075443	0.000403973	1.035885512

Table 4: $\gamma = 4.5,\ N = 10000,\ R = 0.3$

$R = 0.3$					
n=m	UMVUE		MLE		$e_{\hat{R},\tilde{R}}$
	$\bar{\tilde{R}}$	$\widehat{MSE}(\tilde{R})$	$\bar{\hat{R}}$	$\widehat{MSE}(\hat{R})$	
5	0.296872619	0.025408893	0.287073599	0.019896770	0.783063229
6	0.300113926	0.021019930	0.291196503	0.017083109	0.812710063
7	0.300667283	0.018255329	0.292798374	0.015255130	0.835651899
8	0.298696893	0.016043637	0.291807129	0.013724330	0.855437620
9	0.299821243	0.013995597	0.293504456	0.012166139	0.869283352
10	0.299779049	0.012428744	0.293990368	0.010948896	0.880933413
15	0.299262754	0.008531467	0.295251008	0.007840245	0.918979678
20	0.299880526	0.006128631	0.296777791	0.005748820	0.938026761
30	0.299364740	0.004188645	0.297256181	0.004014880	0.958515092
40	0.300307335	0.003101096	0.298688676	0.003002087	0.968072876

Table 5: $\gamma = 4.5,\ N = 10000,\ R = 0.6$

$R = 0.6$					
n=m	UMVUE		MLE		$e_{\hat{R},\tilde{R}}$
	$\bar{\tilde{R}}$	$\widehat{MSE}(\tilde{R})$	$\bar{\hat{R}}$	$\widehat{MSE}(\hat{R})$	
5	0.599389361	0.024387944	0.564472844	0.023398279	0.959419910
6	0.599688353	0.019630258	0.570183433	0.019129410	0.974485892
7	0.600578072	0.016886692	0.575131398	0.016559874	0.980646452
8	0.599963775	0.014397779	0.577551599	0.014264567	0.990747739
9	0.599991750	0.012655836	0.579975345	0.012571633	0.993346729
10	0.600430646	0.011223082	0.582347269	0.011178991	0.996071413
15	0.600478265	0.007443647	0.588302793	0.007449730	1.000817231
20	0.600086320	0.005312875	0.590917543	0.005334720	1.004111740
30	0.600607212	0.003530196	0.594446145	0.003536547	1.001799057
40	0.600846263	0.002605547	0.596235883	0.002608081	1.000972441

Table 6: $\gamma = 4.5,\ N = 10000,\ R = 0.9$

$R = 0.9$					
n=m	UMVUE		MLE		$e_{\hat{R},\tilde{R}}$
	$\bar{\tilde{R}}$	$\widehat{MSE}(\tilde{R})$	$\bar{\hat{R}}$	$\widehat{MSE}(\hat{R})$	
5	0.899390142	0.003219176	0.879422528	0.004501926	1.398471344
6	0.899063717	0.002563739	0.882758182	0.003463913	1.351117541
7	0.900024744	0.001992949	0.886356606	0.002590682	1.299923891
8	0.900258673	0.001690681	0.888467592	0.002135483	1.263090530
9	0.899443294	0.001464135	0.888995104	0.001825548	1.246843941
10	0.899677805	0.001296352	0.890367763	0.001580357	1.219080431
15	0.900546980	0.000773028	0.894525921	0.000879501	1.137734760
20	0.900092707	0.000575634	0.895614830	0.000637670	1.107769709
30	0.900036277	0.000376961	0.897085859	0.000403962	1.071627494
40	0.900065063	0.000286674	0.897866175	0.000301752	1.052595974

