Analysis of Students' Misconceptions and Error Patterns in Mathematics: The Case of Fractions

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Abstract: This study analyzed three fifth grade students' misconceptions and error patterns when working with equivalence, addition and subtraction of fractions. The findings revealed that students used both conceptual and procedural knowledge to solve the problems. They used pictures, gave examples, and made connections to other mathematical concepts and to daily life topics. Error patterns found include using addition and subtraction of numerators and denominators, and finding the greatest common factor.

Understanding fractions is one of the foundational skills for more complex mathematics such as algebra. Often, students at all levels (elementary to college) struggle to compute (add, subtract, multiply and divide), compare and order fractions correctly. Ashlock (2006) wrote about the different error patterns students make in computation while Siegler, Fazio, Bailey, and Zhou (2013) explained how a deep understanding of fractions is important for more advanced mathematics, and how difficult it is for children to acquire this understanding. Jordan, Hansen, Fuchs, Siegler, Gersten and Micklos (2013) also found in their research that students often struggle with fractions due to different methods of instruction and/or skills students develop during their schooling. Thus, research in how students form these concepts and what error patterns they develop is important in order to create better opportunities for them to use fractions both in the classroom and outside of it.

Researchers (Ashlock, 2006; Voza, 2011) have found error patterns in fraction computation and suggest that appropriate tools, such as diagnostic interviews, be used to assess students' math understanding. This study analyzed how students used conceptual and procedural knowledge when working with fractions through the examination of student tasks and interviews. For the purpose of this study conceptual knowledge is defined as knowledge of math facts and properties that are recognized as being related in some way; and procedural knowledge is defined as the set of rules and algorithms used to solve math problems (Hiebert & Wearne, 1986).

Purpose of the Study and Research Questions

When students work out problems in mathematics they usually use a combination of conceptual and procedural knowledge. Thus, the purpose of the study was to analyze how students used conceptual and/or procedural knowledge to work out problems involving equivalence, addition and subtraction of fractions. These concepts are essential for understanding higher mathematics such as algebra as well as to understand daily tasks such as measuring during cooking (3/4 cup of sugar), telling time (a quarter past 10), or using money (half a dollar). By analyzing how students use their knowledge, either conceptual or procedural, or a combination of both, instruction can be improved to meet the needs of the students who create erroneous patterns in computation. In order to investigate these issues, the research questions that guided this study were: (1) What error patterns or common misconceptions do students portray when working with fractions? (2) When doing so, what type(s) of knowledge, conceptual and/or procedural, do students use to explain equivalence, addition, and subtraction of fractions?

Relevant Literature

In the past decade, the Trends in International Mathematics and Science Study (TIMSS) has conducted several studies in order to compare student achievement in fourth grade mathematics. There were three content domains covered in the assessment including number, geometry and measure, and data; out of which, number, including fraction concepts, was found to be one of the areas that needed improvement (Gonzales et. al, 2009). The TIMSS results showed that U.S. fourth graders performed better on the lower function cognitive domain so-called *knowing*[1] than on *applying*[2] and *reasoning*[3] domains. Furthermore, the National Council for Teachers of Mathematics (NCTM) has stressed the importance of reforming mathematics instruction by moving from teaching only-content-skills objectives to using problem solving, reasoning, communication, connections, and representation. The teaching of fractions, sometimes considered one of the most challenging concepts for students, has gone through this reform but students still struggle to understand them. By understanding the current mathematics curriculum and instructional practices teachers and researcher can investigate and understand how students create concepts and how they apply them.

Curriculum

Curriculum can be explained in terms of national standards and state standards in addition to their alignment to curriculum materials and assessment. The national standards are based on five content standards including number sense and operation, algebra, geometry, measurement, and data analysis and probability, and five process standards consisting of problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000). In an effort to improve student learning in mathematics, the standards require students to build new knowledge through solving problems, to apply their knowledge to new situations, and to monitor and reflect about their solutions (NCTM, 2000). NCTM (2000) states that students should be given opportunities to investigate problems, evaluate results, organize information, and communicate their findings. They should also be able to recognize, apply, and interpret what to do in each problem; and create a system of effective methods to solve math problems (NCTM, 2000).

In the specific topic of fractions, the National Council of Teachers of Mathematics states that students in third grade to fifth grade will be able to the following:

> "develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers; use models, benchmarks, and equivalent forms to judge the size of fractions; recognize and generate equivalent forms of commonly used fractions; develop and use strategies to estimate computations involving fractions in situations relevant to student's experience; and use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions" (NCTM, 2000, p.25).

[1] The cognitive domain of *knowing* addresses facts, procedures, and concepts that students need to know to function mathematically (Gonzales et. al, 2009).
[2] The cognitive domain of *applying* focuses on students' abilities to apply knowledge and conceptual understanding to solve problems or to answer questions (Gonzales et. al, 2009).
[3] The cognitive domain of r*easoning* requires higher order thinking skills and goes beyond the cognitive processes involved in solving routine problems to include unfamiliar situations, complex contexts, and multi-step problems (Gonzales et. al, 2009).

Fractions are first introduced as early as in second grade. At this level, students are asked to use concrete models to show the relationships between wholes and their parts (Florida State Standards, 2013). Students at the intermediate elementary grades (third to fifth grade) are required to use manipulatives, pictorial representations, and procedures to order fractions, to find equivalence, and to add and subtract fractions (Florida State Standards, 2013). Students are required to use a variety of materials and strategies to understand and apply fractions in order to construct conceptualizations that will serve as the foundation for more advanced mathematics. However, Charles (2009) argued that in reality most state standards communicate only mathematical skills or procedural fluency rather than conceptual understanding. Textbooks, being the primary curriculum material in math classrooms, tend to focus on the mastery of skills rather than in concept understanding. Hodges, Landry, and Cady (2009) claimed that conventional math textbooks usually provide a plethora of resources for teachers but are deficient on pedagogical approaches to promote student conceptual learning. Additionally, the need to align high-stakes assessment to the national and state mathematics curricula has had an impact on the instruction of mathematics.

Assessment is an integral part of curriculum and instruction. In traditional classrooms, tests were given at the end of a lesson, chapter, or unit in order to assess students' understanding of math concepts. Nowadays, assessment is as much a tool of final evaluation of student knowledge as well as an ongoing measurement of students' progress. By using questioning and listening, teachers can informally assess students' conceptual development (Borgioli, 2008). Teachers can also use observations, student journal entries, admit-exit slips (Altieri, 2009), small group responses, and activity sheets to evaluate students' understanding of a concept. Ashlock (2006) suggested that diagnostic interviews are also effective tools to assess students' math understanding.

Instruction

In analyzing instruction of the concept of fractions and student understanding and application, it can be observed that students' misconceptions lead to error patterns that students repeatedly use. Ashlock (2006) discussed how students make generalizations in areas such as part-whole relationship, equivalence, addition, subtraction, multiplication, and division of fractions that show error patterns. These mistakes are due in part to students using procedural knowledge rather than a combination of procedural and conceptual knowledge in order to solve fraction problems. Zambo (2008) added that general math misunderstandings are based on perceptual (visual or auditory), memory-based, or integrative (connecting abstract ideas) difficulties. Thus, instruction plays an important part in helping students make meaningful connections to create mathematical conceptions.

Misconceptions and Error Patterns in the Topic of Fractions

Ashlock (2006) explained in his book how students' misconceptions in the area of fractions lead to error patterns. When teachers understand what error patterns students use, they can develop an appropriate activity or lesson that re-teaches the concept successfully. Voza (2011) has researched numerical reasoning and thinking and has found that identifying the types of errors students make when working on a math problem is important for teachers to use the appropriate activities to reteach the concept. The author also argued that when teachers do not recognize the errors patterns students have, they create an unmotivated and discouraging environment for children (Voza, 2011). It is important for students to have conceptual and procedural understanding while working with fractions as to avoid confusing certain steps or operations when trying to solve problems.