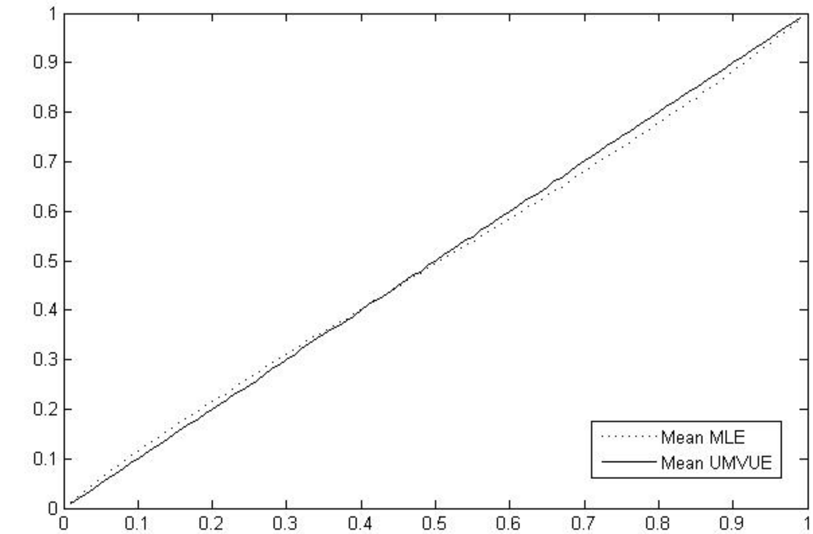

Figure 1: $\gamma = 1.25$, $m = n = 5$, $N = 10000$

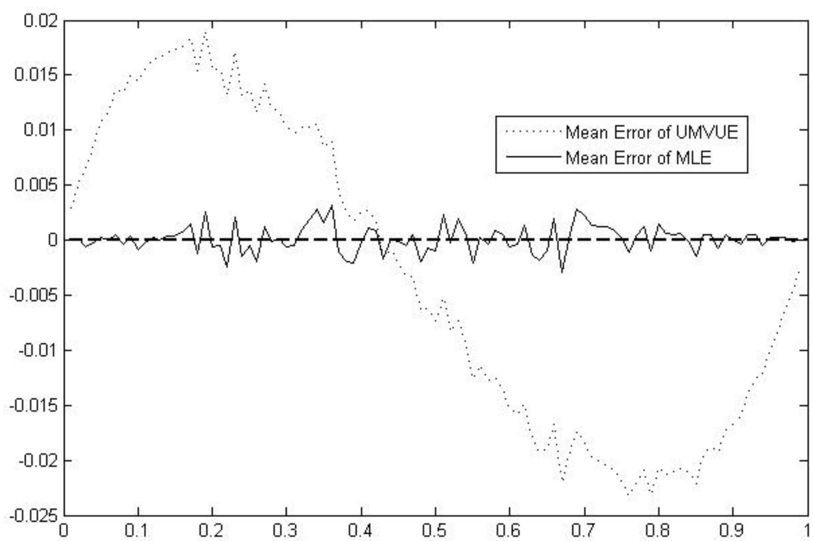

Figure 2: $\gamma = 1.25$, $m = n = 5$, $N = 10000$

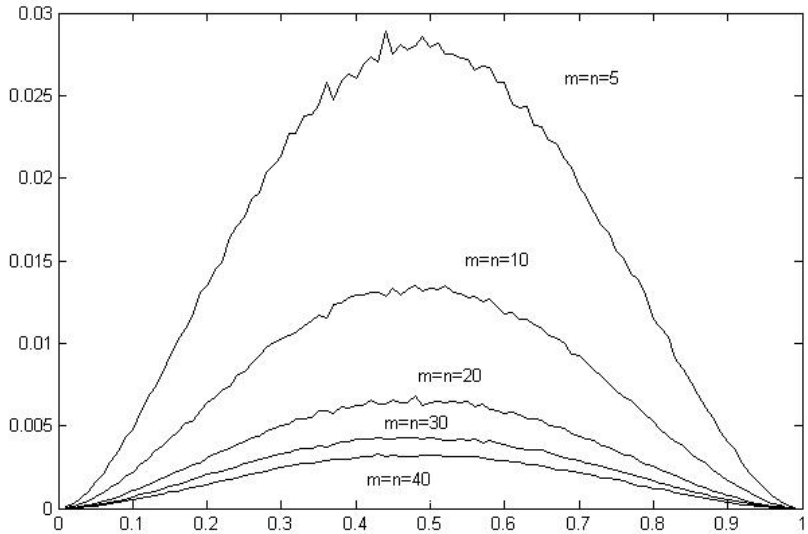

Figure 3: $\gamma = 1.25$, $N = 10000$, UMVUE