Setting and Participants

This study was conducted within a fifth grade class of a southeastern suburban public elementary school during the winter of 2013. A total of 14 students completed the first part of the study which included a completion of a task with 12 fraction questions. From within the 14 students, 3 students were interviewed for the second part of the investigation. They were purposely chosen based on their answers to the 12 fraction questions. Both the class activity and the interviews took place at the school site.

Whole Class

The class that participated in the study had a total of 15 students. The class was described as “gifted” in which the students are receiving a fast-paced curriculum as they are advanced academically. However, the teacher explained that all the students had been placed in this class because they were advanced in the subject of Reading. The children performed on average when compared to other fifth graders in the area of mathematics. The majority of the students were also English Language Learners, but their ESOL (English for Students of Other Languages) levels were not disclosed by the teacher. Five visits were done to the classroom in order to explain to the students what the research study entailed, conduct the first part of the study in which the students completed a worksheet with 12 fraction questions, and finally to interview the students.

Individual Participants

For the second part of the study 3 students were interviewed, the participants were chosen by using purposeful sampling. The students’ task (12-question worksheet) were analyzed and 3 students who had errors in their answers were chosen to be interviewed.

Student #1: Cathy.

The first student who was interviewed was Cathy. She was a Hispanic fifth grade student, mostly quiet in class, raised her hand often to answer her teacher’s questions. Cathy was an English Language Learner who has lived in the U.S. for less than 2 years; only used English to explain her answers. Even when she was once whispering to herself to solve a question, she used English. She was eager to explain her thinking during the interview as she emphasized phrases such as “the higher number” to explain her fraction addition and subtraction answers. She used pictures of what she understood as equivalent fractions to explain what she had done in the first part of the worksheet (Equivalence of Fractions).

Student #2: Sergio.

The second student who was interviewed was Sergio. He was a Hispanic fifth grade student in the ESOL program. He has lived in the United States for about 4 years. He spoke fluent English. Sergio seemed a little nervous at the beginning of the interview. He gave shorter answers to interview questions as well as to follow up questions and to probes.

Student #3: Ivan.

The third student who was interviewed, Ivan, was also a Hispanic fifth grade student in the ESOL program. He has born in the United States; his first language was Spanish. He spoke fluent English and Spanish, and used some Spanish phrases when explaining his thinking. Ivan seemed comfortable during the interview. He gave long responses to the researcher’s questions, many followed by a story.

Description of Documents

For the purpose of investigating the error patterns and misconceptions students have when working with fractions, a worksheet with 12 questions was used. These were divided into three sections: (a) equivalence of fractions, (b) addition of fractions, and (c) subtraction of fractions. The 12 questions were open ended so the students could answer as they chose. Simple directions were given for each section.

For the first section the first 2 fractions were commonly used fractions and were given in simplified form: ½ and ¾. The last two fractions were not simplified: 3/9 and 10/15. For the second and third sections (addition and subtraction of fractions) 2 problems had fractions with equal denominators, and 2 problems had fractions with different denominators.

Methods

The study employed a qualitative research design. Qualitative research is described by Bogdan and Biklen (2007, p. 4) as having "actual settings as the direct source of data and the researcher [as] the key instrument." Commonly, instruments to collect data include observations, document analysis, and interviewing. This study used these three types of data collection methods. The class was observed once for 30 minutes while the students were completing the task.

The first task included a worksheet with 12 questions about equivalence of fractions, addition and subtraction of fractions. These 12 questions included a variety of problems which demonstrated students' understanding of the types of knowledge (conceptual, procedural or both) they used when working with fractions. These questions were open-ended questions and the students showed their work in the space provided. Subsequently, Ashlock's (2006) ideas were used when creating the task. Following his examples the addition and subtraction questions were written horizontally (e.g. 4/3 – 1/6). The questions also included same and different denominators.