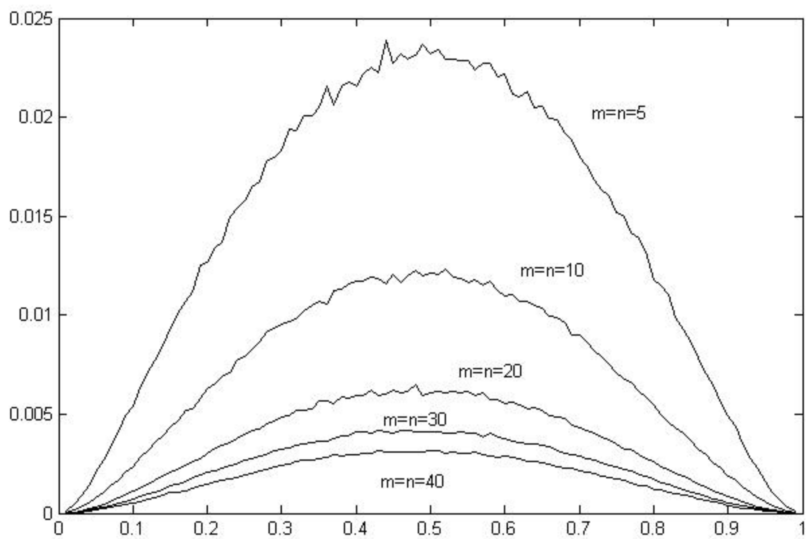

Figure 4: $\gamma = 1.25$, $N = 10000$, MLE

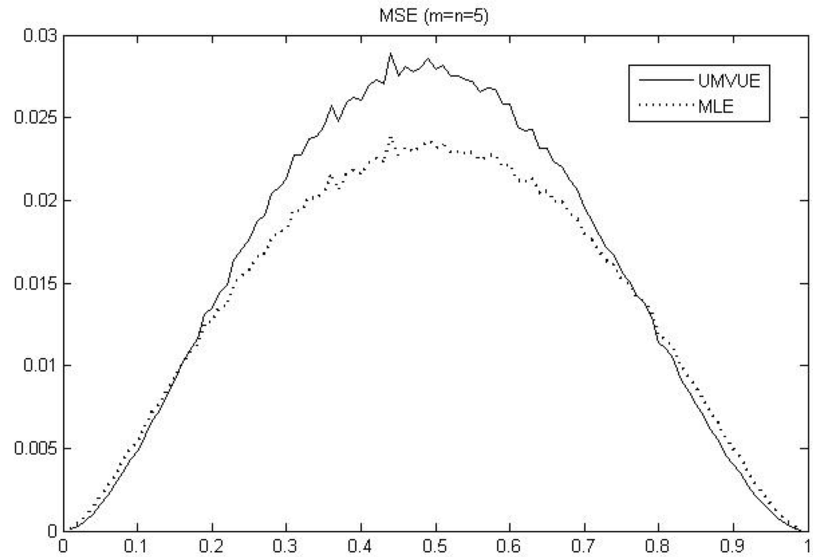

Figure 5: $\gamma = 1.25,\ N = 10000$

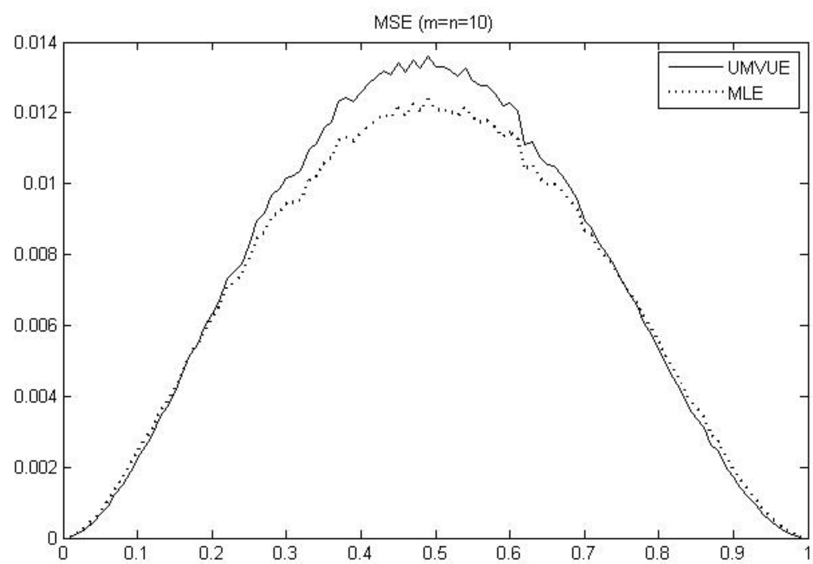

Figure 6: $\gamma = 1.25,\ N = 10000$