When the students completed the task, 3 students were chosen from those who made mistakes in their answers. Each interview lasted approximately 1 hour. This was aimed to find what type(s) of knowledge students used when working with fractions: conceptual and/or procedural as well as to investigate patterns of misconceptions students had. Additionally, random students were asked to explain what they had done in specific questions. Interviewing the students was essential to this research study. Researchers have mentioned that interviews help in understanding important topics (Rubin & Rubin, 2012), in gaining insights into students' understanding of concepts and procedures, in learning how students communicate mathematical ideas, and in discovering students' dispositions towards math (Ashlock, 2006). In this study, the interviews helped in discovering how students think about and understand fractions and how they used both procedural and conceptual knowledge to make sense of this concept. The original interview protocol included 6 questions, an introduction section and closing remarks. After the first interview was transcribed and analyzed 6 more questions were added.

Data Analysis

After all the data was compiled, transcripts of all interviews were used to start analyzing the students' responses. The transcripts were coded and were used together with the task to investigate the students' understanding of fractions. The following sections present the findings for the class as a whole and for the three individual students who were interviewed

Whole Class

Equivalence of fractions. As the class completed the first activity which was the worksheet with 12 fraction questions there was silence. Then, suddenly the students started asking about the first question. The directions were not clear; they read "Write two fractions that are equivalent to the following fractions." The direction had to be read and explained to them. The term "equivalent" was explained. A student said "equivalent" meant an equal fraction so the directions were repeated using the word “equal” instead of “equivalent.” Then, there was silence again as the students stated working to find the equivalent fractions.

The majority of the students answered the 4 questions about equivalence correctly (10 students out of 14 total students). Only 2 students’ papers showed they had used pencil-and-paper multiplication to find the equivalent fractions. Another student made a computation error when dividing to find the second equivalent fraction to ¾, but was able to correct it when was asked to explain it. There were 2 students who made the same mistake on the last question of this section (d) 10/15. They added 5 to the numerator to find the denominator and so the equivalent fraction. These responses showed the use of procedural understanding. In addition, five students drew pictures. However, unless interviewed, it was difficult to understand what some students meant with certain pictures. This method (drawing pictures) is usually connected to conceptual understanding (Davis, 2000; Ashlock, 2006).

When random students were asked to explain what they had done for the first problem (equivalent fractions for 1/2) one student said “I found fractions that were half of the top” (meaning the numerator is half of the denominator). Also it was interesting to observe that most students found the first equivalent fraction and worked on from that one (multiplied or divided mostly) to find the second equivalent fraction instead of going back to the original fraction.

Addition of fractions. When working in solving the addition problems in this section, some students said they did not remember how to add fractions. Most students added the fractions with equal denominators correctly but did not find the least common denominator for unlike denominators. The observed error patterns included adding both numerators and denominators, keeping the same numerator and adding the denominators (3/4 + 3/8 = 3/12), writing the least denominator and adding the numerators (1/5 + 3/8 = 4/5).

Subtraction of fractions. During this section, some students mentioned again they did not remember how to subtract fractions. Fractions had not been taught this school year yet. Similar to the addition of fractions, when students subtracted fractions with like denominators their answers where correct, most mistakes were found with unlike denominators. Some unique error patterns were found in the class. The students either subtracted both numerators and both denominators; if the answer was zero for the denominators some wrote the zero and some did not. These responses show that students at this level still do not understand the concept of undefined fractions. For question (k) 4/3 – 1/6 = 3/6 a student said she “just picked the # that was [will give her] half, didn’t want it to be 1 once [the resulting fraction was] simplified.” She subtracted the numerators and chose the largest denominator out of the two given fractions. Another student wrote 4/3 – 1/6 = 3/0 because “since you can’t do 3-6, I put 0.” This student still does not understand negative numbers.

Student 1: Cathy

Equivalence of fractions. Cathy used pictures of fractional parts to explain how she found equivalent fractions. She defined concepts by giving examples of what she understands to be a fraction. This is evident when she said “that is a fraction because it represents 4 parts of 6 in total.” When explaining what is 2/3 she said “it will be 3 parts and I only color 2 of them.” She uses procedural understanding together with conceptualizations. For instance, when she was explaining how she had solved the fractions equivalent to ½ she said, “I was thinking [of] multiplying by 2 (procedural). 2 times 1 is 2, and 2 times 2 is 4, and that is a half because 2… because out of 4 …[draws four-part rectangle, shades 2 sections] …2 are colored so it’s a half, 2 and 2 is 4 (conceptual).” When using procedures to find the second equivalent fractions she multiplied her answer rather than going back to the original fraction. This may cause her to make errors with computation later on. Interestingly, she did not considered that her two equivalent answers for ¾, 6/8 and 15/20, were equivalent. Based on her procedural knowledge of multiplying numerator and denominator by the same number she stated that they are not equal because "6 times nothing… equals 15, it will have a remainder. And 8 times nothing equals 20, it will have a remainder too." When asked if these 2 fractions were equal to ¾ she answered positively based on the multiplication procedure.

Addition of fractions. When Cathy started explaining her thinking when she answered the second section, she said she added the numerators and found a number that was common to both. She used a number that was the largest factor for both denominators, or the greatest common factor. For example when she worked with ¾ + 3/8, she said the common denominator will be 4 because 4 is the greatest number both denominators have as factors. She used this same error pattern for like and unlike denominators.

Subtraction of fractions. To solve the subtraction problems she followed her same error pattern of finding the greatest common factor but subtracted instead. An example is as follows: to explain her answer to question (k) 4/3 – 1/6 = 3/3 "I did the same thing, I subtracted 4 minus 1 equals 3, and I found the highest number that will give me 6 and 3 so it will be 3 because 3 times 1 is 3 and 3 times 2 is 6."

In conclusion, she uses both conceptual and procedural knowledge to explain her thinking. The error patterns are present only when she uses procedures. She has conceptual understanding of what a fraction is and with more experiences with different representations (e.g. number lines, concrete manipulatives, comparisons) would be able to understand fractions more in depth.

Student 2: Sergio.

Equivalence of fractions. Sergio started out by using procedures to explain how he found the equivalent fractions. He said "So one half is…I know ½ is equal to two fourths 'cause I multiplied it. And then since one half is a half, I divided 16 by 2 and gave me 8." It is common for students to use this procedure (multiply or divide numerator and denominator by the same number) to find equivalent fractions (Ashlock, 2006). However as he continued his explanation he started thinking more conceptually. This is shown when he answered the second problem ¾, and said "I remember the quarters, the dollar and the quarters. And I knew ¾ is 75 hundreds [or 3 quarters]." By making connections to other concepts (e.g. money) the student was conceptualizing fractions and was making meaningful math connections to daily life concepts. At times he also worked form his answer to find the second equivalent fraction rather than going back to the given fraction.

Addition of fractions. He also started this section procedurally. His error pattern was consistent throughout this section as he added numerators and denominators for all his answers.

Subtraction of fractions. During this section Sergio followed the same patterns he used for addition. He subtracted the numerators and then he subtracted the denominators. In this section he wrote 3/0 and 2/0 for questions i and j. He did not understand yet the concept of undefined fractions. However, when asked to explain these answers he treated the answers as improper fractions and converted them into mixed numbers. He commented that the answer was "3 wholes, basically, because nothing goes into 0 but 0," so he treated the zero as a 1. For question k. 4/3 – 1/6, he switched the order and instead of subtracting 3 minus 6, he subtracted 6 minus 3. Again he used the pattern of subtracting numerators and then subtracting denominators to answer the questions.

In conclusion, at the beginning it seemed as if Sergio did not have a clear definition of a fraction as he mentioned that it can be something divided into equal and unequal parts. However, to clarify the unequal parts to a fraction, he explained that an improper fraction when converted into a mixed number can have unequal parts: a whole and another divided into 4 parts for 1 ¼, for instance. As he works throughout the worksheet he uses mostly procedural knowledge to answer the questions. He would benefit from adding and subtracting fractions vertically to find the least common denominator and by estimating answers before computing so than he can see if the answers make sense (Ashlock, 2006).