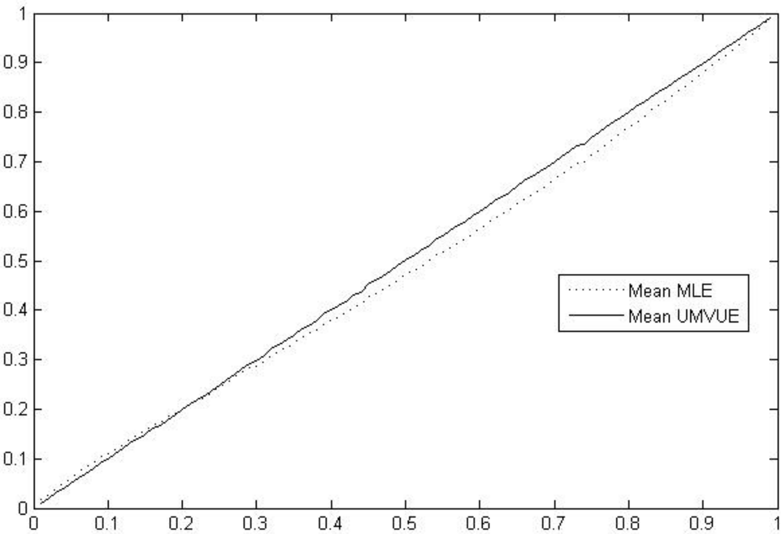

Figure 7: $\gamma = 4.5,\ m = n = 5,\ N = 10000$

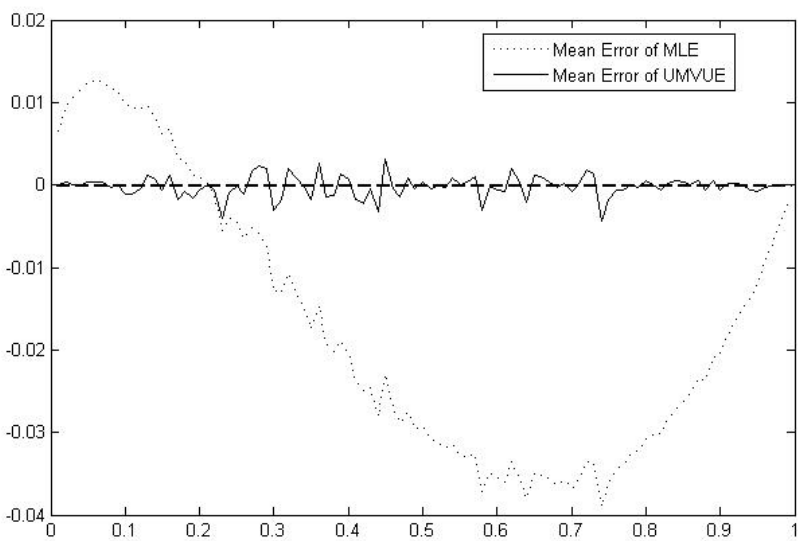

Figure 8: $\gamma = 4.5,\ m = n = 5,\ N = 10000$

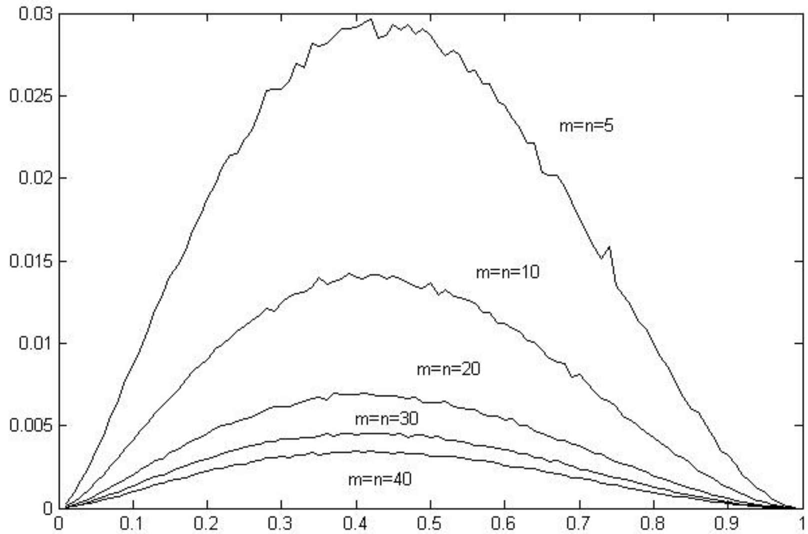

Figure 9: $\gamma = 4.5$, $N = 10000$, UMVUE