Student 3: Ivan

Equivalence of fractions. When Ivan starts explaining how he solved the equivalent fractions he started out by referring to something his teacher has taught him last year. Ashlock (2006) explained that students sometimes use an *external* figure as justification for their thinking being a textbook or authority figure. He described in his book about the two other types of schemes: *empirical*, observed when students use perception or concrete object to show their answer is right; and *analytical* in which the student uses counting strategies or cites mathematical relations (Ahlock, 2006). He stated that as student's thinking develops over time, they moved from external to a more empirical and analytical justifications. Interestingly though after citing his teacher, he used conceptualization of what ½ means to him. He said "pretty much all you have to do is find something that's half. If they give you one half think of other things that are halves like two fourths, 50 one-hundredths, 20 fifty [means 25/50, corrects himself later], like that." He also used pictures to while solving the questions while the other two students who were interviewed used pictures during the interview. This also shows conceptual understanding of fractions. However, when he solved the other questions about equivalence he used procedural knowledge. When explaining how he got 27/36 for ¾ he said "you could multiply, you can use multiplication like I did here, 3 times 9 is 27, 4 times 9, 36." He referred to his teacher again when solving the third question and when he used the procedure of multiplication. He simplified the last question first and then found 2 equivalent fractions from the simplified one.

Addition of fractions. To solve the addition problems Ivan added the numerators and the denominators. He changed his first answer to a mixed number (5/4 became 1 3/4). He also used a procedure to convert the improper fraction into a mixed number.

Subtraction of fractions. For the section in subtraction of fractions he followed the same error pattern he used with addition. He used pictures to explain his answer to problem k and referred to his teacher again by saying "my teacher, she taught it with a circle." Once he uses the circle he talks about degrees of an angle: 60˚, 90˚, 180˚ and combines it with fractions. He says "90 and 90 is 180, and that's half of a circle, and that's how you use the fractions incorporated with a circle." This is an example of conceptual understanding as he connects something learned in geometry and whole number addition to making sense of fractions.

Looking back at Ivan's responses it can be concluded that he used both procedural and conceptual understandings to solve the three sections. He used novel ideas and connections to other mathematics concepts to explain his thinking. He would benefit from using the number line to find fractions and to estimate answers before computing (Ashlock, 2006). Using concrete manipulatives and diagrams could also help him understand how to add and subtract fractions correctly.

The data collected during this research study included the class observation, students' worksheets, and transcripts.

Discussion

The research questions that guided this study were:

1. What error patterns or common misconceptions do students portray when working with fractions?
2. When doing so, what type(s) of knowledge, conceptual and/or procedural, do students use to explain equivalence, addition, and subtraction of fractions?

After analyzing the data, it can be concluded that the students used both conceptual and procedural understanding when working with equivalence, addition and subtraction of fractions. These results support the idea that students use both conceptual and procedural understanding to explain their mathematical ideas (Ashlock, 2006; Hiebert & Wearne, 1986). The students used pictures, gave examples, and made connections to other math concepts and to daily life topics that showed they had conceptualized somehow what a fraction is. When it came to addition and subtraction they reverted to their conceptualization of fractions and used pictures to offer explanations for what they did. Some of the procedures they used included computation of whole numbers: multiplying, dividing, adding and subtracting. Cathy used a very innovative way to find the least common denominator. When she did not remember how to find the addition and subtraction answers, she used the greatest common factor to find the common denominator. The researcher found out from the teacher that this concept had not being taught. It was interesting to see how this student was using this concept to make sense of the problems when she had not learned it formally yet.

Importance and Implications

The information learned from this study can guide teachers to enhance instruction. It is important to give students a variety of methods to discover what fractions are, how they can be computed, and how they can be applied to subjects other than math. These results can also be used to assist students in correcting their error patterns and misconceptions.

If this study is replicated, it may be helpful to have a variety of materials (e.g. number lines, fraction tiles, fraction circles, LEGOS, etc.) available for students to use if they chose to when answering the questions. Also the worksheet directions can include a statement saying "Show your work" to remind the children to use the space provided to draw or replicate what they are doing mentally. This may help in understanding what some students who were not interviewed where thinking about.

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