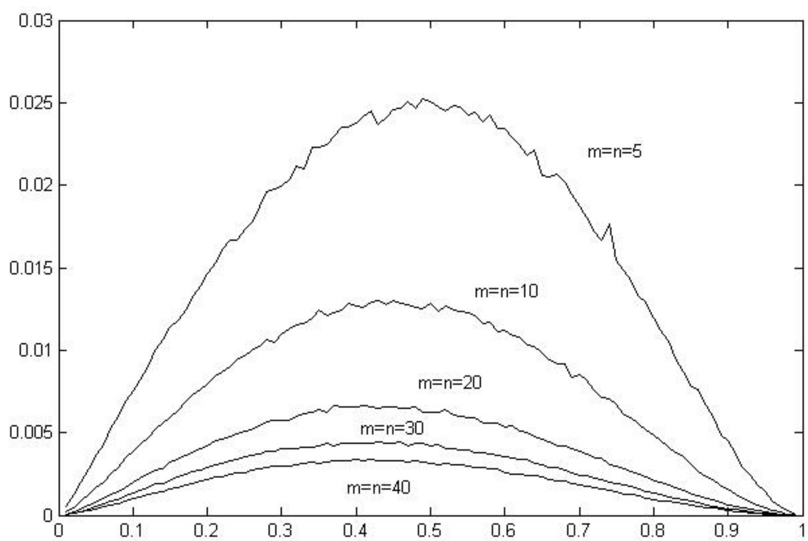

Figure 10: $\gamma = 4.5$, $N = 10000$, MLE

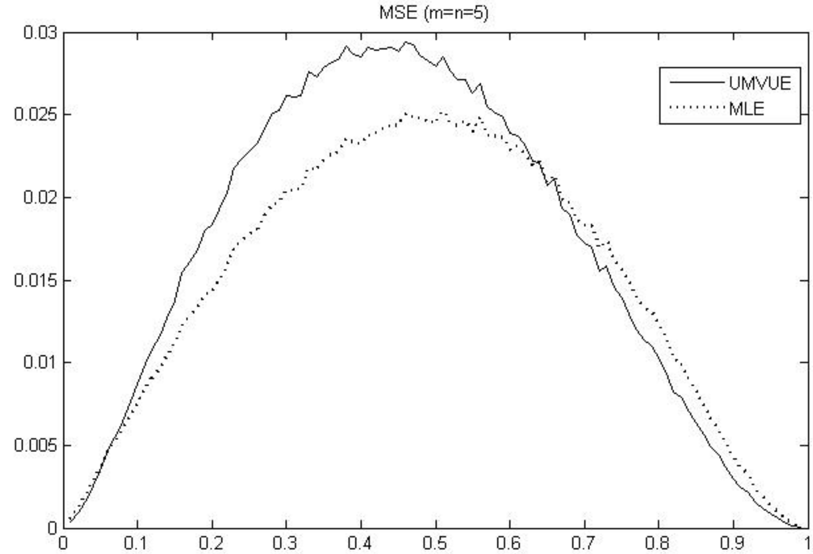

Figure 11: $\gamma = 4.5$, $N = 10000$

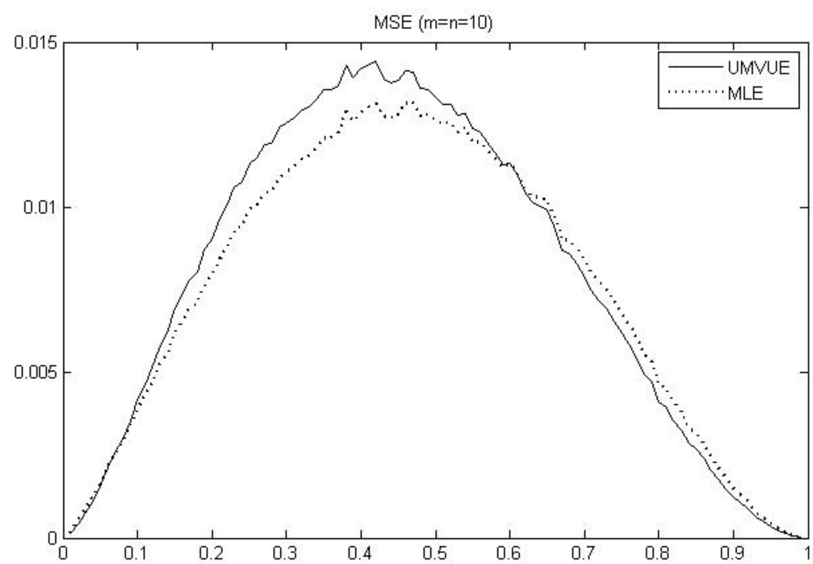

Figure 12: $\gamma = 4.5$, $N = 10000$